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On the Numerical Range of some Bounded Operators

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Abstract. In this paper, we give conditions under which the numerical range of a weighted composition operator, acting on a Hilbert space, contains zero as an interior point and we investigate extreme points of the numerical range of an operator acting on an arbitrary Banach space. Also, we give necessary and sufficient conditions under which the numerical range of an operator on some Banach spaces, to be closed. Finally, we characterize the structure of the numerical range of an operatoracting on Banach weighted Hardy spaces.

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1 Introduction

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^p(\beta)$ denotes the space of such formal power series. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$ and the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_n$ [16]. Also if

$$g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in H^q(\beta^{p/q}),$$

then

$$\|g\|^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p.$$

The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p = 2$ and respectively $\beta(n) = 1$, $\beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{1/2}$. If $\lim_n \frac{\beta(n+1)}{\beta(n)} = 1$ or $\liminf_n \beta(n)^{\frac{1}{n}} = 1$, then $H^p(\beta)$ contains in the set of functions analytic on the open unit disc U. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis for $H^p(\beta)$ such that $\|f_k\| = \beta(k)$. Clearly M_z, the operator of multiplication by z on $H^p(\beta)$ shifts the basis $\{f_k\}_k$. The spaces $H^p(\beta)$ are also called as weighted Hardy spaces. The set of multipliers

$$\{\psi \in H^p(\beta) : \ \psi H^p(\beta) \subseteq H^p(\beta)\}$$

is denoted by $H^p_\infty(\beta)$ and the linear operator of multiplication by ψ on $H^p(\beta)$ by M_ψ.

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then it has been proved that $H^q(\beta^{\frac{p}{q}})$ and $H^q(\beta^{-1})$ are norm isomorphic, where $\beta^{-1} = \{\beta^{-1}(n)\}_{n=0}^{\infty}$ [5, Lemma 2.1]. Hence $H^p(\beta)^* = H^q(\beta^{-1})$.

It is convenient and helpful to introduce the notation $\langle f, g\rangle$ to stand for $g(f)$ where $f \in H^p(\beta)$ and $g \in H^q(\beta^{-1})$ where $\frac{1}{p}+\frac{1}{q}=1$. Note that in this case we have

$$< f, g >= \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

Also, for simplicity we will use $\|\cdot\|_p$ and $\|\cdot\|_q$ instead of $\|\cdot\|_{H^p(\beta)}$ and $\|\cdot\|_{H^q(\beta^{-1})}$ respectively. For some references on weighted Hardy spaces, one can see [14-20].

Let X be a Banach space of analytic functions on a bounded plane domain Ω. Recall that a complex number λ is said to be a bounded point evaluation on X if the functional of point evaluation at $\lambda, e_\lambda : X \to \mathbb{C}$ defined by $e_\lambda(f) = f(\lambda)$, is bounded. If $X = \mathcal{H}$ is a Hilbert space and e_λ is bounded, then there exists a function $k_\lambda \in \mathcal{H}$ such that $e_\lambda(f) = \langle f, k_\lambda\rangle$ for all $f \in \mathcal{H}$ and $\|e_\lambda\| = \|k_\lambda\|$. The function k_λ is called the reproducing kernel at λ. Also, remember that the set of multipliers of X is denoted by $\mathcal{M}(X)$ that consists of functions ψ such that $\psi X \subseteq X$. Each multiplier ψ induces a multiplication operator $M_\psi : X \to X$ defined by $M_\psi f = \psi f$.

A function φ that maps Ω into itself induces a composition operator C_φ on X defined by $C_\varphi f = f \circ \varphi$. Also, if $\psi \in \mathcal{M}(X)$, then the weighted composition operator $C_{\psi,\varphi} : X \to X$ is defined by $C_{\psi,\varphi} f = \psi.f \circ \varphi$.

2 Main Results

The numerical range of operators has been the focus of attention for several decades and many properties of numerical range have been studied. The numerical range of a bounded operator T on a Hilbert space H is the set of complex numbers

$$W(T) = \{\langle Th, h\rangle : h \in H, \|h\| = 1\}.$$

Some properties and further developments of the numerical range of a bounded linear operator on a Hilbert space can be found in [7, 8]. The concept of numerical range on a Banach space X, have extended by Bauer and Lumer in [2, 12], that is not necessarily convex, see [3, Example 21.6]. In [4], P. S. Bourdon and J. H. Shapiro worked on the Hardy space H^2 of the open unit disc U, and they considered the numerical ranges of composition operators C_φ induced by holomorphic self-maps φ of U. Some properties of numerical range of bounded operators acting on weighted Hardy spaces has been considered in [9, 11, 20]. For maps φ that fix a point of U, they determined precisely when 0 belongs to the numerical range W of C_φ. The determination of the numerical range for two by two matrics, the Toeplitz-Hausdorff Theorem establishing the convexity of the numerical range for any Hilbert space operator, and the relationship between the numerical range and the spectrum has been given by J. H. Shapiro in [13]. In [6], G. Gunatillake, M. Jovovic, and W. Smith considered the inclusion of zero in the interior of the numerical range.

In this work we investigate conditions under which the numerical range of a composition operator contains zero as an interior point, and the study of extreme points of the numerical range of an operator on a Banach space has been considered. Also, we represent the necessary and sufficient conditions for the closedness of the numerical range of a compact composition operator acting on Banach spaces $H^p(\beta)$.

We employ the notation $\overline{W}(T)$ for the closure of the numerical range of an operator T and the notation $\overline{W(T)}$ for the complex conjugate of the numerical range of an operator T. Also, we will use the usual notations $\sigma(T), \sigma_p(T)$ and $\sigma_{ap}(T)$ respectively for the spectrum, point spectrum, and approximate point spectrum of an operator T. Also, we use $ext(A)$ to denote the set of extreme points of a set A.

Recall that an eigenvalue λ of a bounded linear operator T on a Hilbert space is called normal if $ker(T-\lambda I) = ker(T^* - \overline{\lambda} I)$. It is proved in [10, Theorem 2] that if $\lambda \in \partial W(T)$, then λ is a normal eigenvalue of T.

As usual, $H(\Omega)$ denotes the set of analytic functions on the open unit disc $\Omega \subset \mathbb{C}$ and $H^\infty(\Omega)$ denotes the set of all bounded analytic functions on Ω. Also, $\sup\{|\phi(z)| : z \in G\}$ is denoted by $\|\phi\|_G$, where $G \subset \Omega$. In the following by $B(z)$, we mean a ball with center $z \in \mathbb{C}$. Also, by a weighted composition operator $C_{\psi,\varphi}$ on a Banach space $X \in H(\Omega)$, we mean automatically that φ is a nonconstant mapping from Ω into itself and $\psi \in \mathcal{M}(X)$.

Let ψ be defined on an open set $\Omega \subset \mathbb{C}$. We say that ψ has boundary limit almost somewhere on a set $A \subset \partial\Omega$, if for some $z_0 \in A$, $\lim_{\lambda\in\Omega,\lambda\to z_0} \psi(\lambda)$ exists finitely (by finitely existence of a limit, we mean that limit exists with finite absolute value).

Theorem 2.1. *Let $C_{\psi,\varphi}$ be bounded on $\mathcal{H} \subset H(\Omega)$, each point of Ω be a bounded point evaluation, and $\lim_{\lambda\to\partial\Omega} \|e_\lambda\| = \infty$. Also, let there exists a ball $B(w_0)$; $w_0 \in \partial\Omega$ such that $\overline{\varphi(B(w_0)\cap\Omega)} \subset \Omega$, and $\lim_{\lambda\to\partial\Omega\cap B(w_0)} \psi(\lambda)$ is finite. If ψ has boundary limit almost somewhere on $\partial\Omega \cap B(w_0)$, then $0 \in \overline{W}(C_{\psi,\varphi})$. Also, if φ has a fixed point u in Ω, then $\psi(u) \in W(C_{\psi,\varphi})$.*

Proof. Put $V = B(w_0)\cap\Omega$. Since ψ has boundary limit almost somewhere on $V\cap\partial\Omega$, there is a $z_0 \in V\cap\partial\Omega$ so that $\lim_{\lambda\in V,\lambda\to z_0} \psi(\lambda)$ exists finitely. Now, for $\lambda \in V$ put

$$\omega_\lambda = \langle C_{\psi,\varphi}\frac{k_\lambda}{\|k_\lambda\|}, \frac{e_\lambda}{\|e_\lambda\|}\rangle,$$

where k_λ is the reproducing kernel at λ. Then $\omega_\lambda \in W(C_{\psi,\varphi})$ and we have:

$$\begin{aligned}\omega_\lambda &= \langle \frac{k_\lambda}{\|k_\lambda\|}, C^*_{\psi,\varphi}\frac{e_\lambda}{\|e_\lambda\|}\rangle \\ &= \frac{1}{\|e_\lambda\|^2}\langle k_\lambda, \overline{\psi(\lambda)}e_{\varphi(\lambda)}\rangle.\end{aligned}$$

Hence we get

$$|\omega_\lambda| \le \|e_\lambda\|^{-1}|\psi(\lambda)| \;\; \|e_{\varphi(\lambda)}\|.$$

Note that since $\overline{\varphi(V)} \subset \Omega$, φ is not the identity mapping. Also, we can find a sequence $\{\lambda_n\}_n \subset V$ such that $\lim_{\lambda_n\to z_0} \varphi(\lambda_n)$ exists and is an element of Ω. Therefore, $\lim_{\lambda_n\to z_0} e_{\varphi(\lambda_n)}$ exists with finite norm (because each point of Ω is a bounded point evaluation). Since $\lim_{\lambda_n\to z_0} \|e_{\lambda_n}\| = \infty$, so we get $\omega_{\lambda_n} \to 0$ as $\lambda_n \to z_0$. This says that $0 \in \overline{W}(C_{\psi,\varphi})$.

Now, assume that φ has a fixed point u in Ω. By above calculations, we have

$$\omega_u = \frac{1}{\|e_u\|^2}\psi(u)k_u(\varphi(u)) = \psi(u).$$

Hence $\psi(u) \in W(C_{\psi,\varphi})$. $\square$

Theorem 2.2. *Let $\sum_{n=0}^{\infty}\beta(n)^{-2}=\infty$, $H^2(\beta)\subset H(U)$, and $C_{\psi,\varphi}$ be bounded on $H^2(\beta)$. If $\psi\in C(\partial U)$ and for some $z_0\in\partial\Omega$, $\lim_{\lambda\to z_0}\varphi(\lambda)$ exists and not equal to z_0, then $0\in\overline{W}C_{\psi,\varphi})$.*

Proof. For $\lambda\in U;\ \lambda\neq 0$, set

$$\omega_\lambda=\langle C_{\psi,\varphi}\frac{k_\lambda}{\|k_\lambda\|},\frac{e_\lambda}{\|e_\lambda\|}\rangle.$$

Then we can see that

$$\begin{aligned}\omega_\lambda &= \frac{1}{\|e_\lambda\|^2}\psi(\lambda)\langle k_\lambda,e_{\varphi(\lambda)}\rangle\\ &= (\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{\beta(n)^2})^{-1}\psi(\lambda)\sum_{n=0}^{\infty}\bar{\lambda}^n\varphi(\lambda)^n.\end{aligned}$$

Therefore,

$$\omega_\lambda=(\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{\beta(n)^2})^{-1}\psi(\lambda)\ \ (1-\bar{\lambda}\varphi(\lambda))^{-1}.$$

Now since $\sum_{n=0}^{\infty}\beta(n)^{-2}=\infty$, $\psi\in C(\partial\Omega)$ and $\lim_{\lambda\to z_0}\varphi(\lambda)\neq z_0$, we get $\omega_\lambda\to 0$ as $\lambda\to z_0$. This implies that $0\in\overline{W}C_{\psi,\varphi})$. $\square$

Corollary 2.3. *Let $C_{\psi,\varphi}$ be bounded on H^2 and $\psi\in C(\partial U)$. If φ is not the identity mapping, then $0\in\overline{W}C_{\psi,\varphi})$.*

Corollary 2.4. *Let C_φ be bounded on H^2 and φ is not the identity mapping. Then $0\in\overline{W}C_\varphi)$.*

Definition 2.5. *Suppose that $\mathcal{H}$ is a Hilbert space of analytic functions on the open unit disc U and ψ is a multiplier of $\mathcal{H}$. A nonconstant holomorphic self map φ of U is called a $\psi-$generator of $\mathcal{H}$, if the set of multiplication by ψ to all polynomials in φ, is dense in $\mathcal{H}$. In the special case of $\psi=1$, we call φ a generator of $\mathcal{H}$.*

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on U such that the set of polynomials is dense in $\mathcal{H}$. If φ is not a $\psi-$generator of $\mathcal{H}$ and $C_{\psi,\varphi}$ is bounded on $\mathcal{H}$, then $0 \in int\ W(C_{\psi,\varphi})$.*

Proof. Since φ is not a $\psi-$generator of $\mathcal{H}$, the set $\{\psi.p\circ\varphi : \ p\ is\ a\ poly.\}$ is not dense in $\mathcal{H}$. But $ran\ C_{\psi,\varphi}$ is equal to the closure of the set $\{\psi.p \circ \varphi : \ p\ is\ a\ poly.\}$, hence $ran\ C_{\psi,\varphi} \neq \mathcal{H}$. This implies that

$$kerC^*_{\psi,\varphi} = (ran\ C_{\psi,\varphi})^{\perp} \neq 0.$$

So 0 is an eigenvalue of $C^*_{\psi,\varphi}$ but not a normal eigenvalue, because $C_{\psi,\varphi}$ is one-to-one. Hence 0 does not belong to the boundary of $W(C^*_{\psi,\varphi})$. Since $0 \in \sigma_p(C^*_{\psi,\varphi}) \subseteq W(C^*_{\psi,\varphi})$, 0 should be belong to the interior of $W(C^*_{\psi,\varphi})$. But $W(C^*_{\psi,\varphi})$ is equal to the conjugate of the set $W(C_{\psi,\varphi})$, hence $0 \in\ int\ W(C_{\psi,\varphi})$ and this completes the proof. □

Corollary 2.7. *Let φ be not a $\psi-$generator of $H^2(\beta)$ and $H^2(\beta) \subset H(U)$. If $C_{\psi,\varphi}$ is bounded on $H^2(\beta)$, then $0 \in\ int\ W(C_{\psi,\varphi})$.*

Corollary 2.8. *Let φ be a nonconstant holomorphic self map of U that is not a generator of $H^2(\beta)$ and let C_φ be bounded on $H^2(\beta)$. Then $0 \in\ int\ W(C_\varphi)$.*

Theorem 2.9. *Let $\mathcal{H}$ be a Hilbert space such that $\mathcal{H} \subset H(U)$, each point of U is a bounded point evaluation, and the set of polynomials is dense in $\mathcal{H}$. If $C_{\psi,\varphi}$ is bounded on $\mathcal{H}$ and ψ has a zero in U or φ is not one-to-one, then φ is not a $\psi-$generator of $\mathcal{H}$.*

Proof. Clearly $C_{\psi,\varphi}$ is injective. Suppose that $\psi(z_0) = 0$, $z_0 \in U_0$. Put

$$\mathcal{M}_{z_0} = \{f \in \mathcal{H} : \ f(z_0) = 0\},$$

and note that $\mathcal{M}_{z_0}$ is a closed subspace of $\mathcal{H}$ that is nontrivial, since $1 \notin \mathcal{M}_{z_0}$. But polynomials are dense in $\mathcal{H}$, hence $\mathcal{M}_{z_0}$ is equal to the closure of $\{p : \ p\ is\ a\ poly.\ and\ p(z_0) = 0\}$. Since $ran\ C_{\psi,\varphi} \subset \mathcal{M}_{z_0}$, φ can not be a $\psi-$generator of $\mathcal{H}$.

Now, if φ is not one-to-one, there exists distinct points z_1, z_2 in U satisfying $\varphi(z_1) = \varphi(z_2)$. If $\psi(z_1) = 0$ or $\psi(z_2) = 0$, then by the first case φ is not a $\psi-$generator of $\mathcal{H}$. Else, let $\psi(z_i) = 0$ for $i = 1, 2$ and note that

$$C^*_{\psi,\varphi}(\overline{\psi(z_1)}e_{z_2} - \overline{\psi(z_2)}e_{z_1}) = \overline{\psi(z_1)}\ \overline{\psi(z_2)}e_{\varphi(z_2)} - \overline{\psi(z_2)}\ \overline{\psi(z_1)}e_{\varphi(z_1)} = 0.$$

Thus

$$\overline{\psi(z_1)}e_{z_2} - \overline{\psi(z_2)}e_{z_1} \in ker(C^*_{\psi,\varphi})$$

and so $ker(C^*_{\psi,\varphi}) \neq 0$. This implies that $ran\ C_{\psi,\varphi} \neq \mathcal{H}$ and so $\{\psi.p \circ \varphi\ :\ p = poly.\}$ can not be dense in $\mathcal{H}$. Hence φ is not a $\psi-$generator of $\mathcal{H}$ $\square$

Corollary 2.10. *Under the conditions of Theorem 2.9,* $0 \in int\ W(C_{\psi,\varphi})$.

Corollary 2.11. *Let* $\liminf \beta(n)^{\frac{1}{n}} = 1$ *and* $C_{\psi,\varphi}$ *be bounded on* $H^2(\beta)$. *If* ψ *has a zero in* U *or* φ *is not one-to-one, then* φ *is not a* $\psi-$*generator of* $H^2(\beta)$, *and so* $0 \in\ int\ W(C_{\psi,\varphi})$.

Corollary 2.12. *Let* φ *be a nonconstant holomorphic self map of* U *and* C_φ *be bounded on* $H^2(\beta)$. *If* φ *is not injective, then* $0 \in\ int\ W(C_\varphi)$.

Now we use a definition of numerical range of an operator acting on Banach spaces which extends the earlier definition of numerical range of an operator acting Hilbert spaces.

Definition 2.13. *Let* X *be an infinite dimensional reflexive Banach space and* T *belongs to* $B(X)$ *(the set of all bounded linear operators on* X*). The numerical range of* T *is defined by* $W(T) = co(V(T))$ *where* $co(V(T))$ *is the convex hull of* $V(T)$ *that is defined as follows:*

$$V(T) = \{x^*(T(x)) : x \in X, x^* \in X^*;\ \|x\| = \|x^*\| = x^*(x) = 1\}.$$

Under Definition 2.13, we extend Theorem 1 of [1] that was stated only for Hilbert spaces.

Theorem 2.14. *Let* X *be an infinite dimensional reflexive Banach space and* $T \in B(X)$ *be compact. Then*
(i) $W(T)$ *is closed if and only if* $0 \in W(T)$.
(ii) If $0 \notin W(T)$, *then* $0 \in ext(\overline{W}(T))$ *and so* $\overline{W}(T) \backslash W(T)$ *consists at most of line segments in* $\partial W(T)$ *which contain* 0 *but no other extreme point of* $\overline{W}(T)$.

Proof. Note that if $\beta \in \overline{V}(T)$ and $\beta \neq 0$, then by Theorem 3.1 in [11], there exists $0 < \alpha \leq 1$ such that $\alpha\beta \in V(T)$. So $\beta \in (0, \alpha^{-1}\beta]$.
(i) Note that $0 \in \sigma(T) \subset \overline{W}(T)$. Now if $W(T)$ is closed, then clearly $0 \in W(T)$. Conversely, if $0 \in W(T)$, then by the convexity of $W(T)$, we get $\beta \in W(T)$ whenever $\beta \in \overline{V}(T)$. Thus $\overline{V}(T) \subset W(T)$ and so we obtain

$$\overline{W}(T) = co\overline{V}(T) \subset W(T).$$

This implies that $W(T)$ is closed.

(ii) First we show that every nonzero extreme point of $\overline{W}(T)$ belongs to $W(T)$. Let $\beta \in ext(\overline{W}(T))$. If $\beta \neq 0$, then as we saw earlier $\beta \in (0, c\beta]$ for some $c \geq 1$ (since $\beta \in \overline{W}(T)$). Thus it should be $\beta = c\beta$ which implies that $c = 1$ and so $\beta \in W(T)$.

Now, note that if $0 \in \overline{W}(T) \backslash W(T)$, then it should be $0 \in \partial W(T)$. If there exists a line segment $[z_1, z_2] \subset \partial W(T)$ such that $0 \in (z_1, z_2)$ and also $[z_1, z_2] \cap \overline{W}(T)$ has two extreme points z_1 and z_2, then as we saw it should be $z_1,\ z_2$ belongs to $W(T)$. Now by the convexity of $W(T)$, we get $0 \in W(T)$ that is a contradiction. So if $0 \notin W(T)$, then $0 \in ext(\overline{W}(T))$. Also, note that if $0 \notin W(T)$ and if there is not a line segment I containing 0 such that $I \subset \partial W(T)$, then every point in $\partial W(T) \backslash \{0\}$ should be an extreme point of the intersection of $\overline{W}(T)$ with a ray from 0. Hence $\overline{W}(T) \backslash W(T) = \{0\}$ and so the proof is complete. □

Corollary 2.15. *Let X be a Banach space and $T \in B(X)$ be compact. If X is not separable, then $W(T)$ is closed.*

Proof. Since T is compact, $\overline{ran(T)}$ is separable. Now if $0 \notin \sigma_p(T^*)$, then $ran(T)$ is dense in X that is a contradiction. Thus

$$0 \in \sigma_p(T^*) \subset W(T^*) = W(T)$$

and so $0 \in W(T)$. Now by Theorem 2.14 (i), $W(T)$ is closed. □

Now we investigate the numerical range of an operator acting on Banach weighted Hardy spaces $H^p(\beta)$, $p > 1$. We will use the facts that

$$e_w(z) = \sum_{n=0}^{\infty} \bar{w}^n z^n \in H^q(\beta^{-1})$$

and

$$\|e_w\|^q = \sum_{n=0}^{\infty} \frac{|w|^{nq}}{\beta(n)^q} < \infty$$

for all w in $\{z : |z| < r\}$ where $r = \liminf_n \beta(n)^{\frac{1}{n}}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus w is a bounded point evaluation on $H^p(\beta)$ if and only if $\{w^n/\beta(n)\} \in \ell^q$.

In the rest of the paper we suppose that $\liminf_n \beta(n)^{\frac{1}{n}} = 1$, so $H^p(\beta) \subset H(U)$ where U is the open unit disc in the complex plane. In the following proposition, the structure of V(T) of an operator T acting on $H^p(\beta)$ has been characterized (see also [9]). This enables calculations in $W(T)$ to be simpler.

Theorem 2.16. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and T be a bounded linear operator on $H^p(\beta)$. Then $V(T) = \{\langle Tf, f^* \rangle : f = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^p(\beta)$ and $\|f\|_p = 1\}$ where $f^*(z) = \sum_{n=0}^{\infty} \hat{f}^*(n) z^n$ satisfies $\hat{f}^*(n) = 0$ whenever $\hat{f}(n) = 0$, and else $\hat{f}^*(n) = \hat{f}(n)|\hat{f}(n)|^{\frac{p}{q}-1}\beta(n)^p$.*

Proof. Clearly if $f \in H^p(\beta)$, then $f^* \in H^q(\beta^{-1})$ and we have $\|f^*\|_q^q = \|f\|_p^p = \langle f, f^* \rangle$. Now, let $h \in H^p(\beta^{-1})$ be such that $\|h\|_q = \|f\|_p = \langle f, h \rangle = 1$. Hence, we can see that

$$\sum_{n=0}^{\infty} [(|\hat{f}(n)|\beta(n))(|\hat{h}(n)|\beta(n)^{-1})] = \|f\|_p \, \|h\|_q = 1.$$

Since equality holds in the Holder inequality (for complex series), it is easy to see that $|\hat{f}(n)|^p \beta(n)^p = |\hat{h}(n)|^q \beta(n)^{-q}$ and $\hat{f}(n)|\hat{h}(n)| = \hat{h}(n)|\hat{f}(n)|$ for all n. If $\hat{f}(n) \neq 0$, we obtain

$$\hat{h}(n) = \hat{f}(n)|\hat{f}(n)|^{-1}|\hat{f}(n)|^{\frac{p}{q}}\beta(n)^{\frac{p}{q}+1} = \hat{f}^*(n).$$

This completes the proof. $\square$

Theorem 2.17. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and T be a bounded linear operator on $H^p(\beta)$. Then $V(T) = \{\langle T(^*g), g \rangle : g = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in$*

$H^q(\beta^{-1})$ and $\|g\|_q = 1\}$ where ${}^*g(z) = \sum_{n=0}^{\infty} \hat{{}^*g}(n)z^n$ satisfies $\hat{{}^*g}(n) = 0$ whenever $\hat{g}(n) = 0$, and else $\hat{{}^*g}(n) = \hat{g}(n)|\hat{g}(n)|^{\frac{q}{p}-1}\beta(n)^{-q}$.

Proof. If $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n \in H^q(\beta^{-1})$, we have

$$ {}^*g \in H^p(\beta) \ \ ; \ \ \|{}^*g\|_p^p = \|g\|_q^q = <{}^* g, g> \ \ ; \ \ {}^*(cg) = c^{\frac{q}{p}}({}^*g). $$

By the same method used in Theorem 2.17, we can complete the proof. □

Remark 2.18. In [11, Theorem 3.1], it has been proved that if T is a compact operator on a reflexive Banach space X, then

$$ (*) \qquad \overline{V}(T) \subseteq \{c\alpha : 0 \le c \le 1, \alpha \in V(T)\}. $$

So if $\alpha \in \overline{V}(T)$, then there exists a sequence $\{x_n^*(Tx_n)\}_n$ in $V(T)$ converges to α and $\|x_n^*\| = \|x_n\| = x_n^*(x_n) = 1$. For the special case $X = H^p(\beta)$, by Theorem 2.16, x_n^* has a certain construction depending on x_n, i.e., if $x_n = \sum_{m=0}^{\infty} \hat{x_n}(m)z^m \in H^p(\beta)$, then

$$ x_n^*(z) = \sum_{m=0}^{\infty} \hat{x_n}(m)|\hat{x_n}(m)|^{\frac{p}{q}-1}\beta(m)^p z^m. $$

This makes that (in the case $X = H^p(\beta)$) the proof of $(*)$ to be easier than the one has been came in [11]. The idea behind the definitions of f^* and ${}^*g$ (where $f \in H^p(\beta)$, $g \in H^q(\beta^{-1})$) is related to the article [20]. Of course, I must point out that a condition of convexity had to be considered in Corollary 6 of [20], since for an operator T acting on a Banach space, $V(T)$ is not necessarily convex.

Corollary 2.19. *Let T be a compact operator on $H^p(\beta)$ and $\beta \in \overline{W}(T)$. Then $\beta = c\alpha$, where $\alpha \in W(T)$ and $|c| = 1$.*

Note that Theorem 3.3 and Corollary 3.4 in [11] have been proved only for composition operators. In the following theorem, we state similar results for weighted composition operators.

Theorem 2.20. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $\sum_{n=0}^{\infty}\beta(n)^{-q}=\infty$, $H^p(\beta)\subset H(U)$, *and* $C_{\psi,\varphi}$ *be bounded on* $H^p(\beta)$. *If* $\psi\in C(\partial\Omega)$ *and for some* $z_0\in\partial\Omega$, $\lim_{\lambda\to z_0}{}^{*}e_\lambda(\varphi(\lambda))$ *exists finitely, then* $0\in\overline{V}(C_{\psi,\varphi})\subset\overline{W}C_{\psi,\varphi})$.

Proof. For $\lambda\in U$, $\lambda\neq 0$, set

$$\omega_\lambda=\langle C_{\psi,\varphi}\frac{{}^{*}e_\lambda}{\|{}^{*}e_\lambda\|},\frac{e_\lambda}{\|e_\lambda\|}\rangle.$$

Note that $({}^{*}e_\lambda)^{*}=e_\lambda$, $\|{}^{*}e_\lambda\|\|e_\lambda\|=\|e_\lambda\|^q$ and $\langle {}^{*}e_\lambda,e_\lambda\rangle=\|e_\lambda\|^{-q}$. Thus by 2.17, $\omega_\lambda\in V(C_{\psi,\varphi})$. Now, we can see that $\omega_\lambda=\psi(\lambda)\|e_\lambda\|^{-q}\,{}^{*}e_\lambda(\varphi(\lambda))$ tends to 0 as $\lambda\to z_0$ and so the proof is complete. □

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