Journal of Mathematical Extension
Vol. 7, No. 1, (2013), 61-67

Approximately Multiplicative Functionals on the Product of Banach Algebras

F. Ershad*
Payame Noor University

L. Bagheri
Payame Noor University

Abstract. In this paper we characterize the conditions under which approximately multiplicative functionals are near multiplicative functionals on the product of Banach algebras.

AMS Subject Classification: 46J05; 46K99
Keywords and Phrases: Commutative Banach algebras, Multiplicative functionals, approximately multiplicative functionals

1. Introduction

Throughout this paper all Banach algebras are commutative. If A is a Banach algebra, then the set of all linear functionals on A is denoted by $A^\star$ and the set of all its nonzero multiplicative functionals is denoted by $\hat{A}$. If $\varphi \in A^\star$, then define

$$\check{\varphi}(a, b) = \varphi(ab) - \varphi(a)\varphi(b)$$

for all $a, b \in A$. If $\delta \in R^+$, we say that φ is δ-multiplicative, whenever $\|\check{\varphi}\| \leqslant \delta$.
Also for each $\varphi \in A^\star$ define

$$d(\varphi) = inf\{\|\varphi - \psi\| : \psi \in \hat{A} \cup \{0\}\}.$$

Received: October 2012; Accepted: January 2013
*Corresponding author

We say that A is an algebra in which approximately multiplicative functionals are near multiplicative functionals, or A is $AMNM$ for short, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $d(\varphi) < \varepsilon$ whenever φ is a δ-multiplicative linear functional.

B. E. Johnson has shown that various Banach algebras are $AMNM$ and some of them fail to be $AMNM$ [5]. Also, this property is still unknown for some Banach algebras such as H^{∞} and Douglas algebras. First author and S. H. Petroudi has shown in their paper that some of weighted Hardy spaces are $AMNM$ [1]. In this paper, first we define a multiplication on $A{\times}B$, where A and B are two Banach algebras and will show that $A{\times}B$ is $AMNM$, where A and B are $AMNM$. Also we show that the converse is true. Then we generalized this to finite product of Banach algebras. For some sources on these topics one can refer to [1-10].
Let A and B be two Banach algebras. Define in $A{\times}B$ addition , multiplication and norm by $(a,b){+}(c,d){=}(a+c,b+d)$,$(a,b)(c,d){=}(ac,bd)$, $\|(a,b)\|{=}\|a\|{+}\|b\|$ for every $a,c \in A$ and $b,d \in B$. Therefore $A{\times}B$ is an algebra such that

$$\begin{aligned} \|(a,b)(c,d)\| &= \|ac\| + \|bd\| \\ &\leqslant \|a\|\|c\| + \|b\|\|d\| \\ &\leqslant (\|a\| + \|b\|)(\|c\| + \|d\|) \\ &= \|(a,b)\|\|(c,d)\|. \end{aligned}$$

Also if $\{(a_n,b_n)\}$ is a cauchy sequence in $A{\times}B$, then$\{a_n\},\{b_n\}$ respectively are cauchy sequence in A and B. Thus there exist $a \in A, b \in B$ such that $a_n \to a, b_n \to b$ and hence

$$\|(a_n,b_n) - (a,b)\| = \|a_n - a\| + \|b_n - b\| \to 0.$$

Therefore $A{\times}B$ is a Banach algebra.

2. Main Results

For the proof of our theorems, we need the following proposition[3].

Proposition 2.1. *Let A be a unital Banach algebra. Then the following are equivalent.*

(i) *A is $AMNM$.*

(ii) *For any sequence $\{\Phi_n\}$ in A^* with $\|\check{\Phi}_n\| \to 0$ there is a sequence $\{\Psi_n\}$ in $\hat{A}\bigcup\{0\}$ with $\|\Phi_n - \Psi_n\| \to 0$.*

(iii) *For any sequence $\{\Phi_n\}$ in A^* with $\|\check{\Phi}_n\| \to 0$ there is a subsequence $\{\Phi_{n_i}\}$ and a sequence $\{\Psi_i\}$ in $\hat{A} \cup \{0\}$ with $\|\Phi_{n_i} - \Psi_i\| \to 0$.*

(iv) *For any sequence $\{\Phi_n\}$ in A^* with $\|\check{\Phi}_n\| \to 0$ and $inf_n\|\Phi_n\| > 0$ there is a sequence $\{\Psi_n\}$ in $\hat{A}$ with $\|\Phi_n - \Psi_n\| \to 0$.*

(v) *For any sequence $\{\Phi_n\}$ in A^* with $\|\check{\Phi}_n\| \to 0$ and $\Phi_n(1) = 1 = \|\Phi_n\|$ there is a sequence $\{\Psi_n\}$ in $\hat{A}$ with $\|\Phi_n - \Psi_n\| \to 0$.*

(vi) *For each $\varepsilon > 0$ there is $\delta > 0$ such that if $\Phi \in A^*$ with $\Phi(1) = 1 = \|\Phi\|$ and $\|\check{\Phi}\| < \delta$ then $d(\Phi) < \varepsilon$.*

Conditions (i) to (iv) are equivalent even if A does not have a unit. If A has an approximate unit of norm 1 then (i) to (iv) are equivalent to the following,

(vii) *For each $\varepsilon > 0$ there is $\delta > 0$ such that if $\Phi \in A^*$ with $\|\Phi\| = 1$ and $\|\check{\Phi}\| < \delta$ then $d(\Phi) < \varepsilon$.*
Also T. M. Rassias proved the following theorem [7].

Theorem 2.2. *Let J be a closed ideal in a Banach algebra A,*

(i) *If A and $\frac{A}{J}$ are $AMNM$, then so is A.*

(ii) *If A is $AMNM$, then so is J.*

(iii) *If A is $AMNM$ and J has a bounded approximate identity, then $\frac{A}{J}$ is $AMNM$.*

Theorem 2.3. *If A and B are two $AMNM$ algebras, then $A{\times}B$ is $AMNM$.*

Proof. We show that the statement (iv) of proposition $2 \cdot 1$ holds for $A{\times}B$. Let $\{\Phi_n\} \subseteq (A{\times}B)^*$, $inf_n\|\Phi_n\| = k > 0$ and $\|\check{\Phi_n}\| \to 0$. Since $inf_n\|\Phi_n\| = k$,for every n, there exist $a_n \in A$ and $b_n \in B$ such that $\|(a_n, b_n)\| = 1$ and $|\Phi_n(a_n, b_n)| > \frac{k}{2}$. If $\Phi_n(a_n, b_n) = \alpha_n$, then for $c_n = \frac{a_n}{\alpha_n}$ and $d_n = \frac{b_n}{\alpha_n}$, we get

$$\|(c_n, d_n)\| < \frac{2}{k} \quad and \quad \Phi_n(c_n, d_n) = 1. \tag{1}$$

For every $n \in \mathbb{N}$, define $\Theta_n \in A^*$ and $\Psi_n \in B^*$ by

$$\Theta_n(a) = \Phi_n(ac_n, d_n), \Psi_n(b) = \Phi_n(c_n, bd_n) \ \ a \in A, b \in B. \tag{2}$$

Therefore for every $a, \acute{a} \in A$, we have

$$\begin{aligned}\check{\Theta}_n(a, \acute{a}) &= \Phi_n(a\acute{a}c_n, d_n) - \Phi_n(ac_n, d_n)\Phi_n(\acute{a}c_n, d_n) \\ &= \Phi_n(a\acute{a}c_n, d_n)\Phi_n(c_n, d_n) - \Phi_n(ac_n, d_n)\Phi_n(\acute{a}c_n, d_n) \\ &\quad +\Phi_n((ac_n, d_n)(\acute{a}c_n, d_n)) - \Phi_n((a\acute{a}c_n, d_n)(c_n, d_n)) \\ &= \check{\Phi}_n((ac_n, d_n), (\acute{a}c_n, d_n)) - \check{\Phi}_n((a\acute{a}c_n, d_n), (c_n, d_n)).\end{aligned}$$

Thus if $\|a\| \leqslant 1$, $\|\acute{a}\| \leqslant 1$, then by (1),

$$\|\check{\Theta}_n(a, \acute{a})\| \leqslant \frac{2}{k^2}\|\check{\Phi}_n\|$$

and hence $\|\check{\Theta}_n\| \to 0$ as $n \to \infty$. Since A is $AMNM$, by Proposition 2.1. (ii), there is a sequence $\{\acute{\Theta}_n\} \subset \hat{A}\bigcup\{0\}$ such that $\|\Theta_n - \acute{\Theta}_n\| \to 0$. Similarly, we can find a sequence $\{\acute{\Psi}_n\} \subset \hat{B}\bigcup\{0\}$ such that $\|\Psi_n - \acute{\Psi}_n\| \to 0$. Define $\acute{\Phi}_n \in (A \times B)^*$, by

$$\acute{\Phi}_n(a, b) = \acute{\Theta}_n(a)\acute{\Psi}_n(b).$$

It is easy to show that $\acute{\Phi}_n \in \widehat{A \times B}\bigcup\{0\}$.
Also we note that

$$\begin{aligned}|\Theta_n(a)\Psi_n(b) - \acute{\Phi}_n(a, b)| \leqslant |\acute{\Psi}_n(b)||\acute{\Theta}_n(a) \\ -\Theta_n(a)| + |\Theta_n(a)||\acute{\Psi}_n(b) - \Psi_n(b)|\end{aligned} \tag{3}$$

$$\Theta_n(a)\Psi_n(b) - \Phi_n(a, b) = \check{\Phi}_n((a, b), (c_n^2, d_n^2))$$

$$-\Phi_n(a,b)\check{\Phi}_n((c_n,d_n),(c_n,d_n))$$
$$-\check{\Phi}_n((ac_n,d_n),(c_n,bd_n)), \qquad (4)$$

and hence $\|\Phi_n - \acute{\Phi}_n\| \to 0$ as $n \to \infty$ and the proof is complete. □

By following lemma we will show that the converse of Theorem 2.3 holds.

Lemma 2.4. *Let A and B be two Banach algebra and $f : A \to B$ be a liner bounded bijection. If $f(ab) = f(a)f(b)$, for every $a, b \in A$, then A is $AMNM$ if and only if B is $AMNM$.*

Proof. Suppose A is $AMNM$ and $\{\Phi_n\}$ be a sequence in B^* with $\|\check{\Phi}_n\| \to 0$. By property of f, for every $a, b \in A$, we have

$$|(\Phi_n\check{o}f)(a,b)| = |\Phi_n(f(a)f(b)) - (\Phi_n of)(a)(\Phi_n of)(b)|$$
$$\leqslant \|\check{\Phi}_n\|\|f\|^2\|a\|\|b\|,$$

and hence $\|(\Phi_n\check{o}f)\| \to 0$. Thus there is a sequence $\{\Psi_n\}$ in $\hat{A}\bigcup\{0\}$ such that $\|\Phi_n of - \Psi_n\| \to 0$. By Open Mapping Theorem, $\|f^{-1}\| < \infty$. Also, note that $\{\Psi_n of^{-1}\} \subset \hat{B}\bigcup\{0\}$ and $\|\Phi_n - \Psi_n of^{-1}\| \leqslant \|f^{-1}\|\|\Phi_n of - \Psi_n\|$. Therefore $\|\Phi_n - \Psi_n of^{-1}\| \to 0$ and so B is $AMNM$. The proof of converse is similar. □

Theorem 2.5. *Let A and B be two Banach algebras . If $A \times B$ is $AMNM$, then A and B are $AMNM$.*

Proof. By definition of multiplication in $A \times B$, it is easy to show that $A \times \{0\}$ is a closed ideal in $A \times B$. Thus by theorem 2.3., $A \times \{0\}$ is $AMNM$. If we define $f : A \times \{0\} \to A$ by $f(a, 0) = a$, then by Lemma 2.4., A is $AMNM$. Similarly, we can show that B is $AMNM$.

Let A_1, A_2, A_3 be three Banach algebras. Define the vector space operations in $A_1 \times A_2 \times A_3$ componentwise, define multiplication in $A_1 \times A_2 \times A_3$ by

$$(a_1, a_2, a_3)(b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3),$$

and define $\|(a_1, a_2, a_3)\| = \|a_1\| + \|a_2\| + \|a_3\|$. Then it is easy to show that $A_1 \times A_2 \times A_3$ by this norm is a Banach algebra. If we define

$f : A_1 \times A_2 \times A_3 \to (A_1 \times A_2) \times A_3$ by $f(a_1, a_2, a_3) = ((a_1, a_2), a_3)$, then by lemma 2.4, $A_1 \times A_2 \times A_3$ is $AMNM$ if and only if $(A_1 \times A_2) \times A_3$ is $AMNM$. Thus by theorem 2.3 and induction we can prove that $A_1 \times A_2 ... \times A_n$ is $AMNM$ if and only if $A_1, ..., A_n$ are $AMNM$. $\square$

References

[1] F. Ershad and S. H. Petroudi, *Approximately multiplicative functionals on the spaces of formal power series,* Abstract and Applied Analysis, 2011 (2011), 1-6.

[2] R. Howey, *Approximately Multiplicative functionals on algebras of smooth functions,* J. London Math. Soc., 68 (2003), 739-752.

[3] K. Jarosz, *Almost multiplicative functionals,* Studia Math., 124 (1997), 37-58.

[4] K. Jarosz, *Perturbations of Banach algebras,* Lecture Notes in Mathematics 1120, Springer, Berlin, 1985.

[5] B. E. Johnson, *Approximately multiplicative functionals,* J. London Math. Soc., 34 (1986), 489-510.

[6] B. E. Johnson, *Approximately multiplicative maps between Banach algebras,* J. London Math. Soc., 37 (1988), 294-316.

[7] T. M. Rassias, *The problem of S. M. Ulam for Approximately multiplicative mappings,* Journal of mathematical analysis and application, 246 (2000), 352-378.

[8] F. Sanchez, *Pseudo-characters and Almost multplicative functionals,* Journal of mathematical analysis and application, 248 (2000), 275-289.

[9] S. Sidney, *Are all uniform algebras AMNM* ?, Bull. London Math. Soc., 29 (1997), 327-330.

[10] B. Yousefi and Y. N. Dehghan, *Reflexivity on weighted Hardy spaces,* Southeast Asian Bulletin of Mathematics, 28 (3) (2004), 587-593.

Fariba Ershad
Department of Mathematics
Assistant Proffessor of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: fershad@pnu.ac.ir

Leila Bagheri
Department of Mathematics
Ph.D Student of Mathematics
Payame Noor University
P.O. Box 19395-3697
Tehran, Iran
E-mail: bagheri@phd.pnu.ac.ir