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Original Research Paper

Using Majorizing Sequences for the Semi-local Convergence of a High-Order and Multipoint Iterative Method along with Stability Analysis

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Abstract. This paper deals with the study of relaxed conditions for semi-local convergence for a general iterative method, k-step Newton's method, using majorizing sequences. Dynamical behavior of the mentioned method is also analyzed via Julia set and basins of attraction. Numerical examples of nonlinear systems of equations will be examined to verify the given theory.

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1 Introduction

One of the most important problems in engineering and sciences is approximation the root x^* of $F(x) = 0$ [9, 11]. Choosing the initial value x_0 suitably guarantees the acceptable solution for the given equation. To

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this end, semi-local convergence analysis is used [21, 24, 30]. In other words, semi-local convergence analysis provides some conditions about how the initial value x_0 to be chosen appropriately [5]. Therefore, in semi-local convergence analysis, it is tired to find as much as possible simple initial conditions in such way that are relaxed compared to the current methods. We emphasize that our approach is totally different from the previous studies. We focus on utilizing majorizing sequences to obtain the initial conditions, which has not been considered in the literature for k-step Newton's method with frozen derivative. In addition, it is attempted to consider the dynamical behavior of the general Traub's iterative method using Julia sets and basins of attraction. The main purpose of studying the dynamical behavior is to find the stable and unstable regions along with the chaos of the given iterative method in the complex plane [14, 15, 28, 27, 36]. This is another superiority and contribution of this study. It is worth noting that the graphical approaches such as basins of attraction and parametric planes have been applied for Newton's and Traub's methods to study the stability of them [16, 7, 13]. We also point out that some of researchers have studied on semi-local convergence of them [29, 10, 5, 26]. However, we focus on this paper from a different point of view. We study the stability and choas of different steps. To this end, based on the different steps of the considered method, some systems of nonlinear equations are verified and analyzed.

Definition 1.1. *(Majorized sequence) Let $\{x_n\}$ be a sequence in a Banach space X, and $\{t_n\}$ be an increasing scalar sequence. We could say $\{x_n\}$ is majorized by $\{t_n\}$ if $||x_{n+1} - x_n|| \leq t_{n+1} - t_n$, for each $n = 0, 1, 2, \ldots$. [18]*

Hence, the convergence of the sequence $\{x_n\}$ is obtained from the convergence of the sequence $\{t_n\}$ [26]. So far, majorized sequences have been used for the study of semi-local analysis for two steps of iterative methods [32, 6, 5, 19]. In this paper, we use it for a general iterative method, say Traub's method or k-step Newton's method with frozen derivatives that is given by

$$\begin{cases} x_n = y_n^0 \\ y_n^1 = y_n^0 - F'(y_n^0)^{-1}F(y_n^0) \\ \\ y_n^2 = y_n^1 - F'(y_n^0)^{-1}F(y_n^1) \\ \vdots \\ x_{n+1} = y_n^k = y_n^{k-1} - F'(y_n^0)^{-1}F(y_n^{k-1}). \end{cases} \tag{1}$$

This method has convergence order $k+1$, and it only uses the first derivative, so its computational cost is less than one step methods such as Chebyshev–Halley-type methods [35, 34]. S. Amat et al. in [2, 1], M. A. Hernández-Verón in [23], and Argyros et al. in [8] studied the semi-local convergence of k-step Newton's method, and completely studied on efficiency index of the method. But, we present semi-local convergence of the method (1) by different technique, majorizing sequences. This technique has been used only for methods with at most two steps. In this paper, by drawing the dynamic planes for different degrees of polynomials and some numerical examples, the influence of the increasing of the steps of the k-step Newton method on the stability and convergence of the method is analyzed. Based on the results of the semi-local convergence, the radius of the convergence can be computed, and the convergence of the method for different initial values can be shown. We introduce it in some tables. For numerical examples, we can examine the results for large steps of the k-step Newton method. Meanwhile, for solving any system of equations with the k-step Newton method, a Mathematica code is presented. One of the advantages of our study compared to the mentioned studies is that it includes less and simpler conditions about the initial guess. Moreover, we were able to check the validity and applicability of our conditions for numerical examples. Meanwhile, we provided a bound for the error $||x_n - x^*||$ in Example 4.1. As we expressed, our initial guess is very simple and can be used effectively as opposed to the current studies.

This paper is organized as follow: In Section 2, by using majorizing sequences semi-local convergence of the k-step Newton's method is obtained. In Section 3, the stability of the k-step Newton method is studied on second, third, and fourth polynomials by basins of attraction and Julia set. In Section 4, some numerical examples are presented to show the applicability of the theoretical results.

2 Semi-local convergence of k-step Newton's method

To obtain the semi-local convergence, we need some auxiliary relations. We intent to gain them by a lemma and majorizing sequences.

Lemma 2.1. *Suppose $L > 0$, $L_0 > 0$, and $s_0^1 > 0$ be parameters. The polynomial q that define by*

$$q(t) = 2L_0 t^{k+1} - L(2 - t^{k-1} - t^k), \tag{2}$$

where the polynomial q has a unique root α in the interval $(0, 1)$. We define the sequence $\{t_n\}$ for $n = 0, 1, 2, \dots$ and $j = 1, 2, \dots, k-1$ by

$$t_0 = 0, \quad s_n^0 = t_n, \quad s_{n+1}^1 = t_{n+1} + \frac{L(t_{n+1} - t_n + s_n^{k-1} - t_n)(t_{n+1} - s_n^{k-1})}{1 - 2L_0(t_{n+1} - t_0)}, \tag{3}$$

$$s_n^k = t_{n+1}, \quad s_n^{j+1} = s_n^j + \frac{L(s_n^j - t_n + s_n^{j-1} - t_n)(s_n^j - s_n^{j-1})}{1 - 2L_0(t_n - t_0)}. \tag{4}$$

Then the sequence $\{t_n\}$ is an increasing and bounded above by $t^{**} = \dfrac{s_0^1}{1-\alpha}$, so that $\{t_n\}$ converges to its least upper bound t^* and we have $t_1 < t^* < t^{**}$. Also, we have the following assumption:

$$0 < \frac{L(t_1 + s_0^{k-1})}{1 - 2L_0 t_1} \leq \alpha < 1 - 2L_0 s_0^1. \tag{5}$$

Moreover, the following estimates are hold for $n = 0, 1, \dots$ and $j = 2, \dots, k$:

$$(s_n^j - s_n^{j-1}) \leq \alpha(s_n^{j-1} - s_n^{j-2}) \leq \alpha^{kn+j-1}(s_0^1 - t_0), \tag{6}$$

$$(s_{n+1}^1 - s_n^k) \leq \alpha(s_n^k - s_n^{k-1}) \leq \alpha^{k(n+1)}(s_0^1 - t_0), \tag{7}$$

$$t_n = s_n^0 \leq s_n^1 \leq s_n^2 \leq \cdots \leq s_n^{k-1} \leq s_n^k = t_{n+1}. \tag{8}$$

Proof. We have $q(0) = -2L$ and $q(1) = 2L_0$. Hence, by intermediate value theorem, $q(t)$ has roots in the interval $(0, 1)$. $q'(t) = 2(k+1)L_0t^k + Lt^{k-2}(tk + k - 1) > 0$ for all points in the interval $(0, 1)$. Hence, the graph of q only intersects the x-axis in the interval $(0, 1)$. So, q has unique root in the interval $(0, 1)$ that we denote it by α.
In what follows, we try to prove that the sequence $\{t_n\}$ is bounded and increasing. This is equivalent to validity of estimates (6)-(8). But they are true if the following relations are true for $m = 0, 1, \ldots$ and $j = 1, \ldots, k - 1$:

$$0 < \frac{L(s_m^j - t_m + s_m^{j-1} - t_m)}{1 - 2L_0t_m} < \alpha, \tag{9}$$

$$0 < \frac{L(s_m^k - t_m + s_m^{k-1} - t_m)}{1 - 2L_0t_{m+1}} < \alpha, \tag{10}$$

$$t_m = s_m^0 \leq s_m^1 \leq s_m^2 \leq \cdots \leq s_m^{k-1} \leq s_m^k = t_{m+1}. \tag{11}$$

We prove (9)-(11) by induction on m. Using the difintion (4) ,we have for $j = 1, \ldots, k - 1$

$$s_0^{j+1} - s_0^j = L(s_0^j + s_0^{j-1})(s_0^j - s_0^{j-1}), \tag{12}$$

that is, if $s_0^j - s_0^{j-1} > 0$, then $s_0^{j+1} - s_0^j > 0$ for every $j = 1, \ldots, k - 1$. Hence, because $s_0^0 = t_0 = 0$ and $s_0^1 > 0$, we conclude that $s_0^{j+1} - s_0^j > 0$ for every $j = 1, \ldots, k - 1$. So as a result, the relation (11) is true for $m = 0$.
Also, using (5), the relations (9) and (10) are true for $m = 0$. Now, we suppose relations (9)-(11) are true for $m = 1, 2, \ldots, n$. We have to prove them for $m > n$. Meanwhile, we have

$$\begin{aligned} 0 <& \frac{L(s_{m-1}^j - t_{m-1} + s_{m-1}^{j-1} - t_{m-1})}{1 - 2L_0t_{m-1}} < \frac{L(s_m^j - t_m + s_m^{j-1} - t_m)}{1 - 2L_0t_m} \\ <& \frac{L(s_m^k - t_m + s_m^{k-1} - t_m)}{1 - 2L_0t_{m+1}}. \end{aligned} \tag{13}$$

Therefore, it would be enough that the correctness of the relation (10) can be proved. Using the hypotheses of induction and relations (6) and

(7), it is deduced that:

$$\begin{aligned}
s_n^j &\le s_n^{j-1} + \alpha^{kn+j-1}(s_0^1 - t_0)\\
&\le s_n^{j-2} + \alpha^{kn+j-2}(s_0^1 - t_0) + \alpha^{kn+j-1}(s_0^1 - t_0)\\
&\le s_n^1 + \alpha^{kn+1}(s_0^1 - t_0) + \cdots + \alpha^{kn+j-1}(s_0^1 - t_0)\\
&\le s_{n-1}^k + \alpha^{kn}(s_0^1 - t_0) + \alpha^{kn+1}(s_0^1 - t_0) + \cdots + \alpha^{kn+j-1}(s_0^1 - t_0)\\
&\le s_1^0 + \alpha(s_0^1 - t_0) + \cdots + \alpha^{kn+j-1}(s_0^1 - t_0)\\
&= \frac{1-\alpha^{kn+j}}{1-\alpha}(s_0^1 - t_0) \le \frac{s_0^1}{1-\alpha} = t^{**}, \text{ for } j = 1, \ldots, k.
\end{aligned}$$

Now, based on the relation (13), the following relation have been shown.

$$\frac{L(s_m^k - s_{m-1}^k + s_m^{k-1} - s_{m-1}^k)}{1 - 2L_0 s_m^k} < \alpha,$$

or,

$$\frac{L(s_0^1 - t_0)(\frac{1-\alpha^{km+k}}{1-\alpha} - \frac{1-\alpha^{k(m-1)+k}}{1-\alpha} + \frac{1-\alpha^{km+k-1}}{1-\alpha} - \frac{1-\alpha^{k(m-1)+k}}{1-\alpha})}{1 - 2L_0 \frac{1-\alpha^{km+k}}{1-\alpha}(s_0^1 - t_0)} < \alpha,$$

or

$$\frac{L\alpha^{km-1}(s_0^1 - t_0)(2 - \alpha^k - \alpha^{k-1})}{1 - \alpha - 2L_0(1 - \alpha^{km+k})(s_0^1 - t_0)} < 1. \tag{14}$$

The function $f_m(t)$ is defined on the interval $(0, 1)$ by the following relation:

$$f_m(t) = Lt^{km-1}(s_0^1 - t_0)(2 - t^k - t^{k-1}) + 2L_0(1 - t^{km+k})(s_0^1 - t_0) + t - 1.$$

The relation (14) is true if $f_m(\alpha) < 0$ for all $m = 1, 2, \ldots$. For this aim, by some algebraic relations, the following recurrent relation is obtained:

$$f_{m+1}(t) - f_m(t) = (1 - t^k)t^{km-1}(s_0^1 - t_0)(2L_0 t^{k+1} - L(2 - t^{k-1} - t^k)).$$

Because α is root of q, we get that

$$f_{m+1}(\alpha) - f_m(\alpha) = (1 - t^k)\alpha^{km-1}(s_0^1 - t_0)q(\alpha) = 0.$$

Therefore, for each $m = 1, 2, \ldots,$ we have

$$f_{m+1}(\alpha) = f_m(\alpha) = f_\infty(\alpha),$$

where $f_\infty(\alpha) = lim_{m\to\infty} f_m(t)$. In the other hand, according to the assumption (5), the relation $f_\infty(\alpha) = 2L_0 s_0^1 + \alpha - 1 < 0$ is established. So, we conclude that $f_m(\alpha) < 0$ and the estimates (6)-(8) are true. Therefore, $\{t_n\}$ is an increasing and bounded sequence by t^{**}, so it converges to least upper bound t^* and the proof is completed. □

Now, we show the semi-local convergence of the k-step Newton's method by using this lemma.

Theorem 2.2. *Let $F : D \subseteq X \to Y$ be a Fréchet- differentiable operator and X and Y are Banach spaces such that D be a convex subset of X. Divided difference of order one for operator F on $D \times D$ defines by $[.,.;F]$. Also, we have $[x,x;F] = F'(x)$. Suppose that there exist $x_0 \in D$ and $0 < L_0 \leq L$ such that for every x, y, z, and $t \in D$*

$$F'(x_0)^{-1} \in L(Y, X), \tag{15}$$

$$||F'(x_0)^{-1}F(x_0)|| \leq s_0^1, \tag{16}$$

$$||F'(x_0)^{-1}([x, y; F] - F'(x_0))|| \leq L_0(||x - x_0|| + ||y - x_0||), \tag{17}$$

$$||F'(x_0)^{-1}([x, y; F] - [z, t; F])|| \leq L(||x - z|| + ||y - t||), \tag{18}$$

and all of the hypotheses of Lemma 2.1 are confirmed. Then, the sequence $\{x_n\}$, generated by the method (1), converges to $x^* \in \overline{U}(x_0, t^*) \subseteq D$ and remains in $\overline{U}(x_0, t^*)$. Moreover, x^* is the unique solution of $F(x) = 0$ in the $\overline{U}(x_0, t^*)$. Also, the following estimate holds for each $n = 0, 1, 2, \ldots$

$$||x^* - x_n|| \leq t^* - t_n. \tag{19}$$

Proof. By induction on n, we shall show that

$$\begin{aligned}
&\Lambda_1 : && ||y_n^1 - x_n|| \leq s_n^1 - t_n, \\
&\Lambda_j : && ||y_n^j - y_n^{j-1}|| \leq s_n^j - s_n^{j-1}, \qquad for\, j = 2\ldots, k-1, \\
&\Lambda_k : && ||x_{n+1} - y_n^{k-1}|| \leq t_{n+1} - s_n^{k-1}.
\end{aligned}$$

For $n = 0$ and first step, by assumption (16), we have

$$||y_0^1 - x_0|| = ||F'(x_0)^{-1}F(x_0)|| \leq s_0^1 \leq t^*.$$

Hence, $y_0^1 \in \overline{U}(x_0, t^*)$, and Λ_1 holds for $n = 0$.

For the jth step of k-step Newton's method, $j = 1, \dots, k-1$, we obtain that

$$F'(x_n)(y_n^j - y_n^{j-1}) = -F(y_n^{j-1}). \tag{20}$$

Also, For the jth step of k-step Newton's method, $j = 2, \dots, k-1$, by using (20) and (17), we get that

$$\begin{aligned} ||y_0^j - y_0^{j-1}|| &= ||F'(x_0)^{-1}(F(y_0^{j-1}) - F(y_0^{j-2}) + F(y_0^{j-2}))|| \\ &= ||F'(x_0)^{-1}(F(y_0^{j-1}) - F(y_0^{j-2}) - F'(x_0)(y_0^{j-1} - y_0^{j-2}))|| \\ &\leq ||F'(x_0)^{-1}([y_0^{j-1}, y_0^{j-2}; F] - F'(x_0))||||y_0^{j-1} - y_0^{j-2}|| \\ &\leq L_0(||y_0^{j-1} - x_0|| + ||y_0^{j-2} - x_0||)||y_0^{j-1} - y_0^{j-2}|| \\ &\leq L(||y_0^{j-1} - x_0|| + ||y_0^{j-2} - x_0||)||y_0^{j-1} - y_0^{j-2}|| = s_0^j - s_0^{j-1}. \end{aligned} \tag{21}$$

Hence, for $j = 2, \dots, k-1$, we have

$$\begin{aligned} ||y_0^j - x_0|| &\leq ||y_0^j - y_0^{j-1}|| + ||y_0^{j-1} - y_0^{j-2}|| + \cdots + ||y_0^1 - x_0|| \\ &\leq s_0^j - s_0^{j-1} + s_0^{j-1} - s_0^{j-2} + \cdots + s_0^1 - t_0 = s_0^j \leq t^*. \end{aligned}$$

Therefore, $y_0^j \in \overline{U}(x_0, t^*)$, and Λ_j holds for $n = 0$, for $j = 2, \dots, k-1$.

Then for last step, kth step, of k-step Newton's method, such as (21) we obtain that

$$\begin{aligned} ||y_0^k - y_0^{k-1}|| &= ||x_1 - y_0^{k-1}|| \\ &\leq ||F'(x_0)^{-1}([x_1, y_0^{k-1}; F] - F'(x_0))||||x_1 - y_0^{k-1}|| \\ &\leq L(||x_1 - x_0|| + ||y_0^{k-1} - x_0||)||x_1 - y_0^{k-1}|| \\ &= t_1 - s_0^{k-1}, \end{aligned}$$

and

$$||x_1 - x_0|| \leq ||x_1 - y_0^{k-1}|| + ||y_0^{k-1} - x_0|| \leq t_1 - s_0^{k-1} + s_0^{k-1} - t_0 = t_1 \leq t^*.$$

So, $x_1 \in \overline{U}(x_0, t^*)$, and Λ_k holds for $n = 0$.

Now, we try to show Λ_i is true for $i = 1, \ldots, k$ when $n = 1$ by induction on j. First, we have to show $F'(x_1)^{-1}$ exists so that using (17) and (5), we can deduce that

$$||F'(x_0)(F'(x_1) - F'(x_0))|| \leq L_0(||x_1 - x_0|| + ||x_1 - x_0||) = 2L_0 t_1 < 1.$$

It follows by the Banach lemma on invertible operators that $F'(x_1)^{-1}$ exists and

$$||F'(x_1)^{-1}F'(x_0)|| \leq \frac{1}{1 - 2L_0||x_1 - x_0||}. \tag{22}$$

Using (17), (22), and the kth step of the method (1), for the first step of k-step Newton's method, we get that

$$\begin{aligned}
||y_1^1 - x_1|| &\leq ||F'(x_1)^{-1}F'(x_0)||||F'(x_0)^{-1}F(x_1)|| \\
&\leq \frac{||F'(x_0)^{-1}(F(x_1) - F(y_0^{k-1}) - F'(x_0)(x_1 - y_0^{k-1}))||}{1 - 2L_0||x_1 - x_0||} \\
&\leq \frac{||F'(x_0)^{-1}([x_1, y_0^{k-1}; F] - F'(x_0))||||x_1 - y_0^{k-1}||}{1 - 2L_0||x_1 - x_0||} \\
&\leq \frac{L_0(||x_1 - x_0|| + ||y_0^{k-1} - x_0||)||x_1 - y_0^{k-1}||}{1 - 2L_0||x_1 - x_0||} \\
&\leq \frac{L(t_1 - t_0 + s_0^{k-1} - t_0)(t_1 - s_0^{k-1})}{1 - 2L_0(t_1 - t_0)} \\
&= s_1^1 - t_1,
\end{aligned}$$

and

$$||y_1^1 - x_0|| \leq ||y_1^1 - x_1|| + ||x_1 - x_0|| \leq s_1^1 - t_1 + t_1 - t_0 = s_1^1 \leq t^*.$$

So, $y_1^1 \in \overline{U}(x_0, t^*)$, and Λ_1 holds for $n = 1$.

Now, we suppose Λ_i is true for every $i < j$. Then using (20) and (18), for jth step of the method (1) for $j = 2, \ldots, k$, it is obtained that

$$\begin{aligned}||y_1^j - y_1^{j-1}|| &\leq ||F'(x_1)^{-1}F'(x_0)||||F'(x_0)^{-1}F(y_1^{j-1})|| \\ &\leq \frac{||F'(x_0)^{-1}(F(y_1^{j-1}) - F(y_1^{j-2}) - F'(x_1)(y_1^{j-1} - y_1^{j-2}))||}{1 - 2L_0||x_1 - x_0||} \\ &\leq \frac{||F'(x_0)^{-1}([y_1^{j-1}, y_1^{j-2}; F] - F'(x_1))||||y_1^{j-1} - y_1^{j-2}||}{1 - 2L_0||x_1 - x_0||} \\ &\leq \frac{L(||y_1^{j-1} - x_1|| + ||y_1^{j-2} - x_1||)||y_1^{j-1} - y_1^{j-2}||}{1 - 2L_0||x_1 - x_0||} \\ &\leq \frac{L(s_1^{j-1} - t_1 + s_1^{j-2} - t_1)(s_1^{j-1} - s_1^{j-2})}{1 - 2L_0(t_1 - t_0)} \\ &= s_1^j - s_1^{j-1},\end{aligned}$$

and by hypotheses of induction, it is deduced that

$$||y_1^j - x_0|| \leq ||y_1^j - y_1^{j-1}|| + ||y_1^{j-1} - x_0|| \leq s_1^j - s_1^{j-1} + s_1^{j-1} - t_0 = s_1^j \leq t^*.$$

So, $y_1^j \in \overline{U}(x_0, t^*)$, and Λ_j holds for $n = 1$ and every $j = 2, \ldots, k$.

If we replace the role of x_1, y_1^1, y_1^2, y_1^3, ...,y_1^k with x_n, y_n^1, y_n^2, y_n^3, ...,y_n^k, the relations Λ_j, $j = 1, \ldots, k$ are true. Also, it is obtained that

$$\begin{aligned}||x_{n+1} - x_n|| = ||y_n^k - y_n^0|| &\leq \Sigma_{i=0}^{i=k-1}||y_n^{k-i} - y_n^{k-i-1}|| \\ &\leq \Sigma_{i=0}^{i=k-1}(s_n^{k-i} - s_n^{k-i-1}) = s_n^k - s_n^0.\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence in the closed subset, $\overline{U}(x_0, t^*)$, of the Banach space X. Therefore, $\{x_n\}$ converges to x^* in the $\overline{U}(x_0, t^*)$. By relation (20), it is deduced that

$$||F'(x_0)^{-1}F(y_n^j)|| \leq L(||y_n^j - x_n|| + ||y_n^{j-1} - x_n||)||y_n^j - y_n^{j-1}||.$$

By letting $k \to \infty$ and for $j = 1, \ldots, k$, we get that

$$F(x^*) = \lim_{k\to\infty} F(y_n^j) = 0,$$

because F is continuous and the sequences $\{y_n^j\}$ are majorized by the sequence $\{x_n\}$, so the point x^* is the solution of $F(x) = 0$.

Also, if y^* is another solution of the equation $F(x) = 0$ in the $\overline{U}(x_0, t^*)$, then by (5), it is obtained that

$$\begin{aligned}||F'(x_0)([x^*, y^*; F] - F'(x_0))|| &\leq L_0(||x^* - x_0|| + ||y^* - x_0||)\\ &\leq L_0(t^* + t^*) \leq L_0(\frac{2s_0^1}{1-\alpha})\\ &\leq 1.\end{aligned}$$

Therefore, $[x^*, y^*; F]^{-1}$ exists by Banach lemma on invertible operators. So, using the relation

$$[x^*, y^*; F](x^* - y^*) = F(x^*) - F(y^*) = 0,$$

we get that $x^* = y^*$.

The sequence $\{x_n\}$ is a Cauchy sequence, so by induction on m for each $m = 1, 2, \ldots,$ we have

$$||x_{m+n} - x_n|| \leq t_{m+n} - t_n,$$

then, by letting $m \to \infty$, it is obtained that

$$||x^* - x_n|| \leq t^* - t_n,$$

where this error bound is always less than $\frac{2s_0^1}{1-\alpha} - t_1$, so the estimate (19) is satisfied. Hence, the proof of Theorem is completed. □

3 Complex dynamics

In this section, we compare the stability of the k-step Newton method for some k based on the concepts of Julia sets and basins of attraction. We first explain some basic concepts according to the Blanchard paper [12].

Definition 3.1. *Let $R : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be the rational function corresponding to the function f. An orbit of $z_0 \in \hat{\mathbf{C}}$ is denoted by*

$$O^+(z_0) = \{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\},$$

where $\hat{\mathbf{C}}$ is Riemann sphere. In addition, the point z_0 is a periodic point of period n if it satisfies $R^n(z_0) = z_0$, and is a fixed point if $n = 1$. Moreover, a fixed point z_0 is called attractor, superattractor, indifferent, or repulsive if $|R'(z_0)| < 1$, $|R'(z_0)| = 0$, $|R'(z_0)| = 1$, or $|R'(z_0)| > 1$, respectively. Any fixed point of the rational function R that is not the root of the function f is denoted by strange fixed point . Each root of the equation $R'(z) = 0$ is denoted by critical points, and free critical points if $f(z)$ is also equal to zero. For more details, one can refer to [30].

Theorem 3.2. *(Fatou-Julia) Let R be a rational function. There exists, at least, one critical point in the connected component of the basin of attraction of an attracting fixed periodic point [20, 25].*

Definition 3.3. *If α is a root of the function f, then the following set is denoted by basin of attraction or convergence region of α:*

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbf{C}} : \lim_{n\to\infty} R^n(z_0) = \alpha\}.$$

Julia and Fatou sets have an important role in the analysis of the dynamical system defined by iterating the rational function R. In the following, we try to introduce these two significant sets. We have to propose some definitions and preliminary propositions. For further descriptions, one can see [17, **?**].

Definition 3.4. *Suppose $\{f_\alpha\}$ is a family of complex analytic functions defined on a domain D. $\{f_\alpha\}$ is called a normal family if every infinite subset of it contains a subsequence which converges uniformly on every compact subset of D.*

Definition 3.5. *Let R be the rational function associated with iteration ϕ. The point z is a stable point for R if there is a neighborhood U of z such that $\{R^n(z)\}$, for $n = 0, 1, \dots,$ form a normal family on U.*

Definition 3.6. *All of the stable points of R belong to stable set of R that is called normal set or Fatou set.*

Definition 3.7. *Julia set is the complement of Fatou set and is the unstable set.*

Fatou set is denoted by $\mathcal{F}(R)$, and Julia set is denoted by $\mathcal{J}(R)$. They are completely invariant under R; that is

$$R(\mathcal{F}(R)) = \mathcal{F}(R) = R^{-1}(\mathcal{F}(R)), \quad R(\mathcal{J}(R)) = \mathcal{J}(R) = R^{-1}(\mathcal{J}(R))$$

In [12, 17, **?**], we see that the behavior of rational map on Julia set is chaotic and on Fatou set is stable.

Remark 3.8. In this paper, for numerical results and plotting the pictures, we work in Mathematica. For plotting the pictures, we have used the codes that base of them presented in [33]. The computer specifications are Intel(R) Xeon(R), CPU E7-4870 2.40 GHz (2 processors), with 16 GB of RAM.

3.1 Conjugacy classes

Theorem 3.9. *(Scaling Theorem for Newton's method [4]) Let $T(x) = \alpha x + \beta$, with $\alpha \neq 0$, be an affine map. Let $g(x) = (foT)(x)$ such that $f(x)$ be a polynomial. Then the fixed point operators of Newton's method on f and g, R_f and R_g, respectively, are affinely conjugated by T, that is, $(ToR_g)(x) = (R_f oT)(x)$ for all x.*

For study the convergence regions of k-step Newton's method for finding the roots of the function f, we can use Theorem 3.9 and study on a family of functions and next extend the results for all of the functions that have the features in family where it is very helpful.
S. Amat et.al in [3, 2] have presented the dynamical studies for k-step Newton's method for some steps. In [3], they presented dynamical behavior for two steps Newton's method. In [2], they presented three basins of attraction for comparing the stability of three, four, and five steps Newton's method. They showed that black regions are bigger in four steps method.
Our aim in this paper is to study the convergence regions and stability behavior of k-step Newton's method for different steps, one, two,...,and eight steps, and compare them with together for two, three, and four

degree polynomials.
Therefore, we consider $p(x) = x^2 + c$ where c is an arbitrary complex number, and second degree polynomials can be parameterized to it. (one can see [12]). So, our results can be acceptable for all of the second degree polynomials. Now, the map $R_{k,p}(x, c)$ is called the rational operator associated with the polynomial $p(x) = x^2 + c$ and k-step Newton's method for $k = 1, \dots, 6$.

Also, Blanchard in [12] showed that by conjugacy map $h(x) = \frac{x-i\sqrt{c}}{x+i\sqrt{c}}$ (a Möbius transformation) rational operator R of Newton's method for quadratic polynomials is conjugate to the rational map x^2. The Möbius transformation has these properties:

$$i)\, h(\infty) = 1, \quad ii)\, h(i\sqrt{c}) = 0, \quad iii)\, h(-i\sqrt{c}) = \infty.$$

In the analogous ways, operators $R_{k,p}(x, c)$ on quadratic polynomials for k-step Newton's method, for $k = 2, 3, 4, 5, 6$, is conjugated to operator $R_{k,p}(x)$. Therefore, we have

$$R_{k,p}(x) = h\, o\, R_{k,p}(x, c)\, o\, h^{-1}(x).$$

One can compute $R_{k,p}(x)$ for each k by a computer software such as Mathematica, easily.

Remark 3.10. We mention that $\sqrt{c}$ is the principal second root of c.

3.2 Strange fixed points

Strange fixed points are the roots of the equation $R_{k,p}(x) = x$ that are not the zeros of $p(x) = 0$, where $R_{k,p}$ is conjugated to rational operator associated with iteration function of k-step Newton's method. By conjugacy relations, the roots of $p(x)$ are equivalent to 0 and ∞. It is clear that ∞ is the fixed point for $R_{k,p}(x)$ for all $k = 1, 2, \dots$. We solve the equations $R_{k,p}(x) = x$ for x and $k = 1, 2, 3, 4, 5, 6, 7, 8$, and obtain all of the fixed points of $R_{k,p}(x)$ except ∞. For all of the k, one of the roots is 0. So others are strange fixed points. In Table 1, we can see the numbers of the roots of the equation $R_{k,p}(x) = x$ for different k.

Remark 3.11. Infinity is the strange fixed point for $R_{k,p}(x, c)$ for each k. Moreover, by Möbius transformation, infinity changes to 1. It is clear that $|R'_{k,p}(1)| > 1$ for each k. Therefore, infinity is a repulsive strange fixed point.

Table 1: fixed points for $R_{k,p}(x)$ except of infinity

k	1	2	3	4	5	6	7	8	9
numbers	2	4	8	16	32	64	128	256	512

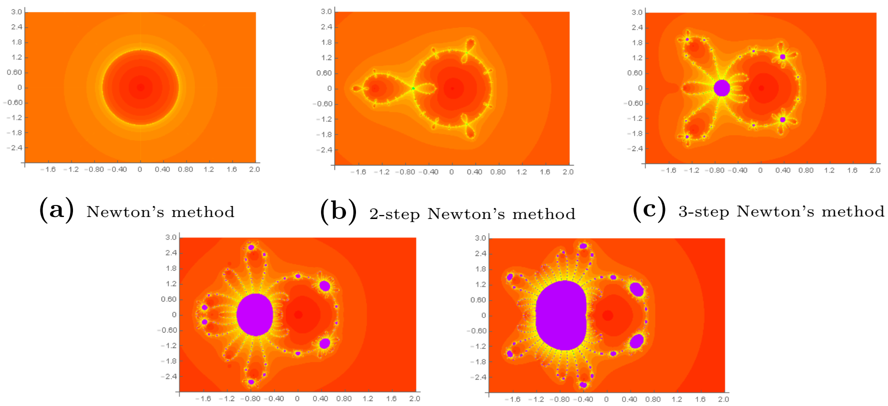
Möbius transformation changes the roots of polynomial $p(x)$ to 0 and ∞ that they are supperattracting points. If we compute the $|R'_{k,p}(x)|$ for strange fixed point x, we find that all of the strange fixed points are repulsive, so, there is no attracting strange fixed point for k-step Newton method on the family of quadratic polynomials. Therefore, there is no free critical point for k-step Newton method on the family of the quadratic polynomials, because by using Theorem 3.2, critical points belong to basin of attraction of the attracting fixed points.

3.3 Julia sets

Because we have not any critical point for quadratic polynomials, we cannot plot parameter planes. Thus, for comparing the behavior of k-step Newton's methods on quadratic polynomials, we use Julia sets. It is noteworthy that for quadratic polynomials, Blanchard in [13] proved that one step Newton method is globally convergent.

Figure 1 shows that when the steps of the k-step Newton's method increases, the Julia set becomes more complicated and purple color becomes bigger.

Red regions are stable points. For any z in the red regions, the orbit of z has a convergent subsequence to 0 or ∞ while yellow regions are unstable points. Purple color is the points that orbit of them does not converge to 0 or ∞. The pictures concluded that increasing in the steps of the k-step Newton method may increase the chaotic in root finding.



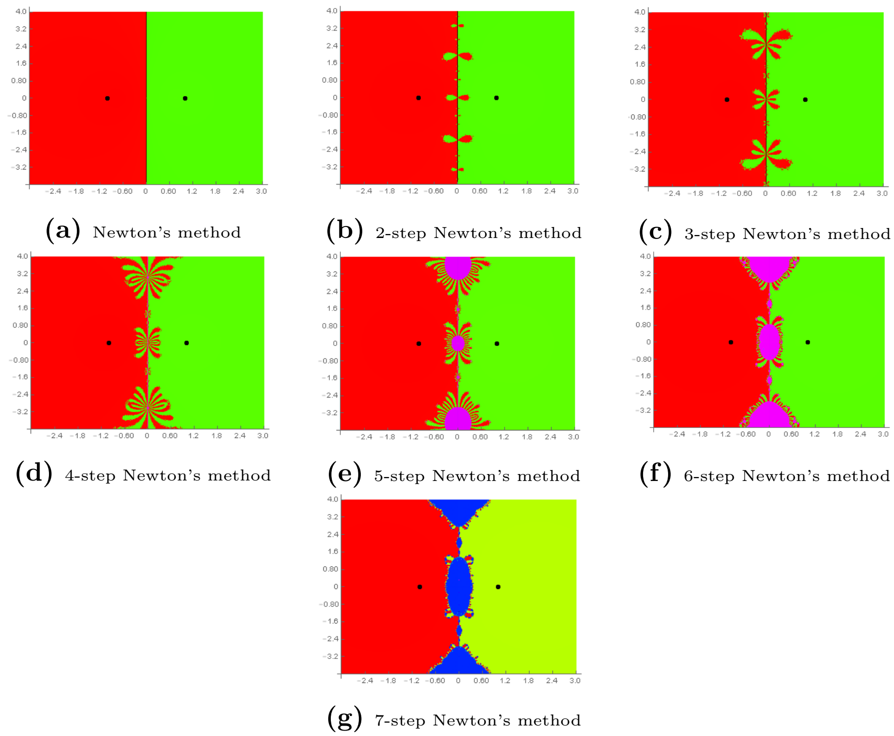
(a) Newton's method **(b)** 2-step Newton's method **(c)** 3-step Newton's method

(d) 4-step Newton's method **(e)** 5-step Newton's method

Figure 1: Comparing the Julia sets for 1,2,3,4,5 steps Newton's method for second degree polynomials

3.4 Basins of attraction

Dynamic planes show basins of attraction of any root along with chaotic behaviors. We plot dynamic planes for $x^2 + c$, $x^3 + c$, $x^4 + c$, where c is arbitrary chosen. In Figures 2-7 we must have two distinct colors for second degree polynomials, three distinct colors for third degree polynomials, and four distinct colors for fourth degree polynomials. But we see that when the step k increases, we have more distinct colors and chaotic behaviors increase, too. Black points are the roots of polynomials. Moreover, these figures show that for high-order polynomials, when the steps of the method (1) increase, the basins of attraction of the roots of polynomial become smaller at a faster rate.



(a) Newton's method

(b) 2-step Newton's method

(c) 3-step Newton's method

(d) 4-step Newton's method

(e) 5-step Newton's method

(f) 6-step Newton's method

(g) 7-step Newton's method

Figure 2: Dynamic planes of k-step Newton's method for $p(x) = x^2 - 1$

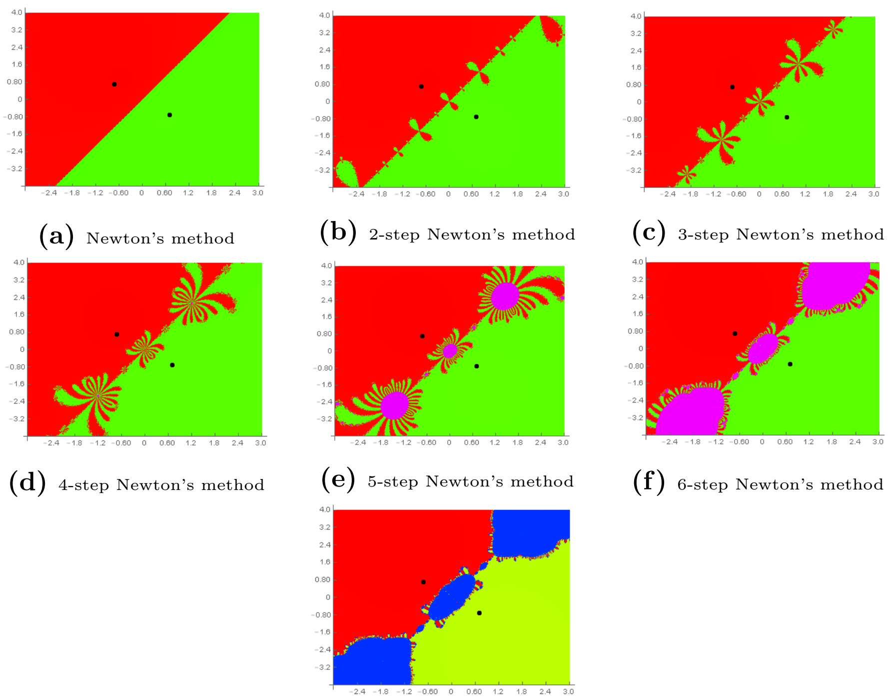
4 Numerical Examples

We want to confirm our theoretical results for semi-local convergence by some numerical examples.

Example 4.1. Let $F : \mathbf{R}^m \to \mathbf{R}^m$ be a differentiable function where

$$F(x_1, x_2, \ldots, x_m) = (e^{x_1} - 1, e^{x_2} - 1, \ldots, e^{x_m} - 1), x_i \in (-1, 1),$$
$$i = 1, \ldots, m.$$

We mention that e is the Exponential number. For $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_m)$, the divided difference $[X, Y; F]$ is given by

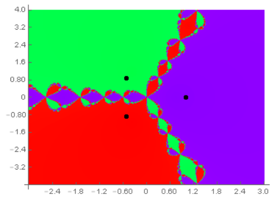


(a) Newton's method (b) 2-step Newton's method (c) 3-step Newton's method

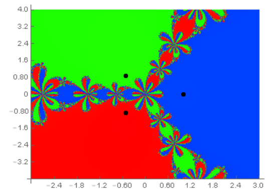
(d) 4-step Newton's method (e) 5-step Newton's method (f) 6-step Newton's method

(g) 7-step Newton's method

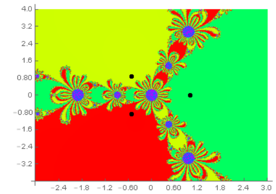
Figure 3: Dynamic planes of k-step Newton's method for $p(x) = x^2 + i$



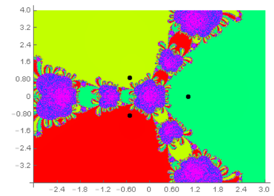
(a) Newton's method



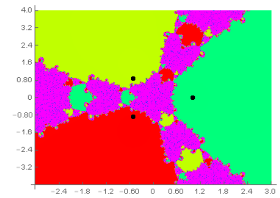
(b) 2-step Newton's method



(c) 3-step Newton's method

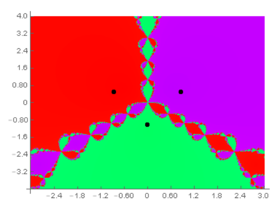


(d) 4-step Newton's method

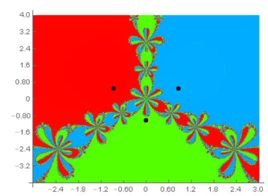


(e) 5-step Newton's method

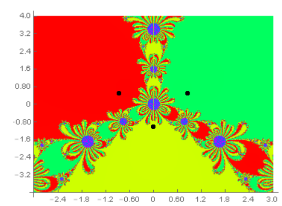
Figure 4: Dynamic planes of k-step Newton's method for $p(x) = x^3 - 1$



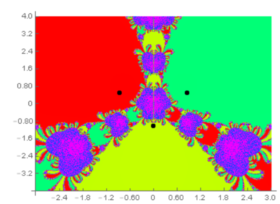
(a) Newton's method



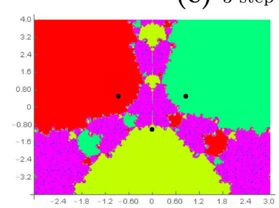
(b) 2-step Newton's method



(c) 3-step Newton's method

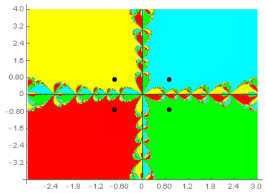


(d) 4-step Newton's method

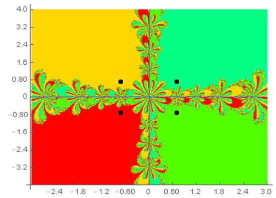


(e) 5-step Newton's method

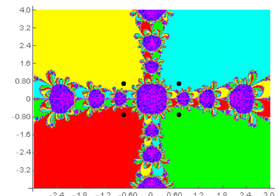
Figure 5: Dynamic planes of k-step Newton's method for $p(x) = x^3 - i$



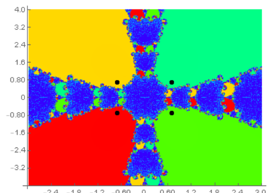
(a) Newton's method



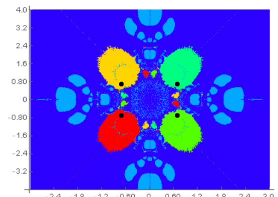
(b) 2-step Newton's method



(c) 3-step Newton's method

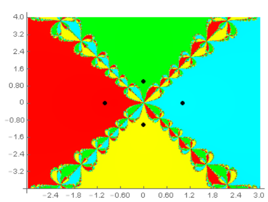


(d) 4-step Newton's method

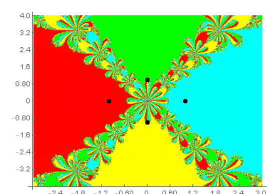


(e) 5-step Newton's method

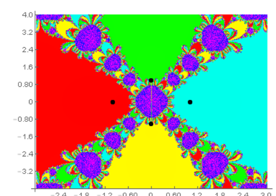
Figure 6: Dynamic planes of k-step Newton's method for $p(x) = x^4 + 1$



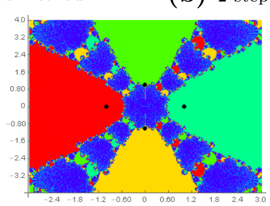
(a) Newton's method



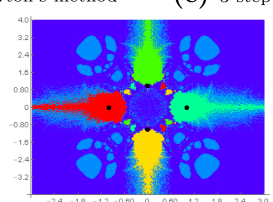
(b) 2-step Newton's method



(c) 3-step Newton's method



(d) 4-step Newton's method



(e) 5-step Newton's method

Figure 7: Dynamic planes of k-step Newton's method for $p(x) = x^4 - 1$

$$\begin{pmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{e^{x_m}-e^{y_m}}{x_m-y_m} \end{pmatrix},$$

and Jacobian derivative of F is equal to

$$\begin{pmatrix} e^{x_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{x_m} \end{pmatrix}.$$

Suppose that $||.||$ is max-norm and $X_0 = (x_1^0, x_2^0, \dots, x_m^0)$, where $x_i^0 = x_0$ for each $i = 1, \dots, m$.
First, we must compute L_0 and L in the relations (17) and (18). Also, we use approximation formula $e^x \approx 1 + x + \frac{x^2}{2}$. So, we have

$$\begin{aligned} &||F'(X_0)^{-1}([X,Y;F] - [X_0,X_0;F])|| \\ &\qquad \le ||F'(X_0)^{-1}|| max_{1\le i\le m}(|\frac{e^{x_i}-e^{y_i}}{x_i-y_i} - e^{x_0}|) \\ &\qquad \le |e^{-x_0}| max_{1\le i\le m}(1 + \frac{1}{2}(x_i+y_i) - 1 - (x_0)) \\ &\qquad \le \frac{|e^{-x_0}|}{2}(||X-X_0|| + ||Y-X_0||), \end{aligned}$$

and

$$\begin{aligned} &||F'(X_0)^{-1}([X,Y;F] - [Z,T;F])|| \\ &\qquad \le ||F'(X_0)^{-1}|| max_{1\le i\le m}(|\frac{e^{x_i}-e^{y_i}}{x_i-y_i} - \frac{e^{z_i}-e^{t_i}}{z_i-t_i}|) \\ &\qquad \le |e^{-x_0}| max_{1\le i\le m}(1 + \frac{1}{2}(x_i+y_i) - 1 - \frac{1}{2}(z_i+t_i)) \\ &\qquad \le \frac{|e^{-x_0}|}{2}(||X-Z|| + ||Y-T||). \end{aligned}$$

Hence, it is obtained $L = L_0 = \frac{e^{-x_0}}{2}$. By relation (16), we consider $s_0^1 = |1 - e^{-x_0}|$. Also, by relations (3) and (4), we have $t_1 = s_0^k$, $s_0^0 = 0$, and for $j = 1, \dots, k-1$

$$s_0^{j+1} = s_0^j + L(s_0^j + s_0^{j-1})(s_0^j - s_0^{j-1}).$$

When the steps of k-step Newton's method increase, then we must choose the initial guess x_0 more close to the solution $x^* = \{\overbrace{0, 0, \ldots, 0}^{m}\}$. In the Table 2, by Mathematica code, for different x_0, we compute L_0, s_0^1, t_1, and α. We compute α by the relation (2). Moreover, for a given x_0, we calculate the maximum value for the error bound $||x^* - x_n||$, (19), by $\frac{2s_0^1}{1-\alpha} - t_1$.
Hence, we conclude that

$$0 < \frac{L(t_1 + s_0^{k-1})}{1 - 2L_0 t_1} < \alpha < 1 - 2L_0 s_0^1,$$

for every step k and and given x_0.
So, we can conclude that when k increases, the initial guess x_0 must be chosen closer to the solution x^*.

Table 2: Results of the semi-local convergence for Example 4.1

k	x_0	$L = L_0$	s_0^1	$s_0^k = t_1$	$\frac{L(s_0^{k-1} + t_1)}{1 - 2L_0 t_1}$	α	$1 - 2L_0 s_0^1$	$\|\|x^* - x_n\|\|$
1	0.1	0.452919	0.0951626	0.0951626	0.0471098	0.5	0.913893	0.190325
2	0.1	0.452919	0.0951626	0.0992596	0.0966399	0.722714	0.913893	0.343193
3	0.1	0.452919	0.0951626	0.09962	0.0988909	0.803761	0.913893	0.484932
4	0.1	0.452919	0.0951626	0.0996525	0.0990894	0.847597	0.913893	0.624415
5	0.1	0.452919	0.0951626	0.0996554	0.0991073	0.875282	0.913893	0.763021
6	0.1	0.452919	0.0951626	0.0996556	0.0991089	0.894404	0.913893	0.901191
7	0.1	0.452919	0.0951626	0.0996557	0.099109	0.908419	0.913893	1.03911
8	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.91914	0.990149	0.123054
9	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.91914	0.990149	0.137447
20	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.96633	0.990149	0.295524
30	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.977341	0.990149	0.439121
50	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.986299	0.990149	0.72626
60	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.988561	0.990149	0.86982
69	0.01	0.495025	0.00995017	0.00999967	0.00999916	0.99004	0.990149	0.999022
70	0.001	0.4995	0.0009995	0.00999967	0.0100905	0.990181	0.999001	0.10612
100	0.001	0.4995	0.0009995	0.00999967	0.0100905	0.993109	0.999001	0.145055
150	0.001	0.4995	0.0009995	0.00999967	0.0100905	0.995397	0.999001	0.217154
200	0.001	0.4995	0.0009995	0.00999967	0.0100905	0.996545	0.999001	0.289254
300	0.001	0.4995	0.0009995	0.00999967	0.0100905	0.997694	0.999001	0.433451
690	0.001	0.4995	0.0009995	0.00999967	0.0100905	0.998996	0.999001	0.995822

4.1 Systems of nonlinear equations

We first analyze some systems of nonlinear equations. They are very important in many areas of mathematics and engineering. For solving the system by k-step Newton's method, similar to Newton's method, we first linearize the system, and then solve it. Suppose that we want to solve the following system of nonlinear equations $F(X) = 0$, which $F = (f_1, f_2, \dots, f_n)^T$ and $X = (x_1, x_2, \dots, x_n)^T$. Let $f_i : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable functions, and D is a convex subset of $\mathbb{R}^n$. Let Y be an approximate solution for $F(X) = 0$. We attempt to compute the $H = (h_1, h_2, \dots, h_n)^T$ such that $(y_1 + h_1, y_2 + h_2, \dots, y_n + h_n)$ will be a better solution by following equation:

$$X^{(k+1)} = X^{(k)} + H^{(k)},$$

where the Jacobian system is

$$F'(X^{(k)})H^{(k)} = -F(X^{(k)}),$$

where

$$F'(X) = J(X) = \begin{bmatrix} \dfrac{\partial f_1(X)}{\partial x_1} & \dfrac{\partial f_1(X)}{\partial x_2} & \cdots & \dfrac{\partial f_1(X)}{\partial x_n} \\ \dfrac{\partial f_2(X)}{\partial x_1} & \dfrac{\partial f_2(X)}{\partial x_2} & \cdots & \dfrac{\partial f_2(X)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n(X)}{\partial x_1} & \dfrac{\partial f_n(X)}{\partial x_2} & \cdots & \dfrac{\partial f_n(X)}{\partial x_n} \end{bmatrix}. \tag{23}$$

Also, for $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ the divided difference $[X, Y; F]$ is matrix $([X, Y; F]_{ij})$, $i, j = 1, \dots, n$, where

$$[X, Y; F]_{ij} = \frac{f_i(x_1, \dots, x_j, y_{j+1}, \dots, y_n) - f_i(x_1, \dots, x_{j-1}, y_j, \dots, y_n)}{x_j - y_j}, \tag{24}$$

see [31].
We can solve the above system by the following Mathematica code. For the examples in the paper, the initial value $X_{0,0}$ is the same X_0, tolerance is considered 10^{-6}, and accuracy is considered 1000, but, this code is applicable for every system with any accuracy and tolerance if the convenient initial value is chosen.

```
Clear["Global‘*"]
nn = Input["accuracy"];
M = Input["tolerance"];
v = Input["number of equations of the system"];
n = Input["steps of the method"];
Y = Table[Subscript[y, i], {i, 1, v}];
For[i=1,i<=v,i++,Subscript[f,i][Y_]=Input[Subscript[f,i]]];
(*Subscript[f,i] is the ith equation of the given system*)
F = Function[Y, Table[Subscript[f, i][Y], {i,1,v}]];
      G[Y_] = Table[
 D[Subscript[f, i][Y],Subscript[y, j]],{i,1,v},{j,1,v}];
 J[c_]:=Flatten[G[Y]/.{Table[Y[[i]]->c[[i]],{i,1,v}]},1];
             Subscript[X,0,0] = Input["initial value"];
             Subscript[X,1,0] = Subscript[X, 0, 0] + 1;
      k = 0;
 While[Norm[Subscript[X,k+1,0]-Subscript[X,k,0], 2] > M,
           k++;For[m = 0, m < n, m++,
                 Subscript[H, k, m] =
                   SetAccuracy[
LinearSolve[J[Subscript[X,k,0]],-F[Subscript[X,k,m]]],nn];
Subscript[X,k,m+1] =Subscript[X,k,m]+Subscript[H,k,m]];
    Subscript[X,k+1,0] =Subscript[X,k,n];]
X^* = Subscript[X, k + 1, 0] // N
```

Example 4.2. Suppose $F(X) = 0$ be the system of nonlinear equations. We consider the system F as follow: (e is the Exponential number)

$$\mathcal{A} : F(X) = \begin{cases} f_1(x_1, x_2, x_3) = x_1x_2 - x_3^2 - 1 = 0, \\ f_2(x_1, x_2, x_3) = x_1x_2x_3 - x_1^2 + x_2^2 - 2 = 0, \\ f_3(x_1, x_2, x_3) = e^{x_1} - e^{x_2} + x_3 - 3 = 0, \end{cases} \tag{25}$$

where $D = [0,2]^3 \subseteq \mathbb{R}^3$. The solution X^* for F in D is (1.777671918, 1.423960598, 1.237471118).
For choosing appropriate initial values, we use the conditions obtained in the section 2. So, we first compute the divided difference $[X,Y;F]$ for $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ by using (24).

$$[X,Y;F] = \begin{pmatrix} y_2 & x_1 & -x_3 - y_3 \\ -x_1 - y_1 + y_2 y_3 & x_2 + y_2 + x_1 y_3 & x_1 x_2 \\ \frac{e^{x_1} - e^{y_1}}{x_1 - y_1} & \frac{-e^{x_2} + e^{y_2}}{x_2 - y_2} & 1 \end{pmatrix}.$$

Also, using (23), we have

$$F'(X) = \begin{pmatrix} x_2 & x_1 & -2x_3 \\ -2x_1 + x_2 x_3 & 2x_2 + x_1 x_3 & x_1 x_2 \\ e^{x_1} & -e^{x_2} & 1 \end{pmatrix}.$$

We consider max-norm or ∞-norm, for computing norm $||.||$. Hence, we have for given X_0 and $X, Y, Z, T \in D$:

$$||F'(X_0)^{-1}([X,Y;F] - [Z,T;F])|| \leq ||F'(X_0)^{-1}||8(||X - Z|| + ||Y - T||),$$

also, we can obtain

$$||F'(X_0)^{-1}([X,Y;F] - F'(X_0))|| \leq ||F'(X_0)^{-1}||5(||X - X_0|| + ||Y - X_0||).$$

Therefore, by relations (17) and (18), we have $L = 8||F'(X_0)^{-1}||$ and $L_0 = 5||F'(X_0)^{-1}||$.
By using, relations (2), (12), and (16), we confirm the condition (5) for different X_0. We show the results in the table 4. We conclude that when the steps of k-step Newton's method increase, we must choose X_0 closer to the solution X^* i.e. $||X_0 - X^*||$ decrease.

Table 3: Results of the semi-local convergence for Example 4.2

k	X_0	$\|\|X_0 - X^*\|\|$	L	L_0	s_0^1	$s_0^k = t_1$	$\frac{L(s_0^{k-1} + t_1)}{1 - 2L_0 t_1}$	α	$1 - 2L_0 s_0^1$
1	(1.77,1.4,1.23)	0.039	3.7091	3.3845	0.0241	0.0241	0.1070	0.5153	0.8366
2	(1.77,1.4,1.23)	0.039	3.7091	3.3845	0.0241	0.0263	0.2277	0.7838	0.8366
3	(1.77,1.4,1.23)	0.039	3.7091	3.3845	0.0241	0.0267	0.2201	0.8135	0.8366
4	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.8557	0.9481
5	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.8821	0.9481
6	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.9003	0.9481
7	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.9137	0.9481
8	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.9237	0.9481
9	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.9318	0.9481
10	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.9383	0.9481
11	(1.77,1.42,1.23)	0.019	3.6892	3.3664	0.0077	0.0079	0.0618	0.9437	0.9481
20	(1.777,1.423,1.237)	0.009	3.6673	3.3464	0.00096	0.00096	0.0071	0.9684	0.9935
40	(1.777,1.423,1.237)	0.009	3.6673	3.3464	0.00096	0.00096	0.0071	0.9840	0.9935
70	(1.777,1.423,1.237)	0.009	3.6673	3.3464	0.00096	0.00096	0.0071	0.9908	0.9935

Example 4.3. For comparing with the system $\mathcal{A}$, we compute the solution of another system by the k-step Newton method. We consider $G : D \subseteq \mathbb{R}^6 \to \mathbb{R}^6$ is a differentiable function, and $D = [-3, 3]^6$ and X^*=(0.2480194, -2.599051, 1.066914, 2.811747, 1.578335, 2.505069).

$$
\mathcal{B}: \begin{cases}
g_1(x_1,x_2,x_3,x_4,x_5,x_6) = x_1x_2 - 5x_3 + x_4^2 - x_6 + x_5 - 1 = 0, \\
g_2(x_1,x_2,x_3,x_4,x_5,x_6) = x_2x_3 - x_5x_1^2 + x_2^2 - 2 + x_6 - x_4 - x_5 = 0, \\
g_3(x_1,x_2,x_3,x_4,x_5,x_6) = e^{x_1} - x_6e^{x_2} + x_3 - 3 + x_4x_6 - x_5^4 = 0, \\
g_4(x_1,x_2,x_3,x_4,x_5,x_6) = x_2x_5 - x_3^2 - 1 + x_5^2x_6, \\
g_5(x_1,x_2,x_3,x_4,x_5,x_6) = x_1x_2x_6 - x_1^2 + x_2 - 2 + x_6^2, \\
g_6(x_1,x_2,x_3,x_4,x_5,x_6) = e^{x_1} + x_3 - 3 + 2x_5 - x_6.
\end{cases}
$$

We have that

$$
\begin{aligned}
G(X) = (&g_1(x_1,x_2,x_3,x_4,x_5,x_6), \\
&g_2(x_1,x_2,x_3,x_4,x_5,x_6), g_3(x_1,x_2,x_3,x_4,x_5,x_6), \\
&g_4(x_1,x_2,x_3,x_4,x_5,x_6), g_5(x_1,x_2,x_3,x_4,x_5,x_6), \\
&g_6(x_1,x_2,x_3,x_4,x_5,x_6)).
\end{aligned}
$$

For system $\mathcal{B}$, by using (24), the divided difference $[X, Y; G]$ for $X = (x_1, \dots, x_6)$ and $Y = (y_1, \dots, y_6)$ is

$$
\begin{pmatrix}
y_2 & x_1 & -5 & x_4 + y_4 & 1 & -1 \\
-(x_1 + y_1)y_5 & x_2 + y_2 + y_3 & x_2 & -1 & -x_1^2 - 1 & 1 \\
\frac{e^{x_1} - e^{y_1}}{x_1 - y_1} & \frac{(-e^{x_2} + e^{y_2})y_6}{x_2 - y_2} & 1 & y_6 & \frac{y_5^4 - x_5^4}{x_5 - y_5} & x_4 - e^{x_2} \\
0 & y_5 & -x_3 - y_3 & 0 & x_2 + (x_5 + y_5)y_6 & x_5^2 \\
-x_1 - y_1 + y_2y_6 & x_1y_6 + 1 & 0 & 0 & 0 & x_1x_2 + x_6 + y_6 \\
\frac{e^{x_1} - e^{y_1}}{x_1 - y_1} & 0 & 1 & 0 & 2 & -1
\end{pmatrix},
$$

and also by using (23), $G'(X)$ is

$$\begin{pmatrix} x_2 & x_1 & -5 & 2x_4 & 1 & -1 \\ -2x_1x_5 & 2x_2+x_3 & x_2 & -1 & -x_1^2-1 & 1 \\ e^{x_1} & -e^{x_2}x_6 & 1 & x_6 & -4x_5^3 & x_4-e^{x_2} \\ 0 & x_5 & -2x_3 & 0 & x_2+2x_5x_6 & x_5^2 \\ x_2x_6-2x_1 & x_1x_6+1 & 0 & 0 & 0 & x_1x_2+2x_6 \\ e^{x_1} & 0 & 1 & 0 & 2 & -1 \end{pmatrix}.$$

By using (17) and (18), we compute L and L_0 for given X_0:

$$L_0 = 25||G'(X_0)^{-1}||, \quad \text{and } L = 49||G'(X_0)^{-1}||,$$

thus, we confirm the condition (5) for different k. We will show the condition (5) is true for every k, but when k increased, we must choose X_0 closer to X^*. We show the results in the table 5.

Table 4: Results of the semi-local convergence for Example 4.3

k	X_0	$\|\|X_0 - X^*\|\|$	L	L_0	s_0^1	$s_0^k = t_1$	$\frac{L(s_0^{k-1}+t_1)}{1-2L_0t_1}$	α	$1-2L_0s_0^1$
1	(0.248, -2.599, 1.067, 2.81, 1.578, 2.505)	2.2e-3	60.4415	30.8375	1.7e-3	1.9e-3	1.183e-1	6.145e-1	8.92e-1
2	(0.248, -2.599, 1.067, 2.81, 1.578, 2.505)	2.2e-3	60.4415	30.8375	1.7e-3	1.9e-3	2.524e-1	8.081e-1	0.892
3	(0.248, -2.599, 1.067, 2.81, 1.578, 2.505)	2.2e-3	60.4415	30.8375	1.7e-3	1.9e-3	0.2687	0.8700	0.892
4	(0.248, -2.599, 1.067, 2.811, 1.578, 2.505)	1.2e-3	60.4194	30.8262	7.4e-4	7.8e-4	9.954e-1	0.9014	0.954
5	(0.248, -2.599, 1.067, 2.811, 1.578, 2.505)	1.2e-3	60.4194	30.8262	7.4e-4	7.8e-4	0.9956e-1	0.9205	0.954
6	(0.248, -2.599, 1.067, 2.811, 1.578, 2.505)	1.2e-3	60.4194	30.8262	7.4e-4	7.8e-4	0.9956e-1	0.9334	0.954
7	(0.248, -2.599, 1.067, 2.811, 1.578, 2.505)	1.2e-3	60.4194	30.8262	7.4e-4	7.8e-4	0.9956e-1	0.9427	0.954
8	(0.248, -2.599, 1.067, 2.811, 1.578, 2.505)	1.2e-3	60.4194	30.8262	7.4e-4	7.8e-4	0.9956e-1	0.9497	0.954
9	(0.248, -2.599, 1.067, 2.8117, 1.5783, 2.505)	2.4e-4	60.4221	30.8276	6.8e-5	6.8e-5	8.41e-3	0.95518	0.9957
10	(0.248, -2.599, 1.067, 2.8117, 1.5783, 2.505)	2.4e-4	60.4221	30.8276	6.8e-5	6.8e-5	8.41e-3	0.95958	0.9957
11	(0.248, -2.599, 1.067, 2.8117, 1.5783, 2.505)	2.4e-4	60.4221	30.8276	6.8e-5	6.8e-5	8.41e-3	0.96319	0.9957
20	(0.248, -2.599, 1.067, 2.8117, 1.5783, 2.505)	2.4e-4	60.4221	30.8276	6.8e-5	6.8e-5	8.41e-3	0.97959	0.9957
40	(0.248, -2.599, 1.067, 2.8117, 1.5783, 2.505)	2.4e-4	60.4221	30.8276	6.8e-5	6.8e-5	8.41e-3	0.98974	0.9957
70	(0.248, -2.599, 1.067, 2.8117, 1.5783, 2.505)	2.4e-4	60.4221	30.8276	6.8e-5	6.8e-5	8.41e-3	0.99412	0.9957

Conclusions

In this paper, we presented the semi-local convergence of k-step Newton's method by using majorizing sequences. This is a novelty that we obtained semi-local convergence of a k-step iterative method by majorizing sequences where k can be greater than two. Meanwhile, for quadratic polynomials, by Julia set, we obtained that there exist more chaotic behaviors when the steps of the method (1) increase. Then by using basins of attraction, we obtained the same conclusions for polynomials of the second, third, and fourth degrees. We showed that when k increase, the basin of attractions are smaller. In the end, we confirmed the theoretical results of semi-local convergence and dynamical study in numerical examples. We showed that as the steps of the method (1) increase, the convergence radius becomes smaller. By these applicable examples, we obtained the same results of dynamic study. We can compute the ACOC about $k+1$ for k-step Newton's method (1) by different k. In addition, in Example 4.1, we could find a bound for $||X^* - X_n||$.

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