Journal of Mathematical Extension
Vol. 15, No. 3, (2021) (4)1-23
URL: https://doi.org/10.30495/JME.2021.1393
ISSN: 1735-8299
Original Research Paper

An Efficient Algorithm for Solving Absolute Value Equations

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Abstract. Recently, absolute value equations (AVEs) are lied in the consideration center of some researchers since they are very suitable alternatives for many frequently occurring optimization problems. Therefore, finding a fast solution method for these type of problems is very significant. In this paper, based on the mixed-type splitting (MTS) idea for solving linear system of equations, a new fast algorithm for solving AVEs is presented. This algorithm has two auxiliary matrices which are limited to be nonnegative strictly lower triangular and nonnegative diagonal matrices. The convergence of the algorithm is discussed via some theorems. In addition, it is shown that by suitable choice of the auxiliary matrices, the convergence rate of this algorithm is faster than that of the SOR, AOR, Generalized Newton, Picard and SOR-like methods. Eventually, some numerical results for different size of problem dimensionality are presented which admit the credibility of the proposed algorithm.

AMS Subject Classification: 65xx; 65Hxx; 65H10.
Keywords and Phrases: Absolute value equations, $M-$splitting, Mixed-type splitting method, unique solution, spectral radius.

Received: September 2019; Accepted: September 2020
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1 Introduction and background

In this note, we consider the following system of the absolute value equations (AVEs):

$$Ax - B \mid x \mid = b. \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $\mid . \mid$ denotes the absolute value.

The general form of (1) first was presented by Rohn [30] and then Mangasarian [21, 22, 23] checked it in a more general context. Recently, the AVE (1) has been investigated by numerous researchers; one main reason for the recent interest in this new subject, is the fact that some optimization problems, for example convex quadratic programming, linear complementarity problem (LCP), or linear programming can be equivalently overwritten in the form (1) [32]. Mangasarian expressed the NP-hard n-dimensional knapsack feasibility problem as an equivalent AVE [24]. Lately, in order to solve (1), some numerical methods have been developed, like the Newton-type method [23, 6, 14, 16, 42], the sign accord (SA) method [31] and the AOR method [20] (for other numerical methods, one can see the works done by Noor et al. [27, 28]). In [22] it was shown that determining the existence of a solution for AVEs is NP-hard. Mangasarian and Meyer in [25] proved that the equation $Ax-|x| = b$ is uniquely solvable for each $b \in \mathbb{R}^n$, if the singular values of A exceed 1. In [29] it was demonstrated that we can not provide a polynomially comutable necessary and sufficient criteria on unique solvability of the AVE (1). Prokopyev [29] discussed that AVE (1) is equivalent to LCP, which is one of the most important problems in the applied sciences and engineering, see [2, 8, 9]. Also, he presented equivalent linear mixed 0-1 reformulations of AVE (1), which do not require introduction of large data dependent constants. Mangasarian in [23] proposed a semismooth Newton method for solving AVEs. Yong introduced a particle swarm optimization (PSO) to AVE based on aggregate function [39]. Moreover, he considered the Harmony Search (HS) algorithm for solving AVE in [40]. Moosaei et al. [26] introduced and analyzed two methods for solving the NP-hard absolute value equations, in the case that singular values of A exceed 1. Wu and Li in [38] proposed a special SS iteration method, based on the shift splitting (SS) technique, for solving the absolute value equation, which is resulted using reformulating equivalently the AVE as a two-by-two block nonlinear equation. In [23], Mangasarian presented a generalized Newton method for solving AVE (1) and studied its convergence properties. This method can be presented as

$$x^{(i+1)} = (A - D(x^{(i)}))^{-1}b, \quad i = 0, 1, 2, ..., \tag{2}$$

that $x^{(0)}$ denotes the initial guess and $D(x^{(i)}) = diag(sign(x^{(i)}))$ [23]. In implementation of this method a linear equations system with coefficient matrix $A - D(x^{(i)})$ should be solved in each iteration. Since the coefficient matrices in (2) are changed, in each iteration, the computations of the generalized Newton can be very expensive. In this regard, Rohn et al. in [33] suggested another method to solve AVE (1); in practice their method is reduced to the Picard iteration method

$$x^{(i+1)} = A^{-1}(|x^{(i)}| + b), \quad i = 0, 1, 2, ...,$$

so that $x^{(0)} = A^{-1}b$ denotes an initial value. Actually, in Picard method iterations, a linear system of equations with A as the coefficient matrix (that is constant), have to be solved. Here the basic problem is that, if A is ill-conditioned, in each iteration of the Picard method we are faced with an ill-conditioned linear system of equations. In [12], the SOR-like method is extended for solving the absolute value equation $Ax - |x| = b$. More precisely, the following iterative scheme is developed

$$\begin{cases} x^{(k+1)} = (1-\omega)x^{(k)} + \omega A^{-1}\left(y^{(k)} + b\right) \\ y^{(k+1)} = (1-\omega)y^{(k)} + \omega\left|x^{(k+1)}\right| \end{cases} \tag{3}$$

which can be regarded as an iterative scheme corresponding to the following SOR-like block splitting:

$$\mathcal{A} = \frac{1}{\omega}\begin{bmatrix} A & 0 \\ -\omega D(x) & I \end{bmatrix} - \frac{1}{\omega}\begin{bmatrix} (1-\omega)A & \omega I \\ 0 & (1-\omega)I \end{bmatrix}.$$

The established convergence results for the above method rely on the spectrum of $D(x^{(k+1)})A^{-1}$. In fact, it is proved that if the eigenvalues of $D(x^{(k+1)})A^{-1}$ are real then iterative scheme (3) is convergent for $0 < \omega \leq 1$. In case that all eigenvalues of $D(x^{(k+1)})A^{-1}$ are positive then the method is convergent for $0 < \omega < 2$.

In [6] a smoothing Newton algorithm to solve the AVE was presented. Caccetta proved that this algorithm is globally convergent and its convergence rate is quadratic under the condition that the singular values of A exceed 1. Recently, Yong [41] has proposed an iterative method for absolute value equation $Ax - |x| = b$, where A is an arbitrary square matrix whose singular values exceed 1. He showed that the method converges to the solution of AVE after finite iterations. He also used this method to solve two-point boundary value problem. Furthermore, infeasible AVEs have been investigated by Saeed Ketabchi et al. [15, 16, 17].

In recent years, a lot of efforts have been made in expanding iterative methods for solving (1). For example, Rohn et al. [33] proposed a general preconditioned Richardson iterative method to solve (1). Based on Hermitian and skew-Hermitian splitting of the coefficient matrix A in (1), the Picard-HSS iterative method for AVEs with $B = I$, where I is the identity matrix, has been presented by Salkuyeh [35]. Clearly, the Picard-HSS method falls into category of stationary matrix splitting iteration methods. Whereas, based on our knowledge, little heed has been paid to the classical matrix-splitting iterative methods (such as Gauss-Seidel (GS), Jacobi, Successive Over Relaxation (SOR) and Accelerated Over Relaxation (AOR) methods) for solving (1).

In the current article, we first review the mixed-type splitting (MTS) iterative method and then design it for solving (1). Thereafter, we prove that the MTS algorithm can converges faster than some famous methods by opting appropriate auxiliary matrices. In this regard, the rest of article is organize as follows:
In section 2, we recall some definitions and theorems which will be utilized in the next sections. In Section 3 we review the MTS iterative method for solving linear system $Ax = b$. In section 4, based on mixed-type splitting (MTS) idea, a new iterative algorithm for solving AVEs is presented. Then, in section 5, its convergence analysis is investigated. In order to display the efficacy of the established results, some numerical experiments and comparisons on test examples are presented in section 6 which one of them is related to the real world. Finally, the paper is terminated with some conclusion remarks in section 7.

2 Preliminaries

Suppose $A \in \mathbb{R}^{n\times m}$, is an arbitrary matrix. We say $A \geq 0$ ($A > 0$) when all entries of A are nonnegative (positive). For two matrices A and B in $\mathbb{R}^{n\times m}$, $A \geq B$ ($A > B$) means that $A - B \geq 0$ ($A - B > 0$). For square matrix A, the maximum modulus of the eigenvalues is called spectral radius and is denoted by $\rho(A)$. Let $A = [a_{ij}] \in \mathbb{R}^{n\times m}$ be an arbitrary matrix then $|A| \in \mathbb{R}^{n\times m}$ such that $|A|_{ij} = |a_{ij}|$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$. For $A \in \mathbb{R}^{n\times n}$, the decomposition $A = M - N$ is named a splitting if $M, N \in \mathbb{R}^{n\times n}$ and M is nonsingular. In the following, we recall some definitions and results which are utilized throughout this paper.
We remind that $f(n) = O(g(n))$, if there exists a positive constant k so that $f(n) \leq kg(n)$, for all $n \in \mathbb{R}++$.

Definition 2.1. [5] $A \in \mathbb{R}^{n\times n}$ is named a $Z-$matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, 3, ..., n$ $(i \neq j)$; a $Z-$matrix having positive diagonal elements is called an L-matrix.

Definition 2.2. [5] Assume that A is an $L-$matrix; the matrix A is called an $M-$matrix if A is nonsingular and $A^{-1} \geq 0$.

Definition 2.3. A splitting $A = M - N$ is named an $M-$splitting of A if M is an $M-$matrix and $N \geq 0$.

Definition 2.4. [36] A matrix A is called irreducible if there is no permutation matrix P such that PAP^T is a block upper triangular matrix. Otherwise, it is reducible.

The following theorem prepares another way to check whether a matrix is irreducible or not.

Theorem 2.5. *Let $A \geq 0$ be an $n \times n$ irreducible matrix; then,*

i) A has a positive eigenvalue which is equal to its spectral radius.
ii) There exists an eigenvector $x > 0$ corresponding to the spectral radius of A.
iii) $\rho(A)$ is a simple eigenvalue of A.

Proof. See [36]. □

Theorem 2.6. *Suppose that A is a nonnegative matrix; then,*
a) If $\alpha x \leq Ax$ for some nonnegative vector $x \neq 0$, then $\alpha \leq \rho(A)$.
b) If $Ax \leq \beta x$ for some nonnegative vector $x \neq 0$, then $\rho(A) \leq \beta$. Furthermore, if A is irreducible and $0 \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax$ and $Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

Proof. See [5]. □

Definition 2.7. ([37]) The splitting $A = M - N$ is named

- a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$,
- a nonnegative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
- a weak nonnegative splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type),
- a convergent splitting of A if $\rho(M^{-1}N) < 1$.

Theorem 2.8. *[34] Suppose $A \in \mathbb{R}^{n\times n}$. Then for every natural norm $\|\,.\,\|$ we have $\rho(A) \leq \| A \|$.*

Theorem 2.9. *[34] Suppose A and B are two square matrices such that $0 \leq A \leq B$. Then $\rho(A) \leq \rho(B)$.*

For the choice of optimum parameter of SOR-like method, the following theorem is established.

Theorem 2.10. [12, Theorem 3.2] *Let $A \in \mathbb{R}^{n\times n}$, $\|A^{-1}\| < 1$ and $\rho = \rho(D(x^{(k+1)})A^{-1})$. Suppose that all eigenvalues of $D(x^{(k+1)})A^{-1}$ are positive. Then the optimal parameter ω_o is given by*

$$\omega_o = \frac{2}{1 + \sqrt{1-\rho}}. \tag{4}$$

3 Brief description of MTS method for linear algebraic systems

In [18, 19], MTS method has been introduced for solving the linear system $Ax = b$. Under the assumption that $a_{ii} \neq 0$ for $i = 1, 2, ..., n$, we can consider the following splitting for A:

$$A = M - N, \tag{5}$$

with

$$M = (D + D_1 + L_1 - L), \qquad N = (D_1 + L_1 + U),$$

where $D = diag(A)$, $-L$ and $-U$ are strictly lower and upper triangular matrices gained from A, respectively, $D_1 \geq 0$ is an auxiliary diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and $0 \leq L_1 \leq L$. The MTS iterative algorithm to solve $Ax = b$ is given as follows:

Algorithm 1: Given an initial guess $x^{(0)} \in \mathbb{R}^n$, then for $i = 0, 1, 2...$, until $\{x^{(i)}\}_{i=0}^{\infty}$ converges, compute

$$(D + D_1 + L_1 - L)x^{(i+1)} = (D_1 + L_1 + U)x^{(i)} + b.$$

In this regard, the iteration matrix of the MTS method is expounded by

$$\overline{T} = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U);$$

hence, the above algorithm is a stationary iterative algorithm for solving linear system of equations $Ax = b$. Hadjidimos in [13] showed that when the coefficient matrix A is an M-matrix, the Mixed-Type Splitting iterative method is convergent.

4 New MTS algorithm for solving AVE

As noted above, the main idea of MTS for solving linear systems of algebraic equations, is the decomposition of coefficient matrix into suitable matrices. Based on this idea and by assumption that $a_{ii} \neq 0$ for

$i = 1, 2, ..., n$, we consider the splitting (5) for A in (1). Substituting (5) in (1) gives :

$$(D + D_1 + L_1 - L)x = (D_1 + L_1 + U)x + B \mid x \mid +b,$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular parts of A, respectively, D_1 is an nonnegative auxiliary diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and $0 \leq L_1 \leq L$. Hence, the iterative form of the method is as follows:

$$(D+D_1+L_1-L)x^{(i+1)} = (D_1+L_1+U)x^{(i)}+B \mid x^{(i)} \mid +b, \quad i = 0, 1, 2, ... \tag{6}$$

Since $A = M - N$ is a splitting for matrix A, $M = (D + D_1 + L_1 - L)$ is nonsingular; hence (6) is equivalent to

$$\begin{aligned} x^{(i+1)} &= (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U)x^{(i)} \\ &+ (D + D_1 + L_1 - L)^{-1}B \mid x^{(i)} \mid \\ &+ (D + D_1 + L_1 - L)^{-1}b. \end{aligned} \tag{7}$$

The MTS iterative algorithm to solve (1) can be summarized as following:

Algorithm2: Given an initial guess $x^{(0)} \in \mathbb{R}^n$. for $i = 0, 1, 2...$, until $\{x^{(i)}\}_{i=0}^{\infty}$ converges, compute

$$(D + D_1 + L_1 - L)x^{(i+1)} = (D_1 + L_1 + U)x^{(i)} + B \mid x^{(i)} \mid +b.$$

It is necessary to mention that our MTS algorithm can covers SOR and AOR methods for solving AVEs, by special selection of matrices D_1 and L_1 in (5). If we choose

$$D_1 = \frac{1}{\omega}(1 - \omega)D \quad and \quad L_1 = 0,$$

we are faced with SOR method and if we select

$$D_1 = \frac{1}{\omega}(1 - \omega)D \quad and \quad L_1 = \frac{1}{\omega}(\omega - r)L,$$

we are faced with AOR method ([20]), where ω and r are positive real parameters with $\omega \neq 0$. The convergence of the new algorithm and some of its related advantages will be investigated in the following section.

5 Convergence analysis and theoretical comparisons

In this section we will investigate the convergence of the algorithm for different acceptable situation of matrix A in (1); we will show that the algorithm can convergent to unique solution of (1) for any initial guess $x^{(0)} \in \mathbb{R}^n$ which is a very important feature of this algorithm. This facts are shown by proving some lemmas and theorems as follow.

Lemma 5.1. *[7] Let $A \in R^{n\times n}$ be an $L-$matrix. Consider the splitting (5) for the matrix A, where $0 \leq L_1 \leq L$. Then the matrix $M^{-1}N = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U)$ is nonnegative.*

Lemma 5.2. *[20] Suppose that $A = M - N$ with $det(M) \neq 0$ and $x^{(0)} \in \mathbb{R}^n$ be an arbitrary initial guess. Then for $\rho(\mid M^{-1}N \mid + \mid M^{-1}B \mid) < 1$, the iterative sequence $x^{(i)}$ given by*

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}B \mid x^{(i)} \mid +M^{-1}b, \qquad i = 1, 2, 3, ..., \tag{8}$$

converges to the unique solution x^* of AVE (1).

Corollary 5.3. *Suppose that $A = M - N$ with $det(M) \neq 0$ and $x^{(0)} \in \mathbb{R}^n$ is an arbitrary initial guess. If $\| \mid M^{-1}N \mid \| + \| \mid M^{-1}B \mid \| < 1$, where $\| \,.\, \|$ is an consistent matrix norm, then the given iterative sequence $x^{(i)}$ by (8) converges to the unique solution x^* of (1).*

Proof. Using triangle inequality, we have:

$$\| \mid M^{-1}N \mid + \mid M^{-1}B \mid \| \leq \| \mid M^{-1}N \mid \| + \| \mid M^{-1}B \mid \| < 1 \tag{9}$$

From (9) and Theorem 2.8, we have

$$\rho(\mid M^{-1}N \mid + \mid M^{-1}B \mid) \leq \| \mid M^{-1}N \mid + \mid M^{-1}B \mid \| < 1 \tag{10}$$

Now based on (10), the sequence (8) converges to unique solution of (1) by Lemma 5.2. □

Corollary 5.4. *Suppose that $A = M - N$ is an $M-$splitting and $x^{(0)} \in \mathbb{R}^n$ is an arbitrary initial guess. Then for $\rho(M^{-1}N + M^{-1} \mid B \mid) < 1$, the iterative sequence $x^{(i)}$ given by (8) converges to the unique solution of (1).*

Proof. Since A is an $M-$splitting, $M^{-1} \geq 0$ and $N \geq 0$. Hence $M^{-1}N \geq 0$ and $M^{-1}N + M^{-1} \mid B \mid \geq 0$. Besides

$$0 \leq\mid M^{-1}N \mid + \mid M^{-1}B \mid\leq M^{-1}N + M^{-1} \mid B \mid . \tag{11}$$

From (11) and Theorem 2.9 we have

$$\rho(\mid M^{-1}N \mid + \mid M^{-1}B \mid) \leq \rho(M^{-1}N + M^{-1} \mid B \mid).$$

Now by assumption we have

$$\rho(\mid M^{-1}N \mid + \mid M^{-1}B \mid) \leq \rho(M^{-1}N + M^{-1} \mid B \mid) < 1. \tag{12}$$

Therefore using (12) and Lemma 5.2, the result is obtained. □

To present the next theorems we define

$$T \equiv (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U) + (D + D_1 + L_1 - L)^{-1} \mid B \mid . \tag{13}$$

Note that for AOR and SOR iterative methods, T changes to $T_{r,\omega}$ and T_ω respectively as mentioned in the following:

$$T_{r,\omega} \equiv (D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U] + \omega(D - rL)^{-1} \mid B \mid;$$

$$T_\omega \equiv (D - \omega L)^{-1}[(1 - \omega)D + \omega U] + \omega(D - \omega L)^{-1} \mid B \mid.$$

Theorem 5.5. *Let $A = M - N = D - L - U$ be an $M-$matrix, $D_1 \geq 0$ and $0 \leq L_1 \leq L$, where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular parts of A, respectively. If $\rho(T) < 1$, then the mixed-type splitting iterative method defined by (7) converges to the unique solution of (1) for an arbitrary initial guess $x^{(0)} \in \mathbb{R}^n$.*

Proof. Let us first remind that

$$M = D + D_1 + L_1 - L, \qquad N = D_1 + L_1 + L.$$

Since A is an $M-$matrix and $0 \leq L_1 \leq L$, we obtain
$M^{-1} = (D + D_1 + L_1 - L)^{-1} = [(D + D_1) - (L - L_1)]^{-1} \geq 0$,
$N = D_1 + L_1 + L \geq 0$.
Furthermore, diagonal part of M, (i.e. $D + D_1$) is nonnegative, strictly lower triangular part of M, (i.e. $L_1 - L$) is negative, and strictly upper triangular parts of M is zero. Therefore M is $M-$matrix. Hence A is an $M-$splitting. By using (13) and Corollary 5.4, the proof is completed. □

Remark 5.6. Suppose that AVE (1) is consistent and the iterative sequence generated by Algorithm 2 is converged in k iterations to a solution of (1) with the given error. It can be shown that the complexity of one iteration of Algorithm 2 is bounded from above by $f(n) = \frac{1}{2}(n^3 + 5n^2 + 7n)$. So we can easily see that the complexity of Algorithm 2 is above bounded by $O(kn^3)$. Therefore the proposed algorithm is tractable.

5.1 Theoretically convergence comparison with AOR and SOR

First we remind that (asymptotic) rate of the convergence is defined by $R_\infty = -Ln(\rho)$ (see [36]). The convergence analysis of an iterative method is based on the spectral radius of the iteration matrix. For large number of iterations, the corresponding error remarkably decreases using the spectral radius factor of the iteration matrix; that is, when the spectral radius is smaller, the convergence is faster. The following theorems demonstrate that, when the matrix A in (1) is an irreducible $M-$matrix, then by appropriate choices of the auxiliary matrices D_1 and L_1, the Algorithm 2, converges faster than the AOR and SOR methods.

Theorem 5.7. *Let A be an irreducible $M-$matrix. Also assume that*

$$0 \leq D_1 \leq (\frac{1}{\omega} - 1)D, \quad 0 \leq L_1 \leq (1 - \frac{r}{\omega})L$$

where $0 \leq r < \omega \leq 1$. If T and $T_{r,\omega}$ are irreducible, then

i) $\rho(T) < \rho(T_{r,\omega})$, *if* $\rho(T_{r,\omega}) < 1$,
ii) $\rho(T) = \rho(T_{r,\omega})$, *if* $\rho(T_{r,\omega}) = 1$,
iii) $\rho(T) > \rho(T_{r,\omega})$, *if* $\rho(T_{r,\omega}) > 1$.

Proof. Since A is an $M-$matrix, $U \geq 0$; therefore by assumption, $N = D_1 + L_1 + U \geq 0$. On the other hand, from Lemma 5.1 we have, $M^{-1}N = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U) \geq 0$. Hence $(D + D_1 + L_1 - L)^{-1}$ is nonnegative. Therefore, it is easy to conclude that $T \geq 0$. Also it is clear that $T_{r,\omega}$ is nonnegative; hence, by Theorem 2.5 there exists a positive vector x such that $T_{r,\omega}x = \rho(T_{r,\omega})x$, or equivalently,

$$((D - rL)^{-1}[(1 - \omega)D + (\omega - r)L + \omega U] + \omega(D - rL)^{-1} \mid B \mid)x = \rho(T_{r,\omega})x;$$

$\omega(D-rL)^{-1}[\frac{1}{\omega}[(1-\omega)D+(\omega-r)L+\omega U]+\mid B\mid]x=\rho(T_{r,\omega})x;$

$[\frac{1}{\omega}[(1-\omega)D+(\omega-r)L+\omega U]+\mid B\mid]x=\frac{1}{\omega}\rho(T_{r,\omega})(D-rL)x.$

Thus we have

$$Ux=\frac{1}{\omega}\rho(T_{r,\omega})(D-rL)x+(1-\frac{1}{\omega})Dx-(1-\frac{r}{\omega})Lx-\mid B\mid x. \quad (14)$$

By applying (14), we have

$$\begin{aligned}
Tx-T_{r,\omega}x &= (D+D_1+L_1-L)^{-1}[(D_1+L_1+U)x \\
&\quad +\mid B\mid x-\rho(T_{r,\omega})(D+D_1+L_1-L)x] \\
&= (D+D_1+L_1-L)^{-1}[(D_1+L_1)x+\frac{1}{\omega}\rho(T_{r,\omega})(D-rL)x \\
&\quad +(1-\frac{1}{\omega})Dx-(1-\frac{r}{\omega})Lx-\mid B\mid x+\mid B\mid x \\
&\quad -\rho(T_{r,\omega})(D+D_1+L_1-L)x] \\
&= (1-\rho(T_{r,\omega}))(D+D_1+L_1-L)^{-1}[D_1+(1-\frac{1}{\omega})D \\
&\quad +L_1-(1-\frac{r}{\omega})L]x. \qquad (15)
\end{aligned}$$

From the assumptions of the theorem, we have

$$[D_1+(1-\frac{1}{\omega})D+L_1-(1-\frac{r}{\omega})L]\leq 0;$$

therefore

$$(D+D_1+L_1-L)^{-1}[D_1+(1-\frac{1}{\omega})D+L_1-(1-\frac{r}{\omega})L]\leq 0. \quad (16)$$

i) If $0<\rho(T_{r,\omega})<1$, since $T_{r,\omega}x=\rho(T_{r,\omega})x$, then by (15) and (16) we have $Tx\leq\rho(T_{r,\omega})x$. By Theorem 2.6, we get $\rho(T)<\rho(T_{r,\omega})$.
ii) If $\rho(T_{r,\omega})=1$, then $Tx=\rho(T_{r,\omega})x$. By Theorem 2.6, we have $\rho(T)=\rho(T_{r,\omega})$.
iii) If $\rho(T_{r,\omega})>1$, then $Tx\geq\rho(T_{r,\omega})x$. By Theorem 2.6, we obtain $\rho(T)>\rho(T_{r,\omega})$.

□

Theorem 5.8. *Suppose A is an irreducible $M-$matrix. Also assume that*

$$0 \leq D_1 \leq (\frac{1}{\omega} - 1)D, \quad L_1 = 0$$

where $0 < \omega < 1$. If T and T_ω are irreducible, then

i) $\rho(T) < \rho(T_\omega)$, *if* $\rho(T_\omega) < 1$,
ii) $\rho(T) = \rho(T_\omega)$, *if* $\rho(T_\omega) = 1$,
iii) $\rho(T) > \rho(T_\omega)$, *if* $\rho(T_\omega) > 1$.

Proof. It is enough to choose $r = \omega$ in Theorem 5.7. □
One of our motivations for using this method is that it covers the famous AOR and SOR methods and by comparing this method with both mentioned methods, we find that under some assumptions, the MTS method converges to unique solution of AVE with less number of iterations and less CPU time.

6 Numerical experiments and comparisons

In this section, we report some numerical experiments to prove the performance and efficiency of the proposed algorithm for solving the AVE (1). In Examples 6.1, 6.2 and 6.3, the initial guess is supposed to be $x^{(0)} = (1, 0, 1, 0, ..., 1, 0, ...)^T \in \mathbb{R}^n$, while in Example 6.4 the initial guess is taken to be zero vector and all the iterations are terminated as soon as we reach to

$$\delta_i = \frac{\| Ax^{(i)} - B \mid x^{(i)} \mid -b \|_2}{\| b \|_2} \leq 10^{-6}, \tag{17}$$

where $x^{(i)}$ is the obtained solution by each of the methods at iterate i. It is also assumed that the maximum number of iterations for all methods is 2000. Notation † in tables means the iterations have been stopped after 2000 iterations while the computed approximate solution does not hold in (17). In all of the following examples, the auxiliary matrices are selected from [3] as follow:

$$D_1 = 0.9(1 - \omega)D \quad \text{and} \quad L_1 = 0.8(1 - \tfrac{r}{\omega})L.$$

Table 1: Randomly calculated parameters r and ω for Examples 6.1 and 6.2.

n	25	100	400	900	1600	4900	10000
r	0.7	0.7	0.6	0.4	0.2	0.7	0.5
ω	0.8	0.8	0.7	0.6	0.4	0.8	0.6

All computations were carried out on a computer with an Intel(R) Core(TM) i5-4200U CPU @ 1.60GHz processor and Memory 4GB using MATLAB R2014a. For the act of the inverse of an matrix in the implementation of iterative methods, we always used the LU factorization.

In this regard, here, four numerical examples are presented and are solved which previously have been examined in literature. Their results are shown in Tables 1,...,7, in which they report the number of iterations (denoted with Iter), the error value (denoted with Err) and CPU times for the convergence (denoted with CPU). It should be noted that the calculated time is in seconds.

Example 6.1. [4] Consider the Poisson equation in two dimensions,

$$\begin{aligned} -\Delta u = f \quad & in \quad \Omega = (0,1) \times (0,1), \\ u = 0 \quad & on \quad \Gamma = \partial\Omega. \end{aligned}$$

Using a finite difference discretization with mesh-width $h = \frac{1}{m+1}$, we arrive at a linear system $Cx = d$ where $C \in \mathbb{R}^{n\times n}$ and $n = m^2$. It can be observed that the matrix C is in the form $C = I_m \otimes M + T \otimes I_m$, where $M = trid(\frac{-1}{4}, 1, \frac{-1}{4})$ and $T = trid(\frac{-1}{4}, 0, \frac{-1}{4})$ are $m\times m$ matrices and $\otimes$ is the symbol of the Kronecker product. We have take the right-hand side vector of AVE (1), such that the vector $x^* = (1, 2, 1, 2, ..., 1, 2, ...)^T \in \mathbb{R}^n$ is the exact solution, this ensures that the mentioned AVE is consistent. For AVE (1) with $B = I$ when $A = C$, we report the numerical results for various values of n in Tables 1 and 2. As the numerical results show, both methods Picard and the generalized Newton fail in convergence. However, we observe that the MTS method gives quite proper results.

Example 6.2. [10] Assume that m be a specified positive integer and $n = m^2$. Let AVE (1) is given, in which $A \in \mathbb{R}^{n\times n}$ is presented by

Table 2: Numerical results of Example 6.1 for different values of n.

	MTS		Gen. Newton		Picard	
n	Iter (CPU)	Err	Iter (CPU)	Err	Iter (CPU)	Err
25	41(0.0057)	7.708e-07	†		†	
100	47(0.0068)	6.057e-07	†		†	
400	39(0.0079)	9.988e-07	†		†	
900	34(0.0122)	8.520e-07	†		†	
1600	18(0.0099)	6.005e-07	†		†	
4900	64(0.0602)	8.612e-07	†		†	
10000	27(0.0482)	9.272e-07	†		†	

$A = M + \mu I$, where

$$\mathbf{M} = \begin{pmatrix} S & -I & 0 & \dots & 0 & 0 \\ -I & S & -I & \dots & 0 & 0 \\ 0 & -I & S & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & S & -I \\ 0 & 0 & 0 & \dots & -I & S \end{pmatrix} \in \mathbb{R}^{n\times n},$$

with

$$\mathbf{S} = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 & 0 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & -1 \\ 0 & 0 & 0 & \dots & -1 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m},$$

where $0 \in \mathbb{R}^{m\times m}$ and $I \in \mathbb{R}^{m\times m}$ are the zero matrix and the identity matrix, respectively. This problem is due to the finite difference discretization on equidistant grid of a free boundary value problem about the flow of water via a porous dam [11]. Let $B = I$ and the right-hand side of (1) is constructed such that $x^* = (1, 2, 1, 2, ..., 1, 2, ...)^T \in \mathbb{R}^n$ satisfies $Ax^* - |x^*| = b$. In Table 3, we report the numerical results for various values of μ ($\mu = 0, -0.5, -0.9$) and n ($n = 25, 100, 400, 900, 1600, 4900, 10000$). As the numerical results show, the Picard and the generalized Newton methods fail in several cases, but the MTS algorithm converges appro-

Table 3: Numerical results of Example 6.2 for different values of n, μ.

		MTS		Gen. Newton		Picard	
μ	n	Iter (CPU)	Err	Iter (CPU)	Err	Iter (CPU)	Err
0	25	50(0.0048)	9.217e-07	2(0.0193)	3.624e-16	†	
	100	41(0.0053)	8.589e-07	14(0.0219)	8.145e-16	†	
	400	44(0.0070)	8.364e-07	†		†	
	900	51(0.0126)	9.089e-07	†		†	
	1600	63(0.0225)	9.979e-07	†		†	
	4900	35(0.0373)	8.439e-07	†		†	
	10000	45(0.0851)	9.673e-07	†		†	
-0.5	25	26(0.0030)	8.616e-07	2(0.0030)	2.825e-16	†	
	100	42(0.0047)	8.988e-07	†		†	
	400	61(0.0093)	8.985e-07	1066(6.5646)	1.129e-15	†	
	900	80(0.0183)	9.811e-07	†		†	
	1600	106(0.0359)	9.130e-07	†		†	
	4900	57(0.0595)	8.756e-07	†		†	
	10000	76(0.1467)	9.775e-07	†		†	
-0.9	25	36(0.0032)	9.639e-07	2(0.0032)	3.103e-16	†	
	100	93(0.0090)	9.124e-07	†		†	
	400	199(0.0276)	9.520e-07	†		†	
	900	305(0.0669)	9.858e-07	†		†	
	1600	430(0.1414)	9.743e-07	†		†	
	4900	237(0.2446)	9.738e-07	†		†	
	10000	325(0.6126)	9.857e-07	†		†	

priately to the solution of problem. Furthermore, the MTS algorithm has the least CPU time among other methods.

Example 6.3. [1] Assume that m be a predetermined positive integer and $n = m^2$; plus, suppose that $B = I$, $A = \hat{M} + I$, where

$$\hat{\mathbf{M}} = \begin{pmatrix} S & -0.5I & 0 & \dots & 0 & 0 \\ -1.5I & S & -0.5I & \dots & 0 & 0 \\ 0 & -1.5I & S & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & S & -0.5I \\ 0 & 0 & 0 & \dots & -1.5I & S \end{pmatrix} \in \mathbb{R}^{n\times n},$$

with

$$\mathbf{S} = \begin{pmatrix} 4 & -0.5 & 0 & \dots & 0 & 0 \\ -1.5 & 4 & -0.5 & \dots & 0 & 0 \\ 0 & -1.5 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & -0.5 \\ 0 & 0 & 0 & \dots & -1.5 & 4 \end{pmatrix} \in \mathbb{R}^{m\times m},$$

and $b = Ax^* - |x^*|$, in which $x^* = (1, 2, 1, 2, ..., 1, 2, ...)^T$. One can easily see that A is an irreducible $M-$matrix. In Tables 4 and 5, we compare the obtained results by the new MTS algorithm with the obtained one from AOR and SOR iterative methods from the point of view the spectral radius, the error, the number of iterations and CPU times for Example 6.3. The numerical results confirm that the mixed-type splitting method has a faster asymptotic rate of convergence. As Table 5 states, the number of iterations and CPU-time in our proposed method is far less than AOR and SOR iterative methods.

Table 4: Comparison results of spectral radius for Example 6.3.

n	r	ω	$\rho(T_\omega)$	$\rho(T_r, \omega)$	$\rho(T)$
25	0.7	0.8	0.7854	0.7948	0.7765
100	0.7	0.8	0.8504	0.8576	0.8445
400	0.6	0.7	0.8932	0.8981	0.8801
900	0.4	0.6	0.9158	0.9228	0.8996
1600	0.2	0.4	0.9490	0.9527	0.9178
4900	0.7	0.8	0.8967	0.9043	0.8916
10000	0.5	0.6	0.9468	0.9513	0.9303

Example 6.4. Let m be a prescribed positive integer and $n = m^2$. Consider AVE (1) with $A = tridiag(-I, S, -I) \in \mathbb{R}^{n\times n}$ where $S = tridiag(-1, 8, -1)$ and $B = I$. The numerical results for Example 6.4 are presented in Tables 6 and 7. In Table 6, r and ω parameters are calculated randomly and the value of ω_0 for SOR-like method, is obtained from (4). The results in Table 7 are reported to illustrate the behavior of SOR-like method in comparison with the other methods. To this end, we only work with the dimensions used in [12, Example 2] and the

Table 5: Comparison results for Example 6.3

	SOR		AOR		MTS	
n	Iter (CPU)	Err	Iter (CPU)	Err	Iter (CPU)	Err
25	53(0.0081)	9.762e-07	57(0.0086)	8.197e-07	51(0.0061)	9.257e-07
100	91(0.0160)	9.384e-07	97(0.0159)	9.804e-07	88(0.0087)	8.919e-07
400	178(0.0414)	9.117e-07	190(0.0484)	9.611e-07	157(0.0235)	9.658e-07
900	296(0.1067)	9.484e-07	336(0.1364)	9.739e-07	250(0.0541)	9.180e-07
1600	630(0.3477)	9.645e-07	706(0.4536)	9.847e-07	386(0.1391)	9.566e-07
4900	351(0.5382)	9.954e-07	384(0.7058)	9.361e-07	342(0.3797)	9.897e-07
10000	745(2.1390)	9.603e-07	803(2.6923)	9.791e-07	587(1.1907)	9.661e-07

right-hand side b is constructed such that $x^* = (-1, 1, -1, 1, ..., -1, 1)^T$ satisfies in Eq. (1). As the Table 7 show, all of examined iterative methods are convergent and proposed iterative method outperforms other approaches.

Thus according to the numerical results in Tables 1-7, the new MTS algorithm is more powerful and efficient than the generalized Newton, Picard, SOR, AOR and SOR-like methods.

Table 6: The value of parameters for Example 6.4.

n	64	256	1024	4096
r	0.9239	0.9185	0.9007	0.2670
ω	0.9575	0.9729	0.9421	0.5688
ω_0	1.0671	1.0704	1.0714	1.0717

7 conclusion

In this paper, we presented a new algorithm for iteratively solving the AVE (1), in which the coefficient matrix A is an $M-$matrix. This method utilizes two auxiliary matrices and includes the AOR and SOR methods as special cases. Some sufficient conditions for the convergence of the method have been provided. The numerical results show the validity of the theoretical results and the efficiency of the new method. Moreover, it is concluded that by choosing appropriate auxiliary matrices, the new MTS method converges faster than the famous SOR, AOR, SOR-like, Generalized Newton and Picard methods.

Table 7: Numerical results of Example 6.4 for different values of n.

Method	n	64	256	1024	4096
Gen. Newton	IT	2	2	2	2
	CPU time	0.0035	0.0102	0.0538	0.4631
	Err	2.218e-16	3.370e-16	3.166e-16	3.379e-16
Picard	IT	8	8	8	8
	CPU time	0.0021	0.0027	0.0054	0.0307
	Err	6.920e-07	8.228e-07	8.882e-07	9.209e-07
SOR	IT	14	14	15	32
	CPU time	0.0032	0.0040	0.0071	0.0498
	Err	4.386e-07	4.753e-07	5.336e-07	9.808e-07
AOR	IT	14	14	15	35
	CPU time	0.0032	0.0048	0.0084	0.0608
	Err	5.215e-07	6.293e-07	6.548e-07	8.741e-07
SOR-like (ω_0)	IT	12	12	12	12
	CPU time	0.0038	0.0048	0.0095	0.0502
	Err	5.032e-07	7.585e-07	8.774e-07	9.282e-07
MTS	IT	14	14	15	25
	CPU time	0.0025	0.0027	0.0051	0.0212
	Err	4.310e-07	5.468e-07	5.069e-07	9.384e-07

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