Journal of Mathematical Extension
Vol. 14, No. 4, (2020), 113-122
ISSN: 1735-8299
URL: https://www.ijmex.com
Original Research Paper

The Structure of Unit Group of $\mathbb{F}_{2^n}D_{14}$

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Abstract. Let RG be the gruop ring of the group G over ring R and $\mathscr{U}(RG)$ be its unit group. In this paper, The structure of the unit group of a group ring $\mathbb{F}_{2^n}D_{14}$ of the group D_{14} over a field of characteristic 2 is determined.

AMS Subject Classification: 16S34; 20C05; 16U60
Keywords and Phrases: Group, Ring, Unit

1 Introduction

Let RG be the group ring of group G over field F and $\mathscr{U}(FG)$ be its unit group, *i.e.,* multiplicative subgroup containing all invertible elements. The study of unit group is one of the classical topics in ring theory that started in 1940 with a famous paper written by G. Higman [10]. In recent years many new results have been achived; However, only few group rings have been computed. Unit groups are useful, for instance in the investigation of Lie properties of group rings (for example see [2]) and isomorphism problem (for example see [3]).

Up to now, the structure of unit groups of some group rings has been found. For instance, on integral group ring [11], on permutation group ring [19], on commutative group ring [16], on linear group ring [13], on quaternion group ring [4], on modular group ring [17] and on pauli group ring [8]. In [6], the authors proved which groups can be unit groups, moreover, on properties of unit elements themselves instead of their groups structure [1].

In this paper we'll study the unit group of dihedral group ring. Till now some cases have been studied. For instance, in [12], the authors obtained $\mathscr{U}(F_2D_{2p})$, in [7], Gildea calculated the order of $\mathscr{U}(\mathbb{F}_{p^k}D_{2p^m})$ and in [9] determined the structure of the unitary units of the group algebra $F_{2^k}D_8$. In continue, the authors of [14]

Received: September 2019; Accepted: October 2019
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obtained the structure of unit groups of generalized dihedral group rings and recently in [18], the authors determined the structure of the unitary subgroup of the group algebra $\mathbb{F}_{2^n}(QD)_{16}$ where QD is a quasi dihedral group. In this paper we characterize the unit group structure of dihedral group with order 14 over any finite field with characteristic 2.

2 Preliminary and Notations

In this section, we bring some notations and lemma which we need for the proof of our main results. We denote the order of an element g in the group G by $Ord_G(g)$, the sum of all elements of subset X in ring R by $\widehat{X}$, *i.e.,* $\sum_{r\in X} r$. Notice there is no need for X to be a subring or subgroup, it defines for any arbitrary subset. In group ring RG, when X be subset of all different powers of g (an element of group G), we may simply write $\widehat{g}$ instead of $\widehat{X}$. Also when X be right coset of $\langle g\rangle$ with respect to h, we may write $\widehat{g}h$ for $\widehat{X}$. Let $f : X \to Y$ be an arbitrary function, then $Supp_X(f) = \{x \in X \mid f(x) \neq 0\}$. Also, we use the following notations: $\mathrm{Ann}_R(a) = \{r \in R \mid ra = ar = 0\}$, we denote a finite field of characteristic p with order p^n by $\mathbb{F}_{p^n}$. If E is a vector space over F, then $Dim_F(E)$ is the dimension of E over F. Let $\mathscr{U}(R)$ be the unit group of ring R, *i.e.,* $\mathscr{U}(R) = \{u \in R \mid u^{-1} \in R\}$ and $J(R)$ be the jacobson radical of ring R. Now we state a useful definition and recall a lemma.

Definition 2.1. Let RG be group ring of ring R over the group G, p be a prime number and S_p be subset of all $p-elements$ including identity element of G, *i.e.,* $S_p = \{g \in G \mid \exists n \in \mathbb{Z}^{\geqslant 0};\ \mathrm{Ord}_G(g) = p^n\}$. We define a binary map $T : G \to R$ as follows:

$$T(g) = \begin{cases} 1 & \textbf{\textit{If}} \quad g \in S_p \\ 0 & \textbf{\textit{If}} \quad g \notin S_p \end{cases}$$

As we know that T on G is the base of RG, so we can linearly extend it to whole RG, of course no more remains binary. Also if see elements of RG as functions from G to R, that map every group element (g) to its coefficient (r_g), then their supports will be feasible. Now we can define $\mathrm{Krn}(T) := \{\alpha \in RG \mid \forall g \in G;\ \alpha g \in \mathrm{Ker}_{RG}(T)\}$ and $\mathrm{Spr}(\alpha) := \mathrm{Supp}_G(\alpha)$. Also $\mathrm{Anh}(a) := \mathrm{Ann}_{RG}(a)$ and $\mathrm{Dmn}(S) := \mathrm{Dim}_F(S)$.

Lemma 2.2. *Let F be a finite field of characteristic p, G be a finite group, T be a function defined as above and $s = \widehat{S}_p$. Then:*

(1) $J(FG) \subseteq \mathrm{Krn}(T)$.
(2) $\mathrm{Krn}(T) = \mathrm{Anh}(s)$.
(3) $J(FG) \subseteq \mathrm{Anh}(s)$.

Proof. [20, Lemma 2.2 on p. 151].
□

In the next section we bring our main result.

3 Unit Group of $\mathbb{F}_{2^n}D_{14}$

Let D_{2n} be Dihedral group of order $2n$, C_n be cyclic group of order n and $GL_n(R)$ be general linear group of order n on ring R. Our main result is:

Theorem 3.1. *Let G be the Dihedral group of order 14 and $F = \mathbb{F}_{2^n}$. Then the structure of $\mathscr{U}(FG)$ can be obtained as follows:*

1. ***If** $n = 3m$, **Then** $\mathscr{U}(FG) = C_2^n \times C_{2^n-1} \times GL_2(F)^3$*
2. ***If** $n \neq 3m$, **Then** $\mathscr{U}(FG) = C_2^n \times C_{2^n-1} \times GL_2(F_3)$*

Let $p = 2$, x^y be the conjugate of x by y that is $x^y = y^{-1}xy$, s be as defined in 2.2, $\langle x\rangle$ be the cyclic subgroup generated by x and $\langle x\rangle y$ be right coset of $\langle x\rangle$ with respect to y that is $\langle x\rangle y = \{x^i y \mid -3 \leqslant i \leqslant +3\}$, *i.e.*, $\langle x\rangle y = \{x^{-3}y, x^{-2}y, x^{-1}y, y, xy, x^2y, x^3y\}$. By $\widehat{x}$ definition, sum of different powers of x, that is $\widehat{x} = x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + x^3$ and $\widehat{xy} = x^{-3}y + x^{-2}y + x^{-1}y + y + xy + x^2y + x^3y$, so we have:

Proposition 3.2. *Let $p = 2$ and group $G = D_{14}$, Dihedral group of order 14, that is, $D_{14} = \langle x, y \mid x^7 = y^2 = 1, x^y = x^{-1}\rangle$. Then $\mathrm{Anh}(s) = \langle \widehat{G}\rangle$.*

Proof. It's well known that the conjugacy classes of G are as follows:

$$\begin{aligned} \mathscr{C}_0 &= \{1\} \\ \mathscr{C}_1 &= \{x^1, x^{-1}\} \\ \mathscr{C}_2 &= \{x^2, x^{-2}\} \\ \mathscr{C}_3 &= \{x^3, x^{-3}\} \\ \mathscr{C}_4 &= \langle x\rangle y \end{aligned} \tag{1}$$

It is clear that D_{14} has three types of elements: Identity, elements of form $x^i y$ with order 2 and elements of form $x^i (7 \nmid i)$ with order 7. So $S_2 = \mathscr{C}_0 \cup \mathscr{C}_4$, therefore $\widehat{S_2} = \widehat{\mathscr{C}_0} + \widehat{\mathscr{C}_4} = 1 + \widehat{xy}$, sum of $2-$*elements* including identity. Let $\alpha = \sum_{i=0}^4 \alpha_i = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \mathrm{Anh}(s)$ where $\mathrm{Spr}(\alpha_i) \subseteq \mathscr{C}_i$ and $s = \widehat{S}_2$. Then we have:

$$\begin{aligned} 0 = \alpha.s = (\sum_{i=0}^{4} \alpha_i)(1 + \widehat{xy}) \\ &= (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(1 + \widehat{xy}) \\ &= (\alpha_4 + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3))(1 + \widehat{xy}) \\ &= ((\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) + \alpha_4\widehat{xy}) \\ &+ (\alpha_4 + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)\widehat{xy}) \end{aligned} \tag{2}$$

Notice that for every j, we know:

$$(x^j y)(\widehat{xy}) = \widehat{x} \quad \text{and} \quad (x^j)(\widehat{xy}) = \widehat{xy} \tag{3}$$

So the conjugacy classes of two last parentheses of 2 are different, since left side is zero, every parentheses should be zero separately. Hence:

$$(\alpha_0+\cdots+\alpha_3)+\alpha_4\widehat{xy}=0 \quad \text{and} \quad \alpha_4+(\alpha_0+\cdots+\alpha_3)\widehat{xy}=0$$

Now again by using 3 we can conclude that:

$$(\alpha_0+\ldots\alpha_3)+\varepsilon(\alpha_4)\widehat{x}=0 \quad \text{and} \quad \alpha_4+\varepsilon(\alpha_0+\cdots+\alpha_3)\widehat{xy}=0 \tag{4}$$

As mentioned above $\alpha=\sum_{i=0}^{4}\alpha_i=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4$ where $\mathrm{Spr}(\alpha_i)\subseteq\mathscr{C}_i$ and by definition of $\mathscr{C}_i$'s from 1, we can write:

$$\begin{aligned}
\alpha_0 &= a_0\\
\alpha_1 &= a_1x^1+a_{-1}x^{-1}\\
\alpha_2 &= a_2x^2+a_{-2}x^{-2}\\
\alpha_3 &= a_3x^3+a_{-3}x^{-3}\\
\alpha_4 &= b_{-3}x^{-3}y+b_{-2}x^{-2}y+b_{-1}x^{-1}y+b_0y+b_1xy+b_2x^2y+b_3x^3y
\end{aligned}$$

By substitute of each α_i's in 4, we can calculate the coefficients of each element of the group in the left sides of equations and since right sides are zero, so each coefficient must be zero too. Therefore, for every i and j, we have:

$$\begin{array}{cc}
a_i+\varepsilon(\alpha_4)=0 & b_j+\sum_{i=0}^{3}\varepsilon(\alpha_i)=0\\
a_i=-\sum_{j=-3}^{+3}b_j & b_j=-\sum_{i=-3}^{+3}a_i\\
a_{-3}=\cdots=a_0=\cdots=a_3 & b_{-3}=\cdots=b_0=\cdots=b_3\\
a_i=-7b_j & b_j=-7a_i
\end{array}$$

Since we deal with a field of characteristic 2, so $-7=1$, therefore, for every i and j, we have $a_i=b_j$, thus $\alpha=a_0.\widehat{G}$ and therefore:

$$\mathrm{Anh}(s)=\langle\widehat{G}\rangle$$

□

Let s be as was in 3.2, that is $s=\widehat{S}_2$, then we have:

Proposition 3.3. $\mathrm{Anh}(s)$ *is a nilpotent ideal.*

Proof. Let $\alpha,\beta\in\mathrm{Anh}(s)$, according to 3.2:

$$\alpha.\beta=a\widehat{G}.b\widehat{G}=ab.\widehat{G}\widehat{G}=ab|G|\widehat{G}$$

Since F is a field of characteristic 2, and G has 14 elements, that is, $|G|=14=0$, so $\alpha.\beta=0$, thus $\mathrm{Anh}^2(s)=0$, therefore, $\mathrm{Anh}(s)$ is a nilpotent ideal.

□

Let s be as was in 3.3, that is $s=\widehat{S}_2$, then we have:

Proposition 3.4. $\mathrm{Anh}(s)\subseteq J(FG)$.

Proof. Since every nilpotent ideal is a nil ideal, so 3.3 shows $\mathrm{Anh}(s)$ is a nil ideal. On the other hand, by [15, Lemma 2.7.13 on p. 109], Jacobson radical contains all of the nil ideals, so:

$$\mathrm{Anh}(s) \subseteq J(FG)$$

□

In the next corollary, we'll show that the equality hold:

Corollary 3.5. $J(FG) = \mathrm{Anh}(s)$.

Proof. By 3.4, $\mathrm{Anh}(s) \subseteq J(FG)$ and we know from 2.2 part (3) that $J(FG) \subseteq \mathrm{Anh}(s)$, so the equality is hold:

$$J(FG) = \mathrm{Anh}(s)$$

□

We'll need the following proposition in the next steps:

Proposition 3.6. $\mathrm{Dmn}(J(FG)) = \mathrm{Dmn}(\mathrm{Anh}(s)) = 1$

Proof. By 3.2 and 3.5 we have:

$$J(FG) = \mathrm{Anh}(s) = \langle \widehat{G} \rangle \tag{5}$$

That means, $J(FG)$ and $\mathrm{Anh}(s)$ are generated by one elment $\widehat{G}$, hence:

$$\mathrm{Dmn}(J(FG)) = \mathrm{Dmn}(\mathrm{Anh}(s)) = 1$$

□

Let $H := \langle x \rangle = \{x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3\} \trianglelefteq G$, a normal subgroup of G. Also we recall augmentation ideals $\Delta(G,H) := \langle h-1 |\ h \in H \rangle$, that in special case $H = G$, we denote $\Delta(G) := \Delta(G,G)$. Now it's obvious that by [15, Proposition 3.3.3 on p. 135], we have:

$$\mathrm{Dmn}(\Delta(G,H)) = |G| - [G:H] = 14 - 2 = 12$$
$$\mathrm{Dmn}(\Delta(G,G)) = |G| - [G:G] = 14 - 1 = 13$$

Therefore we can bring the following remark:

Remark 3.7. Dimensions of $\Delta(G,H)$ and $\Delta(G)$ can be computed as follows:

$$\mathrm{Dmn}(\Delta(G,H)) = 12$$
$$\mathrm{Dmn}(\Delta(G,G)) = 13$$

We want to represent a decomposition for $\Delta(G)$ over $J(FG)$ and $\Delta(G,H)$. As both of them are included in $\Delta(G)$, first we show they are disjoint:

Proposition 3.8. $J(FG) \cap \Delta(G,H) = 0$.

Proof. Let $\alpha \in J(FG) \cap \Delta(G,H)$. By 5, $J(FG) = \langle \widehat{G} \rangle$. Now we compute $\alpha.\widehat{x}$ in two different ways, according to see α as an element of $J(FG)$ or $\Delta(G,H)$ separately:

$$\begin{array}{cc}
\alpha \in J(FG) = \langle \widehat{G} \rangle & \alpha \in \Delta(G,H) = \langle x-1 \rangle \\
\alpha = a.\widehat{G} & \alpha = \beta(x-1) \\
\alpha\widehat{x} = a\widehat{G}\widehat{x} = a\widehat{G}|\langle x \rangle| & \alpha\widehat{x} = \beta(x-1)\widehat{x} = \beta(x\widehat{x} - 1\widehat{x}) \\
= a.\widehat{G}.n = a.\widehat{G} = \alpha & = \beta.(\widehat{x} - \widehat{x}) = \beta.0 = 0
\end{array}$$

So we conclude that:

$$\alpha = \alpha.\widehat{x} = 0 \tag{6}$$

And therefore we have:

$$J(FG) \cap \Delta(G,H) = 0$$

□

Now the decomposition can be achieved:

Proposition 3.9. $\Delta(G) = J(FG) \oplus \Delta(G,H)$.

Proof. By 3.6 and 3.7, we have:

$$\mathrm{Dmn}(J(FG)) + \mathrm{Dmn}(\Delta(G,H)) = 1 + 12 = 13 = \mathrm{Dmn}(\Delta(G))$$

Now 3.8 together with above equality shows that:

$$\Delta(G) = J(FG) \oplus \Delta(G,H)$$

□

In the next Proposition, we prove that $\Delta(G,H)$ is a semisimple ring:

Proposition 3.10. $\Delta(G,H)$ *is a semisimple ring.*

Proof. By 3.9, we have $\Delta(G,H) = \Delta(G)/J(FG) \subseteq FG/J(FG)$. And moreover with [15, Theorem 6.6.1 on p. 214] any field over a finite group makes an Artinian group ring, so FG is an Artinian group ring, and [15, Lemma 2.4.9 on p. 87], implies its quotient ring, $FG/J(FG)$, is Artinian too. By [15, Lemma 2.7.5 on p. 107] we know Jacobson radical vanishes, $J(FG/J(FG)) = 0$. By [15, Theorem 2.7.16 on p. 111] we can explore that $FG/J(FG)$ is semisimple, and by [15, Proposition 2.5.2 on p. 91], all of its subrings are semisimple too. So $\Delta(G,H)$ is semisimple.
□

By Artin-Wedderburn Theorem, $\Delta(G,H)$ decomposes to its simple components that are division rings of matrices over extensions of F. Now we need to know their numbers and dimensions. First we show that the center of $\Delta(G,H)$ is included in the center of FG:

Proposition 3.11. $Z(\Delta(G,H)) \subseteq Z(FG)$

Proof. For the proof of this proposition, we need show that each element of $Z(FG)$ must commute with all of elements of FG. Since F is commutative and G is generated by x and y, so it suffices to show they commute with x and y. Let $\alpha \in Z(\Delta(G,H))$, so it commutes with $x-1$ as it is in $\Delta(G,H)$:

$$\alpha.(x-1) = (x-1).\alpha$$
$$\alpha.x - \alpha = x.\alpha - \alpha$$
$$\alpha.x = x.\alpha$$

So α commutes with x. Now we show that α also commutes with y. First we show that $\alpha y - y\alpha$ is in $\mathrm{Anh}(x-1)$. Notice we know that $(x-1)y = y(x^{-1}-1) \in \Delta(G,H)$, so:

$$\begin{array}{cc} (x-1)y \in \Delta(G,H) & y(x-1) \in \Delta(G,H) \\ \alpha.(x-1).y = (x-1).y.\alpha & \alpha.y.(x-1) = y.(x-1).\alpha \\ (x-1).\alpha y = (x-1).y\alpha & \alpha y.(x-1) = y\alpha.(x-1) \\ (x-1)(\alpha y - y\alpha) = 0 & (\alpha y - y\alpha)(x-1) = 0 \end{array}$$

So $(\alpha y - y\alpha) \in \mathrm{Anh}(x-1)$ and by [15, Lemma 3.4.3 on p. 139] we know that:

$$\mathrm{Anh}(x-1) = \mathrm{Anh}(\Delta(G,H)) = FG\widehat{x}$$

Now we compute $(\alpha y - y\alpha).\widehat{x}$ in two different ways, directly itself or see $(\alpha y - y\alpha)$ as an element of $FG.\widehat{x}$ separately. At first we compute it directly. Before that notice $\alpha \in Z(\Delta(G,H)) \subseteq \Delta(G,H)$, so by 6, $\alpha.\widehat{x} = 0$, and although x does not commute with y, but $\widehat{x}$ does. So we have:

$$(\alpha y - y\alpha).\widehat{x} = \alpha.y.\widehat{x} - y.\alpha.\widehat{x} = \alpha\widehat{x}.y - y.\alpha\widehat{x} = 0.y - y.0 = 0$$

Now we compute it according to see $(\alpha y - y\alpha)$ as an element of $FG.\widehat{x}$. Before that notice $|\langle x\rangle| = Ord_G(x) = 7 = 1$. So we have:

$$(\alpha y - y\alpha).\widehat{x} = \beta.\widehat{x}.\widehat{x} = \beta.\widehat{x}.|\langle x\rangle| = \beta\widehat{x} = (\alpha y - y\alpha)$$

Hence $\alpha y - y\alpha = (\alpha y - y\alpha).\widehat{x} = 0$, thus $\alpha y = y\alpha$, that means α also commutes with y and therefore:

$$Z(\Delta(G,H)) \subseteq Z(FG)$$

□

In the next proposition, we obtain the exact structure of $Z(\Delta(G,H))$:

Proposition 3.12. $Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}\rangle$

Proof. Let $\alpha \in Z(\Delta(G,H))$, from [15, Theorem 3.6.2 on p. 151] we know center on conjugacy classes, $Z(FG) = \langle \widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}, \widehat{\mathscr{C}_4}\rangle$, so $Z(\Delta(G,H)) \subseteq \langle \widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}, \widehat{\mathscr{C}_4}\rangle$, by using 3.11. So $\alpha = \sum_{i=0}^{4} r_i\widehat{\mathscr{C}_i} = r_0\widehat{\mathscr{C}_0} + r_1\widehat{\mathscr{C}_1} + r_2\widehat{\mathscr{C}_2} + r_3\widehat{\mathscr{C}_3} + r_4\widehat{\mathscr{C}_4}$. By 6, $\alpha.\widehat{x} = 0$ and notice that $x^{\pm i}\widehat{x} = \widehat{x}$, so $(x^i + x^{-i})\widehat{x} = 2\widehat{x} = 0$. Hence:

$$\begin{aligned} 0 = \alpha\widehat{x} &= \sum_{i=0}^{4} r_i\widehat{\mathscr{C}_i}\widehat{x} = r_0\widehat{\mathscr{C}_0}\widehat{x} + (\sum_{i=1}^{3} r_i\widehat{\mathscr{C}_i}\widehat{x}) + r_4\widehat{\mathscr{C}_4}\widehat{x} \\ &= r_0.1.\widehat{x} + (\sum_{i=1}^{3} r_i(x^i + x^{-i})\widehat{x}) + r_4\widehat{x}\widehat{y}\widehat{x} = r_0\widehat{x} + 0 + r_4\widehat{x}\widehat{y} \end{aligned} \tag{7}$$

Since left side of 7 is zero, so right side coefficients should be zero, $r_0 = r_4 = 0$, hence, $\alpha = \sum_{i=1}^{3} r_i \widehat{\mathscr{C}_i} = r_1\widehat{\mathscr{C}_1} + r_2\widehat{\mathscr{C}_2} + r_3\widehat{\mathscr{C}_3}$, that means $Z(\Delta(G,H)) \subseteq \langle \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3} \rangle$. Now it suffices to show that all of these types of elements are included in $\Delta(G,H)$. We must show that there is a β that $\alpha = \beta(x-1)$. It is straightforward to find β's coefficients by solving a system of linear equations. For $\alpha = \beta(x-1)$:

$$\beta = r_1 x + (r_1 + r_2)x^2 + (r_1 + r_2 + r_3)x^3 + (r_1 + r_2)x^4 + r_1 x^5$$

So $\alpha \in \Delta(G,H)$, and therefore:

$$Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3} \rangle$$

□

Now the dimension of the center of $\Delta(G,H)$ can be computed:

Corollary 3.13. $\mathrm{Dmn}(Z(\Delta(G,H))) = 3$

Proof. By 3.12, we know that $Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3} \rangle$, so:

$$\mathrm{Dmn}(Z(\Delta(G,H))) = 3$$

□

Let $M_n(R)$ be the ring of the square matrices of the order n on the ring R and $GL_n(R)$ be its unit group. Also R^n be the direct sum of n copy of the ring R, *i.e.,* $R^n = \oplus_{i=1}^{n} R$ and F_n be the extension of the finite field F of the order n that is $[F_n : F] = n$. Now we are ready to prove 3.1:

Proof.[Proof of 3.1] Let $\alpha \in Z(\Delta(G,H))$. From 3.12, we know elements of center on conjugacy classes, $\alpha = \sum_{i=1}^{3} r_i \widehat{\mathscr{C}_i} = r_1\widehat{\mathscr{C}_1} + r_2\widehat{\mathscr{C}_2} + r_3\widehat{\mathscr{C}_3}$. Since $char(F) = 2$, we have:

$$\begin{aligned}
\alpha^1 &= r_1^1\widehat{\mathscr{C}_1} + r_2^1\widehat{\mathscr{C}_2} + r_3^1\widehat{\mathscr{C}_3} \\
\alpha^2 &= r_1^2\widehat{\mathscr{C}_2} + r_2^2\widehat{\mathscr{C}_3} + r_3^2\widehat{\mathscr{C}_1} \\
\alpha^4 &= r_1^4\widehat{\mathscr{C}_3} + r_2^4\widehat{\mathscr{C}_1} + r_3^4\widehat{\mathscr{C}_2} \\
\alpha^8 &= r_1^8\widehat{\mathscr{C}_1} + r_2^8\widehat{\mathscr{C}_2} + r_3^8\widehat{\mathscr{C}_3}
\end{aligned}$$

Since $|F| = 2^n$, we know $r_i^{2^n} = r_i$, so $\alpha^{2^n} = r_1\widehat{\mathscr{C}_1}^{2^n} + r_2\widehat{\mathscr{C}_2}^{2^n} + r_3\widehat{\mathscr{C}_3}^{2^n}$. Therefore by 3.7 and 3.13, we conclude that if 3 divides n, then $\alpha^{2^n} = \alpha$ and else $\alpha^{2^{3n}} = \alpha$. Therefore we have:

$$\begin{array}{llll}
\textbf{\textit{If}}\ n = 3m, & \textbf{\textit{Then}}\ \alpha^{2^n} = \alpha & \textbf{\textit{So}}\ \Delta(G,H) \cong M_2(F)^3 \\
\textbf{\textit{If}}\ n \neq 3m, & \textbf{\textit{Then}}\ \alpha^{2^{3n}} = \alpha & \textbf{\textit{So}}\ \Delta(G,H) \cong M_2(F_3)
\end{array}$$

By [15, Proposition 3.6.7 on p. 153], we know $FG \cong F(G/H) \oplus \Delta(G,H)$, and therefore, $\mathscr{U}(FG) \cong \mathscr{U}(F(C_2)) \times \mathscr{U}(\Delta(G,H))$. As $\mathscr{U}(F(C_2))$ was described in the [5, Theorem 1.7 on p. 239], we have:

$$\mathscr{U}(FG) = \begin{cases} C_2^n \times C_{2^n-1} \times GL_2(F)^3 & q = 3m \\ C_2^n \times C_{2^n-1} \times GL_2(F_3) & q \neq 3m \end{cases}$$

□

4 Conclusion

One of the important research works in group ring is obtaining the structure of unit group of some finite group rings. In this paper, we deal with the finite field of even order and we can obtain the structure of unit group of $\mathbb{F}_{2^n}D_{14}$.

Acknowledgements
We would like to thank prof. Ali Rajaei, for his helpful guidance. Also the authors are much grateful to the anonymous referee for his/her valuable comments on our paper, which have considerably improved the presentation of this paper.

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