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# $\alpha$-Prime Ideals

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**Abstract.** Let $R$ be a commutative ring with identity. We give a new generalization to prime ideals called $\alpha$-prime ideal. A proper ideal $P$ of $R$ is called an $\alpha$-prime ideal if for all $a, b$ in $R$ with $ab \in P$, then $a \in P$ or $\alpha(b) \in P$ where $\alpha \in End(R)$. We study some properties of $\alpha$-prime ideals analogous to prime ideals. We give some characterizations for such generalization and we prove that the intersection of all $\alpha$-primes in a ring $R$ is the set of all $\alpha$-nilpotent elements in $R$. Finally, we give new versions of some famous theorems about prime ideals including $\alpha$-integral domains and $\alpha$-fields.

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**Keywords and Phrases:** $\alpha$-Prime ideal; $\alpha$-primary ideal; $\alpha$-nilradical; $\alpha$-integral domain; $\alpha$-field

## 1 Introduction

Throughout this article $R$ will be a commutative ring with nonzero identity and $\alpha : R \longrightarrow R$ a fixed endomorphism on $R$. The notion of a prime ideal plays a key role in the theory of commutative algebra, and it has

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been widely studied. Recall from [4] that a prime ideal $P$ of $R$ is a proper ideal $P$ with the property that for $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$. Recently many generalizations of prime ideals were introduced and studied (see for example [1], [3] and [5]). The radical of an ideal $I$ of a ring $R$ is defined to be $\sqrt{I} = \{r \in R : r^n \in I \text{for some} n \in \mathbb{N}\}$. A proper ideal $P$ is called primary if $ab \in P$ implies $a \in P$ or $b \in \sqrt{P}$ [6]. An integral domain is refereed to us a commutative ring with identity which has no non-zero divisors. For any other concepts see [7]. In this paper, we introduce the notion of $\alpha$-prime ideal, and establish some characterizations of it. We prove and generalize some results of $\alpha$-prime ideals that are analogous to prime ideals.

## 2 Main results

Let $R$ be a ring and let $\alpha \in End(R)$ be a fixed endomorphism. A proper ideal $P$ of a ring $R$ is called an $\alpha$-prime ideal of $R$ if for all $r, s \in R$, $rs \in P$ implies that $r \in P$ or $\alpha(s) \in P$. The definition is equivalent to say $rs \in P$ implies that either $\alpha(r) \in P$ or $s \in P$. In view of the definition of an $\alpha$-prime ideal, we see that in the case when $\alpha$ is the identity map, $\alpha$-prime ideal will be a prime ideal. So $\alpha$-prime ideals are considered as a generalization of prime ideals. For an $\alpha$-prime $P$, if $r \in P$, then $\alpha(r) \in P$ since $r = 1 \cdot r \in P$ implies $1 \in P$ or $\alpha(r) \in P$. Hence we can assume $\alpha(P) \subseteq P$. Also in the case when $\alpha(I) = I$, for all ideals $I$ of $R$, then the prime ideals and $\alpha$-prime ideals will be again the same. Note that every prime ideal is an $\alpha$-prime ideal, where $\alpha$ is the identity map. However, the converse is not true in general as shown in the following example.

**Example 2.1.** Consider the ideal $P =< 2x >$ in the ring $R = \mathbb{Z}_4[x]$ with endomorphism $\alpha$ on $R$ defined by $\alpha(f(x)) = f(0)$. Then $P$ is $\alpha$-prime but not prime, since $2 \cdot x \in P$ and $2, x \notin P$ whereas $\alpha(x) = 0 \in P$.

**Lemma 2.2.** *Let $P$ be an $\alpha$-prime. Then so is $\sqrt{P}$.*

**Proof.** Let $xy \in \sqrt{P}$, for $x, y \in R$. Then $(xy)^n = x^n y^n \in P$ and $P$ being $\alpha$-prime implies $x^n \in P$ or $\alpha(y^n) = (\alpha(y))^n \in P$ which means that $x \in \sqrt{P}$ or $\alpha(y) \in \sqrt{P}$. Hence $\sqrt{P}$ is $\alpha$-prime. $\square$

For a ring $R$ and an $\alpha$-prime ideal $P$ of $R$, we define a subset $S_P$ of $R$ as $S_P = \{r \in R : \alpha(r) \in P\}$. Clearly, $S_P$ is an ideal of $R$ containing $P$. The following is a direct consequence and can be proved easily and so the proof is omited.

**Lemma 2.3.** *Assume $P$ is an $\alpha$-prime ideal of a ring $R$. Then $S_P$ is an $\alpha$-prime ideal of $R$.*

**Lemma 2.4.** *Suppose $P$ is $\alpha$-prime and maximal with respect to the property that $r \in P$ implies $\alpha(r) \in P$. Then $P$ is prime.*

**Proof.** By contrast, suppose $P$ is not prime and so there exist $a, b \in R$ with $ab \in P$ such that $a \notin P$ and $b \notin P$. Consider the ideal $(P, a) = \{m + ra : m \in P, r \in R\}$ and take $x \in (P, a)$. Then $x = m + ra$ and $xb = mb + rab \in P$. Hence $\alpha(x) \in P \subseteq (P, a)$. So by hypothesis, $(P, a) = P$ and hence $a \in P$, which is a contradiction. Therefore $P$ is prime. □

Now we give a charactrization of an $\alpha$-prime ideal.

**Theorem 2.5.** *Let $R$ be a ring and $P$ a proper ideal of $R$. Then $P$ is $\alpha$-prime if and only if for any two ideals $I, J$ of $R$ such that $IJ \subseteq P$, $I \subseteq P$ or $\alpha(J) \subseteq P$.*

**Proof.** Let $P$ be an $\alpha$-prime ideal and $IJ \subseteq P$ with $I \nsubseteq P$. Then there exists $a$ in $I$ such that $a \notin P$. For every $b \in J$, $ab \in IJ \subseteq P$, but $a \notin P$ so $\alpha(b) \in P$, that is, $\alpha(J) \subseteq P$. Conversely, let $ab \in P$, which implies that $\langle a\rangle\langle b\rangle \subseteq P$. Hence we have $\langle a\rangle \subseteq P$ or $\alpha(\langle b\rangle) \subseteq P$. Therefore $a \in P$ or $\alpha(b) \in P$ and $P$ is $\alpha$-prime. □

Let $J$ be a subset of a ring $R$. We show that the $\alpha$-primeness of an ideal $P$ implies the $\alpha$-primeness of the ideal $(P : J)$.

**Proposition 2.6.** *If $P$ is an $\alpha$-prime ideal of a ring $R$ and $I$ a subset of $R$, then so is $(P : I)$.*

**Proof.** Suppose $ab \in (P : I)$ for $a, b \in R$. Then $b \in (P : aI) = (P : a) \cup (P : I)$. Thus $ba \in P$ or $b \in (P : I)$, that is, $b \in P$ or $\alpha(a) \in P$ or $b \in (P : I)$. Therefore $\alpha(a) \in (P : I)$ or $b \in (P : I)$ and $(P : I)$ is $\alpha$-prime ideal. □

**Remark 2.7.** We note that for an $\alpha$-prime ideal $P$ of a ring $R$ and $r \in R$, if $r^n \in P$, then $\alpha(r) \in P$. Thus if we put $r = \alpha(x)$, then $(\alpha(x))^n \in P$ implies that $\alpha \circ \alpha(x) \in P$.

Let $R$ be a ring. An element $a \in R$ is called $\alpha$-nilpotent if $\alpha(a^n) = 0$ for some positive integer $n$. We call the set of $\alpha$-nilpotent elements in a ring $R$ the $\alpha$-nilradical of $R$ and denote by $\mathcal{N}_\alpha$. We know that if $x$ is a nilpotent element in a ring $R$, then $1 - x$ is a unit in $R$. This result can be extended as follows: For an $\alpha$-nilpotent element $r$ in $R$, $1 - \alpha(r)$ is a unit in $R$. In the sight of the definition of $\alpha$-nilpotent elements we can define the $\alpha$-radical of an ideal $I$ to be $\sqrt[\alpha]{I} = \{a \in R : \alpha(a^n) \in I \text{ for some positive integer } n\}$. Thus $\mathcal{N}_\alpha = \sqrt[\alpha]{0}$ and clearly $I \subseteq \sqrt[\alpha]{I}$. Now, we are in a position to characterize the set $\mathcal{N}_\alpha$ as an ideal. First, we have to prove the ideality of $\mathcal{N}_\alpha$.

**Proposition 2.8.** *The set of all $\alpha$-nilpotent elements $\mathcal{N}_\alpha$ is an ideal of $R$.*

**Proof.** Let $x, y \in \mathcal{N}_\alpha$. Then $\alpha(x^n) = \alpha(y^m) = 0$ for some positive integers $n, m$ and by the binomial theorem $(\alpha(x) + \alpha(y))^{n+m-1}$ is a sum where all its monomials contain the product $(\alpha(x))^r(\alpha(y))^s$ with $r + s = m + n - 1$. So the case when $r < n$ and $s < m$ is excluded. Hence each of these monomials is zero. So $(\alpha(x + y))^{n+m-1} = (\alpha(x) + \alpha(y))^{n+m-1} = 0$ and $x + y \in \mathcal{N}_\alpha$. Also, for every $r \in R$, we have $\alpha((rx)^n) = \alpha(r^n) \cdot \alpha(x^n) = 0$. Therefore $\mathcal{N}_\alpha$ is an ideal of $R$. □

Now, we give one of our main results that characterizes the ideal $\mathcal{N}_\alpha$ and it is a generalization of Proposition 1.8 of the Atiyah's book [4]. For this reason we need the following two lemmas.

**Lemma 2.9.** *For $\alpha \in End(R)$, the kernel of $\alpha$ is in the intersection of all $\alpha$-prime ideals.*

**Proof.** Suppose $x \in Ker\alpha$. Then $\alpha(x) = 0$ belongs to every $\alpha$-prime ideal $P$ of $R$. So, $x$ belongs to the inverse image of every $\alpha$-prime ideal which is again an $\alpha$-prime ideal by Proposotion 2.21. Therefore $Ker\alpha \subseteq \bigcap_{P \text{ is } \alpha-\text{ prime in } R} P$. □

**Lemma 2.10.** *Assume that $R$ is an integral domain and $\alpha \in End(R)$. Then the kernel of $\alpha$ is a prime ideal in $R$.*

**Proof.** Suppose $xy \in Ker\alpha$ for $x, y \in R$. Then $\alpha(xy) = \alpha(x)\alpha(y) = 0$ and $R$ being an integral domain implies $\alpha(x) = 0$ or $\alpha(y) = 0$, that is, $x \in Ker\alpha$ or $y \in Ker\alpha$. Therefore $Ker\alpha$ is a prime ideal in $R$. $\square$

**Theorem 2.11.** *The $\alpha$-nilradical $\mathcal{N}_\alpha$ of an integral domain $R$ is the intersection of all the $\alpha$-prime ideals of $R$.*

**Proof.** Suppose $x$ is $\alpha$-nilpotent. Then $\alpha(x^n) = 0$ and $x^n \in Ker\alpha$. Lemma 2.10 implies that $x \in Ker\alpha$ and Lemma 2.9 gives us
$x \in \bigcap_{P \text{ is } \alpha- \text{ prime in } R} P$. Thus $\mathcal{N}_\alpha \subseteq \bigcap_{P \text{ is } \alpha- \text{ prime in } R} P$. For the reverse inclusion, let $x$ be non $\alpha$-nilpotent and define a set $S = \{I : I$ an ideal of $R$ and $\alpha(x^n) \notin I$ for all $n > 0\}$. Clearly 0 belong to $S$ and so $S$ is nonempty. Order $S$ by inclusion and let $\{I_i\}_{i\in I}$ be a chain of ideals of in $S$. Then $I_i \subseteq I_j$ or $I_j \subseteq I_i$ for each pair of indices $i$ and $j$. Set $I = \bigcup_i I_i$, so that it is an ideal in $S$ and becomes an upper bound of the chain. Therefore by Zorn's lemma, $S$ has a maximal element, say $J$. Now to prove that $J$ is $\alpha$-prime, let $\alpha(a), \alpha(b) \notin J$. Then $J \subset J + R\alpha(a)$, $J \subset J + R\alpha(b)$ and so they are not elements of $S$. Thus there exist positive integers $m, n$ such $\alpha(x^m) \in J + R\alpha(a)$, $\alpha(x^n) \in J + R\alpha(b)$. So $\alpha(x^{n+m}) \in J + R\alpha(ab)$. It follows $J + R\alpha(ab) \notin S$ and $\alpha(ab) = \alpha(a)\alpha(b) \notin J$. Therefore by Remark 2.7, $ab \notin J$ and $J$ is an $\alpha$-prime ideal in which $\alpha(x^n) \notin J$, that is $x \notin J$ and so $x \notin \mathcal{N}_\alpha$. $\square$

By taking the quotient ring $R/I$ instead of $R$ in Theorem 2.11 we conclude the following.

**Corollary 2.12.** *For an integral domain $R$ and an ideal $I$ of $R$, the $\alpha$-radical of $I$ is equal to the intersection of all the $\alpha$-prime ideals of $R$ which contains $I$.*

Here are some properties of the $\alpha$-radical of an ideal, which are extended from those of the usual radical of an ideal.

**Proposition 2.13.** *Suppose $I$ and $J$ are two ideals of a ring $R$. Then the following are true.*

1. *If $I \subseteq J$, then $\sqrt[\alpha]{I} \subseteq \sqrt[\alpha]{J}$*
2. *$\sqrt[\alpha]{IJ} = \sqrt[\alpha]{I \cap J} = \sqrt[\alpha]{I} \cap \sqrt[\alpha]{J}$.*
3. *If $\alpha(1) = 1$, then $\sqrt[\alpha]{I} = R$ if and only if $I = R$.*

4. *$\sqrt[\alpha]{I+J} \subseteq \sqrt[\alpha]{\sqrt[\alpha]{I} + \sqrt[\alpha]{J}}$*
5. *If $I$ is an $\alpha$-prime ideal of $R$, then $\sqrt[\alpha]{I^n} = \sqrt[\alpha]{I}$, for all positive integer $n$.*

**Proof.**

1. The proof is clear.
2. To prove the first equality we have $IJ \subseteq I \cap J$, so $\sqrt[\alpha]{IJ} \subseteq \sqrt[\alpha]{I \cap J}$. For the reverse inclusion, let $x \in \sqrt[\alpha]{I \cap J}$. Then $\alpha(x^n) \in I \cap J$ for some positive integer $n$ and so $\alpha(x^{2n}) \in IJ$. Hence $x \in \sqrt[\alpha]{IJ}$. Now to prove the last equality, we have from $I \cap J \subseteq I$ and $I \cap J \subseteq J$ that $\sqrt[\alpha]{I \cap J} \subseteq \sqrt[\alpha]{I} \cap \sqrt[\alpha]{J}$. For other side, let $y \in \sqrt[\alpha]{I} \cap \sqrt[\alpha]{J}$. Then $\alpha(y^r) \in I$ and $\alpha(y^s) \in J$ for some positive integers $r, s$. Hence $\alpha(y^k) \in I \cap J$, for $k = max\{r, s\}$. Thus $y \in \sqrt[\alpha]{I \cap J}$ and the equality holds.
3. Suppose $\sqrt[\alpha]{I} = R$. Then $1 \in \sqrt[\alpha]{I}$ implies that $\alpha(1^n) = \alpha(1) = 1 \in I$ which means that $I = R$. The other implication is obvious.
4. The two inclusions $I \subseteq \sqrt[\alpha]{I}$ and $J \subseteq \sqrt[\alpha]{J}$ together imply that $\sqrt[\alpha]{I+J} \subseteq \sqrt[\alpha]{\sqrt[\alpha]{I} + \sqrt[\alpha]{J}}$.
5. The proof follows from part (2), namely that $\sqrt[\alpha]{I^n} = \sqrt[\alpha]{I.I...I} = \sqrt[\alpha]{I} \cap \sqrt[\alpha]{I} \cap ... \cap \sqrt[\alpha]{I} = \sqrt[\alpha]{I}$.

□

The equality of part (4) is not true in general as it is the case of usual radical. The only thing that we can say is $\alpha(\sqrt[\alpha]{\sqrt[\alpha]{I} + \sqrt[\alpha]{J}}) \subseteq \sqrt[\alpha]{I+J}$.

**Proposition 2.14.** *Let $f : R \to S$ be a ring homomorphism and assume that $\alpha \in End(R) \cap End(S)$ commutes with $f$. Let $P$ and $\bar{P}$ be two ideals of $R$ and $S$ respectively. Then*

1. *$f(\sqrt[\alpha]{P}) \subseteq \sqrt[\alpha]{f(P)}$.*
2. *$\sqrt[\alpha]{f^{-1}(\bar{P})} \subseteq f^{-1}(\sqrt[\alpha]{\bar{P}})$*
3. *If $f$ is an isomorphism, then $f(\sqrt[\alpha]{P}) = \sqrt[\alpha]{f(P)}$.*

**Proof.**

1. Let $x \in f(\sqrt[\alpha]{P})$. Then $x = f(a)$ for some $a \in \sqrt[\alpha]{P}$. Since $a \in \sqrt[\alpha]{P}$, there exists a positive integer $n$ such that $\alpha(a^n) \in P$. Now $\alpha(x^n) = \alpha((f(a)^n) = \alpha(f(a^n)) = f(\alpha(a^n)) \in f(P)$. So $x \in \sqrt[\alpha]{f(P)}$.
2. Let $a \in \sqrt[\alpha]{f^{-1}(\bar{P})}$. Then there exists a positive integer $n$ such that $\alpha(a^n) \in f^{-1}(\bar{P})$. So $f(\alpha(a^n)) \in \bar{P}$. Since $f$ and $\alpha$ commute, $\alpha(f(a)^n) \in \bar{P}$. Hence $a \in f^{-1}(\sqrt[\alpha]{\bar{P}})$. Thus $\sqrt[\alpha]{f^{-1}(\bar{P})} \subseteq f^{-1}(\sqrt[\alpha]{\bar{P}})$.
3. The proof is obtained from part (1) and $f$ being an isomorphism.

□

We know that a proper ideal $P$ of a ring $R$ is prime if and only if $R/P$ has no zero divisors and that $P$ is $\alpha$-prime if and only if every zero divisor of $R/P$ is in $Ker\alpha$. Also, from Lemma 2.9 we have the isomorphism $R/P \cong \dfrac{R/Ker\alpha}{P/Ker\alpha}$. Hence $P$ is $\alpha$-prime if and only if $\dfrac{R/Ker\alpha}{P/Ker\alpha}$ has no zero divisors if and only if $P/Ker\alpha$ is a prime ideal. Therefore we deduce the main connection between prime ideals and $\alpha$-prime ideals.

**Theorem 2.15.** *Let $P$ be a proper ideal of $R$. Then $P$ is an $\alpha$-prime ideal in $R$ if and only if $\frac{P}{Ker\alpha}$ is prime in $\frac{R}{Ker\alpha}$.*

A ring $R$ is called an $\alpha$-integral domain if for all $a, b \in R$ with $ab = 0$, $a = 0$ or $\alpha(b) = 0$ for some endomorphism $\alpha$ on $R$. It is clear that every integral domain is an $\alpha$-integral domain, but the converse is not true as shown in the following example.

**Example 2.16.** Consider the ring $R = \dfrac{\mathbb{Z}[x]}{< x^2 - x >}$ and endomorphism $\alpha$ on $R$ defined by $\alpha(f(x)) = x \cdot f(x)$. Then $R$ is $\alpha$-integral domain but not integral domain, since $x(x-1) = 0$ and $x, 1-x \neq 0$ but $\alpha(1-x) = x(1-x) = 0$

The next theorem characterizes $\alpha$-prime ideals in the sense of quotient rings.

**Proposition 2.17.** *Let $R$ be a commutative ring. Then $P$ is an $\alpha$-prime ideal if and only if $R/P$ is an $\alpha$-integral domain.*

**Proof.** Let $R/P$ be an integral domain. Let $a, b \in R$ such that $ab \in P$. Then $ab + P = P$. Since $R/P$ is an $\alpha$-integral domain, $a + P = P$ or $\alpha(a + P) = P$. So $a \in P$ and $\alpha(a) \in P$. Thus $P$ is an $\alpha$-prime ideal . Conversely, Let $P$ be $\alpha$-prime ideal. Let $a, b \in R$ with $ab \in P$. Then $a \in P$ or $\alpha(a) \in P$. So $a + P = P$ or $\alpha(a) + P = P$. Hence $a + P = P$ or $\alpha(a + P) = P$. Therefore $R/P$ is an $\alpha$-integral domain.
□

An $\alpha$-integral domain $R$ is called an $\alpha$-field if $\dfrac{R}{Ker\alpha}$ is a field. Clearly every field is an $\alpha$-field and the converse is true in the case where $Ker\alpha = 0$.

It is well-known that if $K$ is a field, then $K[x]$ is a principal ideal domain but not a field. Define a ring homomorphism $\alpha : K[x] \to K[x]$ by $\alpha(f(x)) = f(0)$. Then $Ker\alpha =< x >$ and $\dfrac{K[x]}{Ker\alpha} = \dfrac{K[x]}{< x >} \cong K$ is a field. Similarly, for $K[x_1, \dots, x_n]$, we can define an endomorphism $\alpha$ on $K[x_1, \dots, x_n]$ by $\alpha(f(x-_1, \dots, x_n)) = f(0, 0, \dots, 0)$. So, $\dfrac{K[x_1, \dots, x_n]}{Ker\alpha} = \dfrac{K[x_1, \dots, x_n]}{< x_1, \dots, x_n >} \cong K$ is a field. Thus we conclude the following theorem.

**Theorem 2.18.** *For any field $K$, the polynomial ring $K[x_1, \dots, x_n]$ in $n$ indeterminates is $\alpha$-field but not field.*

We know that every finite integral domain is a field. Here we generalize this result to $\alpha$-integral domains.

**Proposition 2.19.** *Every finite $\alpha$-integral domain is an $\alpha$-field*

**Proof.** Suppose $R$ is a finite $\alpha$-integral domain, say $R = \{x_1, x_2, ..., x_n\}$ Then for any $x$ in $R$ with $\alpha(x) \neq 0$, the elements $xx_i$, $i = 1, 2, \dots, n$ are all distinct else for if $xx_i = xx_j$, then $x(x_i - x_j) = 0$ and as $R$ is $\alpha$-integral domain, $x_i - x_j = 0$ or $\alpha(x) = 0$. Hence, $\alpha(x) \neq 0$ implies $x_i = x_j$ and as $R$ has identity, there exists $s \in \{1, 2, ...n\}$ such that $xx_s = 1$. Therefore $x$ has an inverse $x_s$ and $R$ is an $\alpha$-field. □

A proper ideal $I$ of a ring $R$ is called an $\alpha$-primary ideal if for all $a, b \in R$ such that $ab \in I$, $a \in I$ or $\alpha(b^n) \in I$ for some positive integer $n$. Clearly every $\alpha$-prime ideal is $\alpha$-primary. Now we have the following lemma.

**Lemma 2.20.** *If $P$ is an $\alpha$-primary ideal of $R$, then $\sqrt[\alpha]{P}$ is an $\alpha$-prime ideal.*

**Proof.** Assume $ab \in \sqrt[\alpha]{P}$. Then $\alpha((ab)^n) = \alpha(a^n b^n) = \alpha(a^n)\alpha(b^n) \in P$. As $P$ is an $\alpha$-primary ideal, $\alpha(a^n) \in P$ or $\alpha(\alpha(b^n)) = \alpha([\alpha(b)]^n) \in P$. Thus $a \in \sqrt[\alpha]{P}$ or $\alpha(b) \in \sqrt[\alpha]{P}$. Therefore $\sqrt[\alpha]{P}$ is an $\alpha$-prime ideal. □

From Atiyah's book [4], we know that the inverse image of a prime ideal under a ring homomorphism is again prime ideal. Next, we prove that the inverse image of an endomorphism of $\alpha$-prime ideal is $\alpha$-prime in a generalized form.

**Proposition 2.21.** *Let $f : R \to S$ be a ring homomorphism and assume that $\alpha \in End(R) \cap End(S)$ commutes with $f$. Then for any $\alpha$-prime ideal $Q$ of $S$, $f^{-1}(Q)$ is an $\alpha$-prime ideal of $R$.*

**Proof.** Let $Q$ be an $\alpha$-prime ideal of $S$. Then for any two elements $a$ and $b$ in $R$ with $ab \in f^{-1}(Q)$, we have $f(a)f(b) \in Q$ and $Q$ being an $\alpha$-prime ideal implies that $f(a) \in Q$ or $\alpha(f(b)) = f(\alpha(b)) \in Q$, that is, $a \in f^{-1}(Q)$ or $\alpha(b) \in f^{-1}(Q)$ and this is what we want to prove. □

A subset $S$ of a ring $R$ is called an $\alpha$-multiplicative system if $a\alpha(b) \in S$ for all $a, b \in S$. Thus from this definition we have the following lemma.

**Lemma 2.22.** *Let $R$ be a commutative ring with identity. Then $P$ is an $\alpha$-prime ideal if and only if $R - P$ is an $\alpha$-multiplicative system .*

**Proposition 2.23.** *Suppose $S$ is a multiplicative subset of a ring $R$ and $\overline{\alpha} : S^{-1}R \to S^{-1}R$ is the induced map of $\alpha$. Then there is a one-to-one correspondence between $\alpha$-prime ideals $P$ of $R$ with $S \cap P = \emptyset$ and $\alpha$-prime ideals of $S^{-1}R$.*

**Proof.** Suppose $P$ is an $\alpha$-prime ideal in $R$ and $(\frac{a}{s})(\frac{b}{t}) \in S^{-1}P$ for $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$. Then the exists $u \in S$ such that $uab \in P$. So $ua \in P$ or $\alpha(b) \in P$. Thus, $\frac{ua}{us} = \frac{a}{s} \in S^{-1}P$ or $\frac{\alpha(b)}{t} \in S^{-1}P$, that is, $\frac{a}{s} \in S^{-1}P$ or $\overline{\alpha}(\frac{b}{t}) \in S^{-1}P$. Hence $S^{-1}P$ is $\overline{\alpha}-$prime ideal in $S^{-1}R$. The other side is obtained from Proposition 2.21. □

From Proposition 2.21 and Proposition 2.23, we can conclude the following.

**Proposition 2.24.** *Let $f : R \to S$ be a ring homomorphism and assume that $\alpha \in End(R) \cap End(S)$ commutes with $f$. Then an ideal $I$ containing $Ker\alpha$ is an $\alpha$-prime ideal if and only if $f(I)$ is an $\alpha$-prime ideal.*

**Corollary 2.25.** *Let $I$ and $J$ be two ideals of a ring $R$ such that $I \subseteq J$. Then $J/I$ is $\bar{\alpha}$-prime ideal in $R/I$ if and only if $J$ is $\alpha$-prime in $R$, where $\bar{\alpha}$ is the induced map on $R/I$ from $\alpha$.*

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