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New Topologies on the Rings of Continuous Functions

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Abstract. Two new topologies are defined on $C(X)$. These topologies make $C(X)$ to be a zero-dimensional (completely regular) Hausdorff space. $C(X)$ endowed by these topologies is denoted by $C_o(X)$ and $C_{o^{-1}}(X)$. The relations between X, $C_o(X)$ and $C_{o^{-1}}(X)$ are studied and closedness of z-ideals and maximal ideals are investigated in $C_o(X)$ and $C_{o^{-1}}(X)$.

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1. Introduction

In this paper, X assumed to be completely regular Hausdorff space and $C(X)\,(C^*(X))$ stands for the ring of all real valued (bounded) continuous) functions on X. Whenever $C(X) = C^*(X)$, we call X a pseudocompact space. An ideal I in $C(X)$ is said to be a z-ideal if $Z(f) \subseteq Z(g)$, $f \in I$ and $g \in C(X)$ imply that $g \in I$, where $Z(f) = \{x \in X : f(x) = 0\}$. Equivalentely, I is a z-ideal if $M_f \subseteq I$ for each $f \in I$, where M_f is intersection of all maximal ideals containing f, see [2] and [5,4]. Therefore every maximal ideal is a z-ideal. A space X is called a P-space if every G_δ-set (every zero set) in X is open and it is called zero-dimensional if it contains a base of closed-open sets.
In this article we define two topologies on $C(X)$ and $C(X)$ endowed by these topologies denote by $C_o(X)$ and $C_{o^{-1}}(X)$. We show that these

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spaces are Hausdorff, completely regular and zero-dimensional spaces. We study the relations between topological properties of the space X, $C_o(X)$ and $C_{o^{-1}}(X)$. For example we have shown that X is a P-space if and only if $C_o(X)$ is discrete, and X is pseudocompact if and only if the set of units of $C(X)$ (those members u of $C(X)$ with $Z(u) = \emptyset$) is a discrete subspace of $C_{o^{-1}}(X)$. Finally we have investigated the closed ideals of $C_o(X)$ and $C_{o^{-1}}(X)$ and we observed that z-ideals and maximal ideals are closed in $C_o(X)$ and z-ideals are open in $C_{o^{-1}}(X)$ as well. We have also observed that real maximal ideals are closed in $C_{o^{-1}}(X)$ and it turns out that whenever X is pseudocompact then every maximal ideals is closed in $C_{o^{-1}}(X)$ and whenever X is normal, the converse is also true. For the definition of real maximal ideals and undefined terms and notations, the reader is refered to [4].

2. $C_o(X)$ and $C_{o^{-1}}(X)$

Several topologies are defined on $C(X)$ and are studied by topologists, such as pointwise convergence which $C(X)$ considered as the subspace of $\mathbb{R}^X$ with product topology [1], compact open topology or uniform topology [6], m-topology which is finer than uniform topology [4] and [5] and many other topologies on $C(X)$, for example see [3]. Here we introduce two new topologies on $C(X)$.

For each $f \in C(X)$ and each open subset G in X, such that $Z(f) \subseteq G$, we define

$$B(f,G) = \{g \in C(X) : G^c_f \subseteq Z(f-g)\},$$

where $G^c_f = Z(f) \cup G^c$.

It is evident that the collection $\{B(f,G) : G \text{ is open in } X, \text{and } Z(f) \subseteq G\}$ is the base for the neighborhood system at f, for each $f \in C(X)$. In fact $f \in B(f,G)$, for all open set G which $Z(f) \subseteq G$, and $B(f, G \cap H) \subseteq B(f,G) \cap B(f,H)$, for all open sets G, H such that $Z(f) \subseteq G$ and $Z(f) \subseteq H$. Finally for open set G that $Z(f) \subseteq G$, whenever $g \in B(f,G)$, then $B(g,G) \subseteq B(f,G)$. We call the topology generated by this base, open-topology and $C(X)$ endowed with this topology denotes by $C_o(X)$. To introduce another topology on $C(X)$, let r be a positive rational number, and $f \in C(X)$. We consider the set $G_{r,f} = f^{-1}(\,(-r,+r)\,)$, and define

$$B(f, G_{r,f}) = \{g \in C(X) : G^c_{r_f} \subseteq Z(f-g)\},$$

where $G^c_{r_f} = Z(f) \cup G^c_{r,f}$.
The collection $\{B(f, G_{r,f}) : r \in \mathbb{Q}^+\}$ is also the base for neighborhood system at f, for each $f \in C(X)$. In fact $f \in B(f, G_{r,f})$, for all $r \in \mathbb{Q}^+$, $B(f, G_{r,f}) \cap B(f, G_{s,f}) = B(f, G_{r,f})$, for all $r, s \in \mathbb{Q}^+$ such that $r \leqslant s$, and finally for $r \in \mathbb{Q}^+$, whenever $g \in B(f, G_{r,f})$, then $B(g, G_{r,g}) \subseteq B(f, G_{r,f})$, for in this case $G^c_{r,f} \subseteq G^c_{r,g}$. We call the topology generated by this base, invers open-topology and $C(X)$ endowed with this topology denotes by $C_{o^{-1}}(X)$.

Proposition 2.1. *The following statements hold:*
(a) $C_o(X)$ *and* $C_{o^{-1}}(X)$ *are Hausdorff spaces.*
(b) $C_o(X)$ *and* $C_{o^{-1}}(X)$ *are zero-dimensional spaces.*
(c) $C_o(X)$ *and* $C_{o^{-1}}(X)$ *are completely regular spaces.*

Proof. We prove the properties for $C_{o^{-1}}(X)$, the proof for $C_o(X)$ is similar. To prove (**a**) let $f, g \in C(X)$ and $f \neq g$. There exists $x_0 \in X$, such that $f(x_0) \neq g(x_0)$. Now consider three cases:

Case 1: $x_0 \notin Z(f) \cup Z(g)$. Then there exists $i \in \mathbb{Q}^+$ such that $x_0 \notin f^{-1}(\,(-i,i)\,)$ and $x_0 \notin g^{-1}(\,(-i,i)\,)$. Hence $x_0 \in G^c_{i_f} \cap G^c_{i_g}$, and therefore $B(f, G_{i,f}) \cap B(g, G_{i,g}) = \phi$.

Case 2: $x_0 \in Z(g) \backslash Z(f)$. Then there exists $i \in \mathbb{Q}^+$ such that $x_0 \notin f^{-1}(\,(-i,i)\,)$. Hence $x_0 \in G^c_{i_f} \cap G^c_{i_g}$, and therefore $B(f, G_{i,f}) \cap B(g, G_{i,g}) = \phi$.

Case 3: $x_0 \in Z(f) \backslash Z(g)$. This is similar to case 2.

To prove **(b)**, it is sufficient to show that $B(f, G_{r,f})$ is closed, for all $f \in C(X)$ and $r \in \mathbb{Q}^+$. Let $g \notin B(f, G_{r,f})$, then there exists $x_0 \in G^c_{r_f}$ such that $g(x_0) \neq f(x_0)$. Now consider two cases:

Case 1: $x_0 \notin Z(g)$. Then there exists $i \in \mathbb{Q}^+$ such that $x_0 \notin g^{-1}(\,(-i,i)\,)$. Hence $x_0 \in G^c_{r_f} \cap G^c_{i_g}$, and therefore $B(g, G_{i,g}) \subseteq B(f, G_{r,f})^c$.

Case 2: $x_0 \in Z(g)$. This implise that $x_0 \in G^c_{r_f} \cap G^c_{r_g}$ and therefore $B(g, G_{r,g}) \subseteq B(f, G_{r,f})^c$.

Finally it is clear that part (b) implise part(c). □

One of the our goals is to find the relationship between topological structures of the spaces X, $C_o(X)$ and $C_{o^{-1}}(X)$. For this purpose, we give the following propositions.

Proposition 2.2. *X is a P-space if and only if $C_o(X)$ is discrete.*

Proof. Let X be a P-space and $f \in C(X)$. Then $Z(f)$ is an open set in X. Take $G = Z(f)$, hence $G^c_f = Z(f) \cup Z(f)^c = X$. Therefore $B(f,G) = \{f\}$ and this means that $C_o(X)$ is discrete.
Conversely, suppose that $C_o(X)$ is discrete and $Z(f)$ is a zeroset in X. Then there exists open subset G of X such that $Z(f) \subseteq G$ and $B(f,G) = \{f\}$. We claim that $Z(f) = G$. For the otherwise, there exists $t \in G \backslash Z(f)$. Hence $t \notin G^c_f$. But G^c_f is closed and X is a completely regular space, therefore there exists $g \in C(X)$ such that $g(t) = 0$ and $g(G^c_f) = \{1\}$. Then $G^c_f \subseteq Z(fg - f)$, thus $fg \in B(f,G)$. But $fg \neq f$ and this is a contradiction. □

Proposition 2.3. *X is connected if and only if every nonzero isolated point in $C_o(X)$ is a unit in $C_o(X)$.*

Proof. Let X is connected and $f \in C_o(X)$ is a nonzero isolated point. Then $Z(f)$ is open in $C_o(X)$, by the proof of Proposition 2.2. Thus $Z(f)$ is an open-closed subset of X, therefore $Z(f) = \phi$, i.e., f is a uint.
Conversely, suppose that f is a idempotent of $C(X)$. Then $Z(f)$ is an open subset in X, for $Z(f) = f^{-1}((-1,+1))$. Hence f is is isolated in $C_o(X)$. Therefore $f = 0$ or $f = 1$, i.e., X is connected, by [4]. □

In the following proposition, $U(X)$ is the set of all units of $C(X)$.

Proposition 2.4. *The following statements hold:*

(a) *X is pseudocompact space if and only if $U(X)$ is a discrete subspace of $C_{o^{-1}}(X)$.*

(b) *X is finite if and only if $C_{o^{-1}}(X)$ is discrete.*

Proof. Let X be pseudocompact and $f \in U(X)$. Then there exists $i \in \mathbb{Q}^+$ such that $|f(x)| > i$, for all $x \in X$. We take $G_{i,f} = f^{-1}((-i,i))$, hence $G^c_{i_f} = X$. Therefore $B(f, G_{i,f}) = \{f\}$, i.e., f is an isolated point.

Conversely, suppose that $f \in C(X)$ and $g = \frac{1}{|f|+1}$, so g is uint. Hence there exists $i \in \mathbb{Q}^+$ such that $B(g, G_{i,g}) \cap U(X) = \{g\}$, for $U(X)$ is a discrete subspace of $C_{o^{-1}}(X)$. The function h defined by

$$h(x) = \begin{cases} g(x) & G^c_{i_g} \\ i & o.w. \end{cases}$$

is continuous and clearly $h \in B(g, G_{i,g}) \cap U(X)$. Hence $h = g$ and this means that g is bounded away from zero. So f is bounded.
To prove (b), we let $X = \{x_1, x_2, ..., x_n\}$ and $f \in C(X)$. If $f = 0$, then it is clearly that f is an isolated point, for $B(0, G_{r,0}) = \{0\}$, for all $r \in \mathbb{Q}^+$. But if $f \neq 0$, then there exists $r \in \mathbb{Q}^+$ such that $r < \text{Min}\{|f(x_i)| : x_i \in Coz(f)\}$. Hence $G^c_{r_f} = X$, so $B(f, G_{r,f}) = \{f\}$.
Conversely, Suppose that $C_{o^{-1}}(X)$ is discrete. Then $C_o(X)$ is also discrete, for $C_o(X)$ is finer than $C_{o^{-1}}(X)$. Hence X is a P-space, by Proposition 2.2. On the other hand X is pseudocompact by (a), therefore X must be finite. □

It is not hard to show that whenever X is countably compact, then $C_o(X) = C_{o^{-1}}(X)$ and whenever $C_o(X) = C_{o^{-1}}(X)$, then X is pseudocompact. The next proposition provides necessary and sufficient condition for the concidence of two spaces. At first, we define strongly pseudocompact space.

Definition 2.5. *A topological space X is strongly pseudocompact if for every closed subset $F \subseteq X$ and for every $f \in C(X)$, whenever $f|_F$ is unit in $C(F)$, then $f|_F$ is bounded away from zero.*

Clearly, every countably compact space is a strongly pseudocompact space and every strongly pseudocompact space is a pseudocompact space.

Proposition 2.6. *A topological space X is strongly pseudocompact if and only if $C_o(X) = C_{o^{-1}}(X)$.*

Proof. Let X be strongly pseudocompact and $B(f, U)$ be a nhood base at $f \in C_o(X)$, where U is an open subset in X such that $Z(f) \subseteq U$. If $g \in B(f, U)$, then $g|_{U^c}$ is unit in $C(U^c)$, for $Z(g) \subseteq U$. Hence $g|_{U^c}$ is bounded away from zero, i.e., there exists $i \in \mathbb{Q}^+$ such that $|g(x)| > i$,

for each $x \in U^c$. Take $G_{i,g} = g^{-1}((-i,i))$, then $U^c \subseteq G_{i,g}$ and $Z(f) \subseteq Z(g)$, hence we have $B(g, G_{i,g}) \subseteq B(f,U)$, therefore $B(f,U)$ is open in $C_{o^{-1}}(X)$. This means that $C_o(X) = C_{o^{-1}}(X)$.
Conversely, suppose that $F \subseteq X$ is closed and $f \in C(X)$ such that $f|_F$ is uinit element in $C(F)$. We consider the nhood base $B(f, F^c)$ at $f \in C_o(X)$ (note that $Z(f) \subseteq F^c$). Then there exists $i \in \mathbb{Q}^+$ such that $B(f, G_{i,f}) \subseteq B(f, F^c)$, for $C_o(X) = C_{o^{-1}}(X)$. Now we have $G_{i,f} \subseteq F^c$, for if $x_0 \in G_{i,f} \backslash F^c$, then there exists $h \in C(X)$ such that $h(G^c_{i,f}) = \{1\}$ and $h(x_0) = 0$. Hence $fh \in B(f, G_{i,f})$, but $fh \notin B(f, F^c)$, a contradiction. Therefore $F \subseteq G^c_{i,f}$ and this means that $|f(x)| \geqslant r$, for all $x \in F$, i.e., $f|_F$ is bounded away from zero. □

3. Maximal Ideals in $C_o(X)$ and $C_{o^{-1}}(X)$

We know that maximal ideals are closed in $C(X)$ with $m-$topology $(C_m(X))$, see 2N in [4]. In this section we investigate the closedness maximal ideals in $C_o(X)$ and $C_{o^{-1}}(X)$, and we will observe that the maximal ideals in $C_o(X)$ and the real maximal ideals in $C_{o^{-1}}(X)$ are closed. But at first, in the next proposition we show that maximal ideals are also open.

Proposition 3.1. *Every $z-$ideal is open in $C_o(X)$ and in $C_{o^{-1}}(X)$.*

Proof. Let I be a $z-$ideal of $C(X)$ and $f \in I$. We show that $B(f,X) \subseteq I$. In fact if $g \in B(f,X)$, then $Z(f) \subseteq Z(f-g)$ and hence $Z(f) \subseteq Z(g)$, therefore $g \in I$, for I is $z-$ideal. Thus I is an open subset in $C_o(X)$. Similarly, I is open in $C_{o^{-1}}(X)$. □

Proposition 3.2. *Every maximal ideal is closed in $C_o(X)$.*

Proof. Let M be a maximal ideal of $C_o(X)$ and $g \in \mathrm{cl}_o M \backslash M$, where cl_o means the closure with respect to the topology of $C_o(X)$. Then there exists $k \in M$ such that $Z(k) \cap Z(g) = \emptyset$, by Theorem 2.6 in [4], and hence $Z(g) \subseteq Coz(k) = X \backslash Z(k)$. Now we consider nhood base $B(g, Coz(k))$ at g. Clearly $B(g, Coz(k)) \cap M \neq \emptyset$, for $g \in \mathrm{cl}_o M$. So there exists $h \in M$ such that $Coz(k)^c_g \subseteq Z(g-h)$, hence $Z(g) \cup Z(k) \subseteq Z(g-h)$ and therefore $Z(k) \cap Z(h) = \emptyset$. This is a contradiction, because $k, h \in M$. $\square$

Proposition 3.2. shows that every maximal ideal in $C_o(X)$ is closed, however maximal ideals are not necessarily closed in $C_{o^{-1}}(X)$. But real maximal ideals are closed in $C_{o^{-1}}(X)$.

Proposition 3.3. *Every real maximal ideal is closed in $C_{o^{-1}}(X)$.*

Proof. Let M be a real maximal ideal in $C_{o^{-1}}(X)$ and $f \in \mathrm{cl}_{o^{-1}} M$, where $\mathrm{cl}_{o^{-1}}$ means the closure with respect to the topology of $C_{o^{-1}}(X)$. Consider $G_{\frac{1}{n},f} = f^{-1}((-\frac{1}{n}, \frac{1}{n}))$, for all $n \in \mathbb{N}$, then there exists $g_n \in C(X)$ such that $g_n \in B(f, G_{\frac{1}{n},f}) \cap M$, for all $n \in \mathbb{N}$. Since M is a real maximal ideal, $\bigcap_{n\in\mathbb{N}} Z(g_n) \in Z[M]$ by Theorem 5.14 in [4], hence there exists $l \in M$ such that $Z(l) = \bigcap_{n\in\mathbb{N}} Z(g_n)$. Now we claim that $Z(l) \subseteq Z(f)$. In fact if $x_0 \in Z(l) - Z(f)$ then $f(x_0) \neq 0$, hence there exists $n_0 \in \mathbb{N}$ such that $|f(x_0)| > \frac{1}{n_0}$ and therefore $x_0 \in G^c_{\frac{1}{n}f}$. Since $g_{n_0} \in B(f, G_{\frac{1}{n_0},f})$, $x_0 \in Z(f - g_{n_0})$. So $x_0 \in Z(f)$, a contradiction. But M is a $z-$ideal, then $f \in M$ and hence M is closed. $\square$

Corollary 3.4. *If X is pseudocompact, then every maximal ideal in $C_{o^{-1}}(X)$ is closed.*

Now it is natural to ask that " is the converse of the above result true?" The next proposition shows that the answer is positive, whenever X is normal or a P-space. In the proof of this proposition we have used the notation $Neg(f) = \{x \in X : f(x) < 0\}$, $f \in C(X)$. We could not yet setteld this question in general.

Proposition 3.5. *The following statements hold:*

(a) *If X is normal, then hyper real maximal ideals in $C_{o^{-1}}(X)$ are not closed.*

(b) *If X is P-space, then hyper real maximal ideals in $C_{o^{-1}}(X)$ are not closed.*

Proof. Let X be normal and M be a hyper real maximal ideal in $C_{o^{-1}}(X)$. Then there exists $g \in C(X)$ such that Mg is infinitely small, see Theorem 5.6 in [4]. This means that $g \notin M$ and $M|g| < \frac{1}{n}$, for all

$n \in \mathbb{N}$. Hence there exists $h_n \in M$ such that $Z(h_n) \subseteq \mathrm{Neg}(|g| - \frac{1}{n})$, for all $n \in \mathbb{N}$, by Theorem 5.4 in [4]. We consider the continuous functions $k_n : X \backslash [\mathrm{Neg}(|g| - \frac{1}{n})] \to \mathbb{R}$, defined by $k_n(x) = \frac{1}{h_n(x)}$, for all $n \in \mathbb{N}$. Since X is normal and $X \backslash [\mathrm{Neg}(|g| - \frac{1}{n})]$ is a closed subset in X, there exists $\tilde{k_n} \in C(X)$ such that $\tilde{k}_n|_{X \backslash [\mathrm{Neg}(|g| - \frac{1}{n})]} = k_n$. Now we see that $gh_n\tilde{k}_n \in M \cap B(g, G_{g, \frac{1}{n}})$, for all $n \in \mathbb{N}$, hence $g \in \mathrm{cl}_{o^{-1}} M \backslash M$ and therefore M is not closed. Part (b) will be proved by a similar method. □

Corollary 3.6. (a) *If X is a normal space , then X is pseudocompat if and only if every maximal ideal in $C_{o^{-1}}(X)$ is closed.*
(b) *If X is P-space, then every maximal ideal in $C_{o^{-1}}(X)$ is closed if and only if X is finite.*

References

[1] A. V. Arkhangelskii, Cp-theory, in: Recent Progress in Topology, *North-Holland, Amsterdam,* (1992), 1-56.

[2] F. Azarpanah and R. Mohamadian, $\sqrt{z}$-ideals and $\sqrt{z^0}$-ideals in $C(X)$, *Acta Mathematica Sinica, English Series*, 23(6) (2007), 989-996.

[3] G. Dimaio, L. Hola, D. Holy, and D. McCoy, Topology on the space of continuous functions, *Topology Appl.,* 86 (1998), 105-122.

[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, springer, 1976.

[5] E. Hewitt, Rings of real-valued continuous functions I, *Trans. Amer. Math. Soc.,* 48(64) (1948), 54-99.

[6] J. R. Munkres, *Topology, a First Course*, Prentic-Hall, 1974.

[7] S. Willard, *General Topology*, Reading, Massachusetts, Addison-Wesley, 1970.

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