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Some Cohomological Properties of Banach Algebras

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Abstract. In this manuscript, we investigate and study some cohomological properties of Banach algebras. Let A be a Banach algebra with a bounded left approximate identity, and let B be a Banach $A - bimodule$. We show that if AB^{**} and $B^{**}A$ are subset of B, then $H^1(A, B^{(2n+1)}) = 0$ for all $n \geq 0$, whenever $H^1(A, B^*) = 0$.

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1 Introduction

Let B be a Banach $A-bimodule$. A derivation from A into B is a bounded linear mapping $D : A \to B$ such that,

$$D(xy) = xD(y) + D(x)y, \quad (x, y \in A).$$

The space of continuous derivations from A into B is denoted by $Z^1(A, B)$. Easy example of derivations are the inner derivations, which are given for each $b \in B$ by

$$\delta_b(a) = ab - ba, \quad (a \in A).$$

The space of inner derivations from A into B is denoted by $N^1(A, B)$. The first cohomology group of A with coefficients in B is defined to be the quotient space $H^1(A, B) = Z^1(A, B)/N^1(A, B)$. Thus $H^1(A, B) = \{0\}$ if and only if each continuous derivation from A into B is inner. According to the classical definition, a Banach algebra A is amenable if and only if for each Banach A-bimodule B, every derivation from A into B^* is inner i.e. A is amenable if and only if $H^1(A, B^*) = \{0\}$, for every Banach A-bimodule B. The concept of amenability for a Banach algebra A, was introduced by Johnson in 1972, has proved to be of enormous importance problems in Banach algebra theory, see [4].
Let X, Y, Z be normed spaces and let $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens [1] offers two natural extensions m^{***} and m^{t***t} from $X^{**} \times Y^{**}$ into Z^{**} that he called m is Arens regular whenever $m^{***} = m^{t***t}$, for more information see [1] or [7]. Regarding A as a Banach $A-bimodule$, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Suppose that A is a Banach algebra and B is a Banach $A-bimodule$. Since B^{**} is a Banach $A^{**}-bimodule$, where A^{**} is equipped with the first Arens product, we define the topological center of the right module action of A^{**} on B^{**} as follows:

$$Z^\ell_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : \text{ the map } \ a'' \to \pi_r^{***}(b'', a'') \ : \\ A^{**} \to B^{**} \text{ is } \ \text{weak}^* - \text{weak}^* \text{ continuous}\}.$$

In this way, we write $Z^{\ell}_{B^{**}}(A^{**}) = Z(\pi_\ell)$, $Z^{r}_{A^{**}}(B^{**}) = Z(\pi^t_\ell)$ and $Z^{r}_{B^{**}}(A^{**}) = Z(\pi^t_r)$, where $\pi_\ell : A \times B \to B$, $\pi_r : B \times A \to B$ are the left and right module actions of A on B, for more information, see [3]. If we set $B = A$, we write $Z^{\ell}_{A^{**}}(A^{**}) = Z_1(A^{**}) = Z^{\ell}_1(A^{**})$ and $Z^{r}_{A^{**}}(A^{**}) = Z_2(A^{**}) = Z^{r}_2(A^{**})$, for more information, see [5]. Let B be a Banach $A-$*bimodule* and $n \geq 1$. Suppose that $B^{(n)}$ is an $n-th$ *dual* of B. Then $B^{(n)}$ is also Banach $A-$*bimodule*, that is, for every $a \in A$, $b^{(n)} \in B^{(n)}$ and $b^{(n-1)} \in B^{(n-1)}$, we define

$$\langle b^{(n)}a, b^{(n-1)}\rangle = \langle b^{(n)}, ab^{(n-1)}\rangle,$$
$$\langle ab^{(n)}, b^{(n-1)}\rangle = \langle b^{(n)}, b^{(n-1)}a\rangle.$$

Let $A^{(n)}$ and $B^{(n)}$ be $n-th$ *dual* of A and B, respectively. By [8], for an even number $n \geq 0$, $B^{(n)}$ is a Banach $A^{(n)}-$*bimodule*. Then for $n \geq 2$, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for all $b^{(n)} \in B^{(n)}$, $b^{(n-1)} \in B^{(n-1)}$ and $a^{(n-2)} \in A^{(n-2)}$ we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-2)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, then for $n \geq 1$, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $b^{(n)} \in B^{(n)}$, $b^{(n-1)} \in B^{(n-1)}$ and $a^{(n-1)} \in A^{(n-1)}$ we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-1)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-1)}\rangle.$$

and if $n = 0$, we take $A^{(0)} = A$ and $B^{(0)} = B$.
So we can define the topological centers of module actions of $A^{(n)}$ on $B^{(n)}$ similarly.

2 Cohomological Properties of Banach Algebras

Let A be a Banach algebra and $n \geq 1$. Then A is called $n-$*weakly* amenable if $H^1(A, A^{(n)}) = 0$, and is called permanently weakly amenable when A is $n-$*weakly* amenable for each $n \geq 1$. In [2] Dales, Ghahramani, and Gronbaek introduced the notion of n-weak amenability for Banach algebras. They established some relations between m- and n-weak amenability. In particular, they proved that, for every n, (n + 2)-weak amenability always implies n-weak amenability.

Theorem 2.1. *Let B be a Banach $A-bimodule$ and let $n \geq 1$. If $H^1(A, B^{(n+2)}) = 0$, then $H^1(A, B^{(n)}) = 0$.*

Proof. Let $D \in Z^1(A, B^{(n)})$ and $i : B^{(n)} \to B^{(n+2)}$ be the canonical linear mapping as $A-bimodule$ homomorphism. Then $\widetilde{D} = ioD$ can be viewed as an element of $Z^1(A, B^{(n+2)})$. Since $H^1(A, B^{(n+2)}) = 0$, there exists a $b^{(n+2)} \in B^{(n+2)}$ such that

$$\widetilde{D}(a) = ab^{(n+2)} - b^{(n+2)}a, \quad (a \in A).$$

Set a $A - linear$ mapping P from $B^{(n+2)}$ into $B^{(n)}$ such that $Poi = I_{B^{(n)}}$. Then we have $Po\widetilde{D} = (Poi)oD = D$, and so $D(a) = Po\widetilde{D}(a) = aP(b^{(n+2)}) - P(b^{(n+2)})a$, for all $a \in A$. It follows that $D \in N^1(A, B^{(n)})$. Consequently $H^1(A, B^{(n)}) = 0$. $\square$

Theorem 2.2. *Let B be a Banach $A-bimodule$ and $D : A \to B^{(2n)}$ be a continuous derivation. Assume that $Z^\ell_{A^{(2n)}}(B^{(2n)}) = B^{(2n)}$. Then there is a continuous derivation $\widetilde{D} : A^{(2n)} \to B^{(2n)}$ such that $\widetilde{D}(a) = D(a)$ for all $a \in A$.*

Proof. By [[2], Proposition 1.7], the linear mapping $D'' : A^{**} \to B^{(2n+2)}$ is a continuous derivation. Take $X = B^{(2n-2)}$. Since $Z_{A^{(2n)}}(X^{**}) = Z_{A^{(2n)}}(B^{(2n)}) = B^{(2n)} = X^{**}$, by [[2], Proposition 1.8], the canonical projection $P : X^{(4)} \to X^{**}$ is a $A^{**} - bimodule$ morphism. Set $\widetilde{D} = PoD''$. Then $\widetilde{D}$ is a continuous derivation from A^{**} into $B^{(2n)}$, satisfying $\widetilde{D}(a) = D(a)$, $(a \in A)$. $\square$

Corollary 2.3. *Let B be a Banach $A-bimodule$ and $n \geq 1$. If $Z^\ell_{A^{(2n)}}(B^{(2n)}) = B^{(2n)}$ and $H^1(A^{(2n+2)}, B^{(2n+2)}) = 0$, then $H^1(A, B^{(2n)}) = 0$.*

Proof. By [[2], Proposition 1.7] and preceding theorem the result follows. $\square$

Corollary 2.4. *[2]. Let A be a Banach algebra such that $A^{(2n)}$ is Arens regular and $H^1(A^{(2n+2)}), A^{(2n+2)}) = 0$ for each $n \geq 1$. Then A is $2n-$ weakly amenable.*

Assume that A is Banach algebra and $n \geq 1$. We define $A^{[n]}$ as the linear span of

$$\{a_1 a_2 ... a_n : \ a_1, a_2, ..., a_n \in A\},$$

in A.

Theorem 2.5. *Let A be a Banach algebra and $n \geq 0$. Let $A^{[2n]}$ be dense in A and suppose that B is a Banach $A-$bimodule. Assume that AB^{**} and $B^{**}A$ are subsets of B. If $H^1(A, B^*) = 0$, then $H^1(A, B^{(2n+1)}) = 0$.*

Proof. For $n = 0$ the result is clear. Let $B^{\perp}$ be the space of functionals in $B^{(2n+1)}$ which annihilate $i(B)$ where $i : B \to B^{(2n)}$ is the canonical mapping. Its easily verified that,

$$B^{(2n+1)} = i(B)^* \oplus B^{\perp}.$$

It follows that

$$H^1(A, B^{(2n+1)}) = H^1(A, i(B)^*) \oplus H^1(A, B^{\perp}).$$

Since $i(B)^* \cong B^*$ and by assumption $H^1(A, B^*) = 0$, it is enough to show that $H^1(A, B^{\perp}) = 0$.
Now, take the linear mappings L_a and R_a from B into itself by $L_a(b) = ab$ and $R_a(b) = ba$ for all $a \in A$. Since $AB^{**} \subseteq B$ and $B^{**}A \subseteq B$, $L_a^{**}(b'') = ab''$ and $R_a^{**}(b'') = b''a$ for every $a \in A$, respectively. Consequently, L_a and R_a are weakly compact. It follows that for each $a \in A$ the linear mappings $L_a^{(2n)}$ and $R_a^{(2n)}$ from $B^{(n)}$ into $B^{(n)}$ are weakly compact and for every $b^{(2n)} \in B^{(2n)}$, we have $L_a^{(2n)}(b^{(2n)}) = ab^{(2n)} \in B^{(2n-2)}$ and $R_a^{(2n)}(b^{(2n)}) = b^{(2n)}a \in B^{(2n-2)}$. Set $a_1, a_2, ..., a_n \in A$ and $b^{(2n)} \in B^{(2n)}$. Then $a_1 a_2 ... a_n b^{(2n)}$ and $b^{(2n)} a_1 a_2 ... a_n$ are belong to B. Suppose that $D \in Z^1(A, B^{\perp})$ and let $a, x \in A^{[n]}$. Then for every $b^{(2n)} \in B^{(2n)}$, since $xb^{(2n)}, b^{(2n)}a \in B$, we have the following equality

$$\begin{aligned}
\langle D(ax), b^{(2n)} \rangle &= \langle aD(x), b^{(2n)} \rangle + \langle D(a)x, b^{(2n)} \rangle \\
&= \langle D(x), b^{(2n)}a \rangle + \langle D(a), xb^{(2n)} \rangle \\
&= 0.
\end{aligned}$$

It follows that $D \mid_{A^{[2n]}} = 0$. Since $A^{[2n]}$ dense in A, it follows that $D = 0$. Hence $H^1(A, B^{\perp}) = 0$ and the result follows. $\square$

Corollary 2.6. 1. *Let A be a Banach algebra with a bounded left approximate identity, and let B be a Banach $A-bimodule$. Suppose that AB^{**} and $B^{**}A$ are subset of B. Then $H^1(A, B^{(2n+1)}) = 0$ for all $n \geq 0$, whenever $H^1(A, B^*) = 0$.*

2. *Let A be an amenable Banach algebra and B be a Banach $A-bimodule$. If AB^{**} and $B^{**}A$ are subset of B, then $H^1(A, B^{(2n+1)}) = 0$.*

Example 2.7. Suppose that G is a compact group. We know that $L^1(G)$ is $M(G)-bimodule$ and $L^1(G)$ is an ideal in the second dual $M(G)^{**}$ of $M(G)$. By [[6],corollary 1.2], we have $H^1(L^1(G), M(G)^*) = 0$. Then by corollary 2.6, for every $n \geq 1$, we have

$$H^1(L^1(G), M(G)^{(2n+1)}) = 0.$$

Corollary 2.8. *Let A be a Banach algebra and let $A^{[2n]}$ be dense in A. Suppose that AB^{**} and $B^{**}A$ are subset of B. Then the following are equivalent.*

1. $H^1(A, B^*) = 0$.
2. $H^1(A, B^{(2n+1)}) = 0$ *for some* $n \geq 0$.
3. $H^1(A, B^{(2n+1)}) = 0$ *for each* $n \geq 0$.

Proof. $3 \Rightarrow 1 \Rightarrow 2$ is trivial by Theorem 2.5.
$2 \Rightarrow 3$: Suppose that $H^1(A, B^{(2n_0+1)}) = 0$ for some $n_0 \geq 0$. Let $D \in Z^1(A, B^*)$. By considering the injective linear mapping

$$\iota : B^* \to (B^*)^{(2n_0)} = B^{(2n_0+1)},$$

with

$$\langle \iota(b'), b^{(2n_0)} \rangle = \langle b^{(2n_0)}, b' \rangle = b^{(2n_0)}(b'),$$

so $B^* \cong \iota(B^*) \subseteq B^{(2n_0+1)}$ and D can be viewed as an element of $Z^1(A, B^{(2n_0+1)})$. By assumption, there is a $f \in B^{(2n_0+1)}$ such that $D(a) = \delta_f(a)$, for each $a \in A$. Since $B \cong \tau(B) \subseteq B^{(2n_0)}$. Where τ is the canonical embedding. We define $f_0 = f|_B$. Then $f_0 \in B^*$ and

$$D(a) = D(a)|_B = \delta_f(a)|_B = \delta_{f_0}(a),$$

holds for each $a \in A$. It follows that $H^1(A, B^*) = 0$, and so by using Theorem 2.5, the result follows. □

Corollary 2.9. *[2]. Let A be a weakly amenable Banach algebra such that A is an ideal in A^{**}. Then A is $(2n+1)-$ weakly amenable for each $n \geq 0$.*

Proof. It follows by [[2], Proposition 1.3] and corollary 2.8. □

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