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Fritz-John Type Necessary Conditions for Optimality of Convex Generalized Semi-Infinite Optimization Problems

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Abstract. In this paper, we consider the Abadie and the Basic constraint qualifications (CQ) for lower level problem of convex generalized semi-infinite programming problems, and we derive the Fritz-John necessary optimality conditions for the problem under these constraint qualifications.

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1. Introduction

Main problem of this paper has the following form:

$$\text{CGSIP:} \qquad \inf \{ f(x) \mid x \in S \},$$

where

$$S := \{ x \in \mathbb{R}^n \mid g(x,y) \geqslant 0 \quad \text{for all } y \in Y(x) \},$$
$$Y(x) := \{ y \in \mathbb{R}^m \mid h_i(x,y) \leqslant 0, \quad i \in P \}, \quad \text{(set-valued mapping)}$$
$$P := \{1, 2, ..., p\},$$

$f : \mathbb{R}^n \to \mathbb{R}$ is convex function,

$g,\ h_i : \mathbb{R}^{n+m} \to \mathbb{R}$ are convex functions, for all $i \in P$.

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If all emerging functions f, g, and h_is are relaxed to convexity, CGSIP coincides to standard *generalized semi-infinite programming problem* (GSIP, in brief).

Under suitable differentiablity assumptions, the first order optimality conditions for GSIP were proved in various papers. The necessary conditions of Fritz-John (FJ in brief) type of optimality of GSIP, can be established under several lower-level constraint qualifications, e.g., Abadie [11], Kuhn-Tucker [14], and Mangasarian-Fromovitz constraint qualifications [10]. First-order optimality conditions without any constraint qualification at each solution of the lower-level problem were first given in [5] and extended in [13].

The author and his coauthor, considered a nondifferentiable GSIP in [8] for first time. They proved some optimality conditions in type of Fritz-John for GSIP whit Lipschitzian data under Mangasarian-Fromovitz like constraint qualification which is defined by Mordokhvich subdifferential.

In all of above papers, the following uniform boundedness on the set-valued map $Y(x)$ is a standard assumption.

Assumption A: For all $\hat{x} \in S$, the set-valued map $Y(.)$ is uniformly bounded around $\hat{x}$; i.e., there exists a neighborhood U of $\hat{x}$ such that the set $\bigcup_{x \in U} Y(x)$ is bounded.

This assumption, however, is very restrictive for example when the constraint functions $h_i(x,y) \leqslant 0$ $(i \in P)$ are linear. In [12] the smooth GSIPs with completely convex functions $\big($this means $-g(x,y)$ is jointly convex and the graph of the set-valued map Y is convex$\big)$ are considered. In this case the FJ condition has a very simple form and the assumption A may be removed. One of the essential purposes of this paper is to remove this assumption for nonsmooth convex GSIP (CGSIP, shortly).

The organization of the paper is as follows. In Section 2, basic notations and results of convex analysis are reviewed. In Section 3, we consider convex (classic) GSIPs, and we obtain the necessary optimality conditions of FJ type for them. The final section 4 is devoted to extension of the results obtained in the preceding section to the new class of GSIP.

2. Notations and Preliminaries

Given a nonempty set $A \subseteq \mathbb{R}^n$, we denote by $\bar{A}$, $int(A)$, $conv(A)$, and $cone(A)$, the closure of A, the interior of A, convex hull and convex cone containing 0 generated by A, respectively. The negative polar cone $A^{\ominus}$ and indicator function $I_A(.)$ of A are defined as

$$A^{\ominus} := \{u \in \mathbb{R}^n \mid \langle u, x \rangle \leqslant 0, \ \forall\, x \in A\},$$

$$I_A(x) := \begin{cases} 0 & if \quad x \in A \\ +\infty & if \quad x \notin A. \end{cases}$$

The bipolar theorem (see [4]) states that $\left(A^{\ominus}\right)^{\ominus} = \overline{cone}(A)$. Let $D \subseteq \mathbb{R}^n$ be a closed convex set, the normal cone of D at $\bar{x} \in D$ is define as

$$N_D(\bar{x}) := \left\{u \in \mathbb{R}^n \mid \langle u, x - \bar{x} \rangle \leqslant 0 \quad \text{for all } x \in D\right\}.$$

The negative polar cone of $N_D(\bar{x})$ is called the tangent cone of D at $\bar{x}$; $T_D(\bar{x}) := N_D(\bar{x})^{\ominus}$. It is known that $T_D(\bar{x})$ and $N_D(\bar{x})$ are always closed convex cone with contain 0_n (symbol 0_n denotes the origin of $\mathbb{R}^n$).
Given a proper convex function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, i.e.,

$$dom\varphi := \{x \in \mathbb{R}^n \mid \varphi(x) < +\infty\} \neq \emptyset.$$

For any $\hat{x} \in (dom\varphi)$, the subdifferential of φ at $\hat{x}$ is defined as the non-empty closed convex set

$$\partial\varphi(\hat{x}) := \left\{\xi \in \mathbb{R}^n \mid \langle \xi, x - \bar{x} \rangle \leqslant \varphi(x) - \varphi(\bar{x}) \quad \text{for all } x \in \mathbb{R}^n\right\}.$$

As usual, the symbols $\partial_x\varphi(\bar{x}, \bar{y})$ and $\partial_y\varphi(\bar{x}, \bar{y})$ stand for the corresponding partial subdifferential of φ at $(\bar{x}, \bar{y})$.
The following elementary theorem will be used in this paper.

Theorem 2.1. ([4]) *Suppose that φ and ϕ are two proper convex functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, and that $\bar{x} \in int(dom\varphi) \cap int(dom\phi)$. Suppose further that $D \subseteq \mathbb{R}^n$ is a closed convex set.*

- *If $\vartheta(x) = \max\left\{\varphi(x), \phi(x)\right\}$, then*

$$\partial\vartheta(\bar{x}) = conv\big(\partial\varphi(\bar{x}) \cup \partial\phi(\bar{x})\big).$$

- *If φ attains its minimum on D at $\bar{x}$, then*

$$0_n \in \partial\varphi(\bar{x}) + N_D(\bar{x}).$$

Finally in this section, we recall the following notion. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-value mapping. The graph of F is defined as

$$gphF := \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\big\}.$$

3. Main Results

Firstly, the lower level problem of CGSIP at $\hat{x} \in S$ which depends on the parameter $\hat{x}$ is defined as

$$\inf g(\hat{x}, y) \qquad \text{s.t.} \quad y \in Y(\hat{x}), \tag{1}$$

and the set of active constraints at $\hat{x}$ is denoted by

$$Y_0(\hat{x}) := \big\{y \in Y(\hat{x}) \mid g(\hat{x}, y) = 0\big\}.$$

Clearly, $Y_0(\hat{x})$ is just the set of minimizers of the lower level problem (1).

Let

$$\pi := \big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid h_i(x, y) \leqslant 0 \ \ \forall\, i \in P\big\}.$$

For $(x_0, y_0) \in \pi$ set

$$B(x_0, y_0) := \bigcup_{i=1}^{p} \partial h_i(x_0, y_0).$$

The following constraint qualifications were introduced in [1] and [4] as a very general assumption for the necessary conditions in standard optimization problems (differentiable and non-differentiable convex cases). They were deeply extended in [7, 9] for optimization problems with infinite number constraints (convex and non-convex cases).

Definition 3.1. *The convex inequalities system $\{h_i(x, y) \leqslant 0 \mid i \in P\}$ is said to satisfy the*

- *Abadie constraint qualification (ACQ, briefly) at* $(x_0, y_0) \in \pi$ *if*

$$B^{\ominus}(x_0, y_0) \subseteq T_\pi(x_0, y_0).$$

- *Basic constraint qualification (BCQ, shortly) at* $(x_0, y_0) \in \pi$ *if*

$$N_\pi(x_0, y_0) \subseteq cone\big(B(x_0, y_0)\big). \quad \blacktriangle$$

The relationships between the ACQ and BCQ is given in the following lemma.

Lemma 3.2. *Suppose that* $(x_0, y_0) \in \pi$. *Then we have*

$$\text{BCQ at } (x_0, y_0) \implies \text{ACQ at } (x_0, y_0).$$

Furthermore, if the convex cone of $B(x_0, y_0)$ *is closed, these two constraint qualifications are equivalent.*

Proof. Firstly, we suppose that the BCQ verifies. By bipolar Theorem and by definition of tangent cone we give

$$B^{\ominus}(x_0, y_0) = \Big(cone\big(B(x_0, y_0)\big)\Big)^{\ominus} \subseteq N_\pi^{\ominus}(x_0, y_0) = T_\pi(x_0, y_0).$$

Converesly, if ACQ holds and $cone\big(B(x_0, y_0)\big)$ is closed, we conclude that

$$N_\pi(x_0, y_0) = T_\pi^{\ominus}(x_0, y_0) \subseteq \big(B^{\ominus}\big)^{\ominus}(x_0, y_0) = \overline{cone}\big(B(x_0, y_0)\big) = cone\big(B(x_0, y_0)\big). \quad \square$$

We now associate the CGSIP the following Lagrangian function

$$L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^p \longrightarrow \mathbb{R}$$
$$L(x, y, \alpha, \beta) = \alpha g(x, y) + \textstyle\sum_{i=1}^{p} \beta_i h_i(x, y).$$

We denote the set (maybe empty) of Karush-Kahn-Tucker (KKT) multipliers of the problem (1) at $y_0 \in Y_0(x_0)$ by

$$K(x_0, y_0) := \big\{\beta \in \mathbb{R}_+^p \mid 0_m \in \partial_y L(x_0, y_0, 1, \beta), \quad \beta_i h_i(x_0, y_0) = 0 \ \ \forall\, i \in P\big\}. \quad \square$$

Theorem 3.3. (FJ necessary condition for CGSIP under the BCQ). *Let* x_0 *be an optimal solution to the CGSIP. Suppose furthermore that the basic constraint qualification holds for the system* $\big\{h_i(x, y) \leqslant 0 \mid i \in P\big\}$ *at* $(x_0, y_0) \in gphY_0$.

Then, there exist $\lambda_0, \lambda_1 \in [0,1]$ *and* $\beta \in K(x_0, y_0)$ *such that*

$$0_n \in \lambda_0 \partial f(x_0) - \lambda_1 \partial_x L(x_0, y_0, 1, \beta),$$

$$\lambda_0 + \lambda_1 = 1.$$

Proof. With the convention $\inf \emptyset = +\infty$, let $\mu(x_0)$ denote the value function of lower level problem (1)

$$\mu(x_0) := \inf \big\{ g(x_0, y) \mid y \in Y(x_0) \big\}. \tag{2}$$

It is easy to check that the convexity of g and h_is implies the convexity of μ; see [Prop. 2.5][2]. Standard arguments in nonsmooth optimization imply that x_0 is then also a local minimizer of the auxiliary function

$$\theta(x) := \max \big\{ f(x) - f(x_0), -\mu(x) \big\}.$$

The first order necessary optimality condition for convex function now implies that

$$0_n \in \partial\theta(x_0) \subseteq conv\big(\partial f(x_0) \cup \big(-\partial\mu(x_0)\big)\big).$$

Thus, there is a $\tau \in [0,1]$ such that

$$0_n \in \tau \partial f(x_0) - (1-\tau) \partial\mu(x_0). \tag{3}$$

The next step is also efficiently estimate the subdifferential $\partial\mu(x_0)$. Since $y_0 \in Y_0(x_0)$, we observe that $\mu(x_0) = g(x_0, y_0)$ and

$$\mu(x) \leqslant g(x,y), \qquad \forall\, (x,y) \in S \times Y(x).$$

Suppose that $\xi \in \partial\mu(x_0)$. By the definition of the subdifferential, we have

$$\mu(x) - \mu(x_0) \geqslant \langle \xi, x - x_0 \rangle, \qquad \forall\, x \in \mathbb{R}^n.$$

Together these inequalities imply that

$$g(x,y) - g(x_0, y_0) \geqslant \langle \xi, x \rangle - \langle \xi, x_0 \rangle, \qquad \forall\, (x,y) \in S \times Y(x).$$

That is, if and only if (x_0, y_0) is a solution to the following (finite and convex) optimization problem:

$$\begin{aligned} &\min\ g(x,y) - \langle \xi, x\rangle \\ &\text{s.t.}\ \ h_i(x,y) \leqslant 0, \qquad i = 1, 2, ..., p. \end{aligned}$$

By assumption, the basic constraint qualification holds for the above problem at (x_0, y_0), and hence the KKT condition holds at (x_0, y_0); see [4, VII Prop. 2.2.1]. Thus, there is a $\beta := (\beta_1, \beta_2, ..., \beta_p) \in \mathbb{R}^p_+$, such that

$$\begin{aligned} (0_n, 0_m) &\in \partial\Big(g(x,y) - \langle \xi . x\rangle\Big)(x_0, y_0) + \sum_{i=1}^{p} \beta_i \partial h_i(x_0, y_0) \\ &= \partial g(x_0, y_0) - (\xi, 0) + \sum_{i=1}^{p} \beta_i \partial h_i(x_0, y_0), \end{aligned} \tag{4}$$

and

$$\beta_i h_i(x_0, y_0) = 0, \qquad \forall\ i = 1, 2, ..., p.$$

Then we use the following important relationship between the full and partial subdifferentials of convex functions $\Psi(x, y)$ that holds by, e.g., [3, Prop. 2.3.15]:

$$\partial \Psi(x,y) \subseteq \partial_x \Psi(x,y) \times \partial_y \Psi(x,y). \tag{5}$$

Employing (5) in the KKT condition (4), we get

$$\begin{aligned} &\begin{cases} 0_n \in \partial_x g(x_0, y_0) - \xi + \sum_{i=1}^{p} \beta_i \partial_x h_i(x_0, y_0), \\ 0_m \in \partial_y g(x_0, y_0) + \sum_{i=1}^{p} \beta_i \partial_y h_i(x_0, y_0), \\ \beta_i h_i(x_0, y_0) = 0, \quad \forall\ i \in P. \end{cases} \\ \Longleftrightarrow &\begin{cases} \xi \in \partial_x L(x_0, y_0, 1, \beta), \\ 0_m \in \partial_y L(x_0, y_0, 1, \beta), \\ \beta_i h_i(x_0, y_0) = 0, \quad \forall\ i \in P. \end{cases} \\ \Longleftrightarrow &\ \xi \in \bigcup_{\beta \in K(x_0, y_0)} \partial_x L(x_0, y_0, 1, \beta). \end{aligned}$$

Since ξ was an arbitrary element of $\mu(x_0)$, we thus proved

$$\partial\mu(x_0) \subseteq \bigcup_{\beta \in K(x_0,y_0)} \partial_x L(x_0, y_0, 1, \beta). \tag{6}$$

Owning to (3) and (6) the proof is complete. $\square$

Remark 3.4. *It is clear (from definitions) that where h_is are differentiable, then*

- *ACQ and BCQ are equivalent.*
- *In (5), and consequently in (6), equality holds.*

Consequently, the result in [15, Prop. 2.1] is a corollary of Theorem 3.3.
It is known that if h_is are affine, then ACQ holds at each $(x_0, y_0) \in \pi$ (see [4]). Therefore, owning to Remark 3.4, Lemma 3.2, and Theorem 3.3, the following corollary is immediate.

Corollary 3.5. (FJ necessary condition for CGSIP with lower level linear constraints). *Let x_0 be a locally optimal solution of the CGSIP. Suppose that lower level constraint functions h_is (as $i \in P$) are linear on $\mathbb{R}^n \times \mathbb{R}^m$. Then, for all $y_0 \in Y_0(x_0)$ there are $\lambda_0, \lambda_1 \in [0,1]$ and $\beta \in K(x_0, y_0)$ such that*

$$0_n \in \lambda_0 \partial f(x_0) - \lambda_1 \partial_x L(x_0.y_0, 1, \beta),$$

$$\lambda_0 + \lambda_1 = 1.$$

Note that the FJ condition above for the CGSIPs with linear constraint functions $h_i(x, y)$ is very appealing since no uniform boundedness assumption and no constraint qualification are required.
Observe that the necessary optimality condition in Theorem 3.3, can be stated for one $y_0 \in Y_0(x_0)$ only. Since the Slater condition does not depend on (x_0, y_0), we have the following necessary optimality condition for all $(x_0, y_0) \in gphY_0$.

Theorem 3.6. (FJ necessary condition for CGSIP under the Slater like qualification condition). *Let x_0 be an optimal solution to the CGSIP.*

Suppose that $Y_0(x_0) \neq \emptyset$, and that the Slater condition holds for the region

$$\big\{(x, y) \mid h_i(x, y) \leqslant 0 \qquad i \in P\big\},$$

i.e., there exists $(\hat{x}, \hat{y}) \in \pi$ such that $h_i(\hat{x}, \hat{y}) \leqslant 0$ for all $i \in P$. Then, for each choice $y_0 \in Y_0(x_0)$ there exists $\lambda_0, \lambda_1 \in [0, 1]$ and $\beta \in K(x_0, y_0)$ such that

$$0_n \in \lambda_0 \partial f(x_0) - \lambda_1 \partial_x L(x_0.y_0, 1, \beta),$$

$$\lambda_0 + \lambda_1 = 1.$$

Proof. It is well-known that for convex optimization, if the Slater condition holds, then the ACQ holds; see [4]. On the other hand, according to the [6, Theorem 3.10], if the Slater condition holds, then $cone\Big(\bigcup_{i=1}^{p} \partial h_i(x_0, y_0)\Big)$ is closed cone. Owning to the Lemma 3.2, and Theorem 3.3, the proof is complete. □

Note that the Slater condition above is obviously weaker than the Slater condition for the lower level problem (1) which requires the existence of $\hat{y}$ such that $h_i(x_0, \hat{y}) < 0$ as $i \in P$. Theorem 3.6, generalizes the results of [12, Theorem 4.3.5] in that no uniform boundedness assumption is needed and the required Slater condition is weaker.

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