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Some Generalizations of Strongly Prime Ideals

H. Ansari-Toroghy
University of Guilan

F. Farshadifar *
Farhangian University

S. Maleki-Roudposhti
University of Guilan

Abstract. In this paper, we introduce the concepts of strongly 2-absorbing primary ideals (resp., submodules) and strongly 2-absorbing ideals (resp., submodules) as generalizations of strongly prime ideals. Furthermore, we investigate some basic properties of these classes of ideals.

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1 Introduction

Throughout this paper, R will denote an integral domain with quotient field K. Further, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N}$ will denote respectively the ring of integers, the field of rational numbers, and the set of natural numbers.

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*Corresponding Author

A prime ideal I of R is said to be *strongly prime* if, whenever $xy \in I$ for elements $x, y \in K$, then $x \in I$ or $y \in I$ [8]. An ideal I of R is said to be *strongly primary* if, whenever $xy \in I$ for elements $x, y \in K$, then $x \in I$ or $y^n \in I$ for some $n \geq 1$ [4].

The concept of 2-absorbing ideals was introduced in [3]. A proper ideal I of R is said to be a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [5], Badawi, et al. introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is called a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

The purpose of this paper is to introduce the concepts of strongly 2-absorbing primary ideals (resp., submodules) and strongly 2-absorbing ideals (resp., submodules) as generalizations of strongly prime ideals. Furthermore, we investigate basic properties of these classes of ideals.

Let R be an integral domain with quotient field K. An ideal I of R is said to be a *strongly 2-absorbing primary ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in I$ or $(yz)^n \in I$ or $(xz)^m \in I$ for some $n, m \in \mathbb{N}$ (Definition 2.1). A 2-absorbing ideal I of R is said to be a *strongly 2-absorbing ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in I$ or $yz \in I$ or $xz \in I$ (Definition 3.1). Moreover, a submodule N of an R-module M is said to be *strongly 2-absorbing primary* (resp., *strongly 2-absorbing*) if $(N :_R M)$ is a strongly 2-absorbing primary (resp., strongly 2-absorbing) ideal of R (Definition 2.1 and 3.1).

Let R be an integral domain with quotient field K. In Section 2 of this paper, among other results, we prove that if I is a strongly primary ideal of R, then I is a strongly 2-absorbing primary ideal of R (Proposition 2.2). Example 2.3, shows that the converse of Proposition 2.2 is not true in general. In Theorem 2.6, we provide a useful characterization for strongly 2-absorbing primary ideals of R, where R is a rooty domain. In Theorem 2.8, we show that for a strongly 2-absorbing primary ideal I of R:

(a) If J and H are radical ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$;

(b) If J and I are prime ideals of R, then J and I are comparable.

Furthermore, it is shown that if P and Q are non-zero strongly primary ideals of R, then $P \cap Q$ is a strongly 2-absorbing primary ideal of R (Theorem 2.10).

In Section 3 of this paper, among other results, we prove that if I is a strongly prime ideal of R, then I is a strongly 2-absorbing ideal of R (Proposition 3.2). But the converse of Proposition 3.2 is not true in general (Proposition 3.5, Example 3.6, and Example 3.7). In Theorem 3.3, we provide a useful characterization for a strongly 2-absorbing ideal of R. Also, we see that if P and Q are non-zero strongly prime ideals of R, then $P \cap Q$ is a strongly 2-absorbing ideal of R (Theorem 3.16). Finally, it is proved that if M is a Noetherian R-module, then M contains a finite number of minimal strongly 2-absorbing submodules (Theorem 3.29).

2 Strongly 2-absorbing Primary Ideals and Submodules

Definition 2.1. Let R be an integral domain with quotient field K. We say that an ideal I of R is a *strongly 2-absorbing primary ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in I$ or $(yz)^n \in I$ or $(xz)^m \in I$ for some $n, m \in \mathbb{N}$. Also, we say that a submodule N of an R-module M is a *strongly 2-absorbing primary* if, $(N :_R M)$ is a strongly 2-absorbing primary ideal of R.

Proposition 2.2. Let R be an integral domain with quotient field K and let I be a strongly primary ideal of R. Then I is a strongly 2-absorbing primary ideal of R.

Proof. Let $xyz \in I$ for some $x, y, z \in K$. Then by assumption, either $xy \in I$ or $z^n \in I$ for some $n \geq 1$. If $xy \in I$, then we are done. If $z^n \in I$, then $(zx)^n(zy)^n = (zxy)^n z^n \in I$. Thus again by assumption, either $(zx)^n \in I$ or $(yz)^{ns} \in I$ for some $s \geq 1$ as desired. $\square$

Recall that a *discrete valuation ring (DVR)* is a principal ideal domain (PID) with exactly one non-zero maximal ideal.

The following example shows that the converse of Proposition 2.2 is not true in general.

Example 2.3. Let K be a field of characteristic 2 and assume that $I = (X^2)K[[X^2, X^3]]$, where $K[[X^2, X^3]]$ is the ring of formal power series over the indeterminates X^2 and X^3. By considering the elements X^3 and $1/X$ in the quotient field $K((X))$, it is clear that I is not strongly primary. Now, let $fgh \in I$, where $f, g, h \in K((X))$. Then there exist units u, v, w of the DVR $K[[X]]$ and integers α, β, γ for which $f = uX^\alpha$, $g = vX^\beta$, and $h = wX^\gamma$. Then $fgh \in I$ implies that $\alpha + \beta + \gamma \geq 2$; hence, $(\beta + \gamma) + (\alpha + \gamma) + (\alpha + \beta) \geq 4$. Now, if one of $\beta + \gamma$ or $\alpha + \gamma$ is at least one, then correspondingly either $(gh)^2 \in I$ or $(fh)^2 \in I$. On the other hand, if both $\beta + \gamma$ and $\alpha + \gamma$ are at most 0, then $\alpha + \beta \geq 4$. However, this would mean that $fg \in I$. Therefore, I must be a strongly 2-absorbing primary ideal of $K[[X^2, X^3]]$.

Remark 2.4. Clearly, every proper strongly 2-absorbing primary ideal of R is a 2-absorbing primary ideal of R. But the converse is not true in general. Because for example, if we consider the integral domain $\mathbb{Z}$, then $K = \mathbb{Q}$ and $(35/3)(15/2)(4/7) = 50 \in 50\mathbb{Z}$ implies that $50\mathbb{Z}$ is not a strongly 2-absorbing primary ideal of $\mathbb{Z}$. But $50\mathbb{Z}$ is a 2-absorbing primary ideal of $\mathbb{Z}$ by [5, Example 2.17].

Notation 2.5. For a subset S of R, we define $E(S)$ by

$$E(S) = \{x \in K : \ x^n \notin S \ for \ each \ n \geq 1\}.$$

Let R be an integral domain with quotient field K. An ideal I of R is called *strongly radical* if whenever $x \in K$ satisfies $x^n \in I$ for some $n \geq 1$, then $x \in I$ [1].

Following [9], an integral domain R is called *rooty* if each radical ideal of R is strongly radical (equivalently, each prime ideal of R is strongly radical. Thus valuation domains are rooty domains [2]).

Theorem 2.6. Let R be an integral domain with quotient field K and let I be an ideal of R. Consider the following statements:

(a) I is a 2-absorbing primary ideal of R and for each $x, y \in K$ with $xy \notin I$ we have $x^{-1}I \cap E(I) = \emptyset$ or $y^{-1}I \cap E(I) = \emptyset$.

(b) I is a strongly 2-absorbing primary ideal of R.

Then $(a) \Rightarrow (b)$. Moreover, if $K \setminus E(I)$ is closed under addition (in particular, if R is rooty), then $(b) \Rightarrow (a)$.

Proof. $(a) \Rightarrow (b)$ Let $xyz \in I$ for some $x, y, z \in K$ and $xy \notin I$. Then by part (a), either $x^{-1}I \cap E(I) = \emptyset$ or $y^{-1}I \cap E(I) = \emptyset$. If $x^{-1}I \cap E(I) = \emptyset$, then $yz = yzxx^{-1} = (yzx)x^{-1} \in x^{-1}I$ implies that $(yz)^n \in I$ for some $n \geq 1$. Similarly, if $y^{-1}I \cap E(I) = \emptyset$, then we have $(xz)^m \in I$ for some $m \geq 1$, as needed.

$(b) \Rightarrow (a)$ Assume on the contrary that $x, y \in K$ with $xy \notin I$ and $x^{-1}I \cap E(I) \neq \emptyset$ and $y^{-1}I \cap E(I) \neq \emptyset$. Then there exist $a, b \in I$ such that $x^{-1}a \in E(I)$ and $y^{-1}b \in E(I)$. Now as I is a strongly 2-absorbing primary ideal of R, we have $(x)(y)(x^{-1}y^{-1}a) = a \in I$ implies that $(y^{-1}a)^n \in I$ for some $n \geq 1$. In a similar way we have $(x^{-1}b)^m \in I$ for some $m \geq 1$. On the other hand,

$$a + b = (x)(y)(x^{-1}y^{-1}(a + b)) \in I$$

implies that either $xy \in I$ or $(x^{-1}(a + b))^s \in I$ or $(y^{-1}(a + b))^t \in I$. Therefore, as $K \setminus E(I)$ is closed under addition, either $xy \in I$ or $x^{-1}a \notin E(I)$ or $y^{-1}b \notin E(I)$, which is a contradiction. □

Theorem 2.7. Let R be an integral domain with quotient field K and I be an ideal of R. Consider the following:

(a) If $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in I$ or $yz \in \sqrt{I}$ or $xz \in \sqrt{I}$.

(b) If $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in I$ or $(yz)^n \in I$ or $(xz)^m \in I$ for some $n, m \geq 1$ (i.e., I is a strongly 2-absorbing primary ideal of R).

Then $(a) \Rightarrow (b)$. Moreover, if R is a rooty domain, then $(b) \Rightarrow (a)$.

Proof. $(a) \Rightarrow (b)$ This is clear.

$(b) \Rightarrow (a)$ Let $xyz \in I$ for elements $x, y, z \in K$. If $xy \notin I$, then we have either $(yz)^n \in I$ or $(xz)^m \in I$ for some $n, m \geq 1$ by part (b). Since R is a rooty domain, $yz \in \sqrt{I}$ or $xz \in \sqrt{I}$, as needed. □

Theorem 2.8. Let R be an integral domain with quotient field K and let I be a strongly 2-absorbing primary ideal of R. Then we have the following:

(a) If J and H are radical ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$.

(b) If J and I are prime ideals of R, then J and I are comparable.

Proof. (a) Suppose that J and H are radical ideals of R such that $JH \not\subseteq I$. Then there exist $a \in J$ and $b \in H$ such that $ab \in JH \setminus I$. Let $x, y \in I$. Then $(xy/ab)(a/x)(b/1) \in I$ implies that either $(a/x)(b/1) \in I$ or $((xy/ab)(a/x))^n \in I$ or $((xy/ab)(b/1))^m \in I$ for some $n, m \geq 1$. Thus either $x(ab/x) \in xR \subseteq I$ or $(b(y/b))^n \in b^n I \subseteq b^n R \subseteq H$ or $(a(xy/a))^m \in a^m I \subseteq a^m R \subseteq J$. Hence, either $ab \in I$ or $y^n \in H$ or $(xy)^m \in J$. Since $ab \notin I$, we have either $y \in \sqrt{H} = H$ or $xy \in \sqrt{J} = J$. Therefore, $xy \in J \cup H$. This implies that $I^2 \subseteq J \cup H$, as desired.

(b) The result follows from the fact that $J^2 \subseteq I$ or $I^2 \subseteq J$ by part (a). □

Corollary 2.9. Let R be an integral domain with quotient field K and Q be a maximal ideal of R. If Q is a strongly 2-absorbing primary ideal of R, then R is a local ring with maximal ideal Q.

Proof. It follows from Theorem 2.8. □

Theorem 2.10. Let R be an integral domain with quotient field K and let P and Q be nonzero strongly primary ideals of R. Then $P \cap Q$ is a strongly 2-absorbing primary ideal of R. In particular, if N_1, N_2 are two strongly primary submodules of an R-module M, then $N_1 \cap N_2$ is a strongly 2-absorbing primary submodule of M.

Proof. Suppose $(xy)z \in P \cap Q$ and $x, y, z \in K$. Then $(xy)z \in P$ and $(xy)z \in Q$. Since P is strongly primary, so either $xy \in P$ or $z^n \in P$ for some $n \geq 1$. If $xy \in P$, then either $x \in P$ or $y^m \in P$ for some $m \geq 1$. Similarly, $x \in Q$ or $y^t \in Q$ or $z^s \in Q$ for some $s, t \geq 1$. First assume that $x \in P$ and $x \in Q$. Then $(xy)y^{-1} = x \in P$ implies that $xy \in P$ or $(y^{-1})^h \in P$ for some $h \geq 1$. Similarly, $xy \in Q$ or $(y^{-1})^g \in Q$ for some $g \geq 1$. If $(y^{-1})^g \in Q \subseteq R$ or $(y^{-1})^h \in P \subseteq R$, then $(xz)^h = (xyz)^h (y^{-1})^h \in P \cap Q$ or $(xz)^g = (xyz)^g (y^{-1})^g \in P \cap Q$ by definition of an ideal. Otherwise, $xy \in P \cap Q$ as requested. If the statements above lead to different elements in P and Q, we still have that the intersection is strongly 2-absorbing primary. For example, if $z^n \in P$ and $y^t \in Q$, then clearly $(zy)^{nt} \in P$ and $(zy)^{nt} \in Q$ by definition of an ideal, thus $(zy)^{nt} \in P \cap Q$. Now the last statement follows from the fact that $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$. $\square$

Proposition 2.11. Let R be an integral domain with quotient field K and S be a multiplicatively closed subset of R. If I is a strongly 2-absorbing primary ideal of R such that $S \cap I = \emptyset$, then $S^{-1}I$ is a strongly 2-absorbing primary ideal of $S^{-1}R$.

Proof. Assume that $a, b, c \in K$ such that $abc \in S^{-1}I$. Then there exists $s \in S$ such that $(sa)(b)c = sabc \in I$. Since I is a strongly 2-absorbing primary ideal of R, this implies that either $(sa)c \in I$ or $((b)c)^n = (bc)^n \in I$ or $((sa)(b))^m = (sab)^m \in I$ for some $n, m \geq 1$. Thus $ac = (sa)c/s \in s^{-1}I$ or $(bc)^n = ((b)c/1)^n \in s^{-1}I$ or $(ab)^m = ((sa)(b)/s)^m \in s^{-1}I$, as needed. $\square$

Corollary 2.12. Let R be an integral domain with quotient field K, N be submodule of a finitely generated R-module M, and let S be a multiplicatively closed subset of R. If N is a strongly 2-absorbing primary submodule and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a strongly 2-absorbing primary $S^{-1}R$-submodule of $S^{-1}M$.

Proof. As M is finitely generated, $(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ by [10, Lemma 9.12]. Now the result follows from Proposition 2.11. $\square$

Proposition 2.13. Let R be an integral domain with quotient field K and M be an R-module. Let N be a strongly 2-absorbing primary submodule of M. Then we have the following.

(a) If $r \in K$ such that $r^{-1} \in R$, then $(N :_M r)$ is a strongly 2-absorbing primary submodule of M.

(b) If $f : M \to \acute{M}$ is a monomorphism of R-modules, then N is a strongly 2-absorbing primary submodule of M if and only if $f(N)$ is a strongly 2-absorbing primary submodule of $f(M)$.

Proof. (a) Let $xyz \in ((N :_M r) :_R M)$ for some $x, y, z \in K$. Then $rxyz \in (N :_R M)$. Thus as N is a strongly 2-absorbing primary submodule, either $rxy \in (N :_R M)$ or $(rxz)^n \in (N :_R M)$ or $(yz)^m \in (N :_R M)$ for some $n, m \geq 1$. Hence either $xy = r^{-1}rxy \in r^{-1}(N :_R M) \subseteq$

$(N :_R M)$ or $(xz)^n = (r^{-1}rxz)^n \in r^{-1}(N :_R M) \subseteq (N :_R M)$ or $(yz)^m \in (N :_R M)$, as needed.

(b) This follows from the fact that $(N :_R M) = (f(N) :_R f(M))$. $\square$

3 Strongly 2-absorbing Ideals and Submodules

Definition 3.1. Let R be an integral domain with quotient field K. We say that a 2-absorbing ideal I of R is a *strongly 2-absorbing ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in I$ or $yz \in I$ or $xz \in I$. Also, we say that a submodule N of an R-module M is *strongly 2-absorbing* if $(N :_R M)$ is a strongly 2-absorbing ideal of R.

Proposition 3.2. Let R be an integral domain with quotient field K and let I be a strongly prime ideal of R. Then I is a strongly 2-absorbing ideal of R.

Proof. Let $xyz \in I$ for some $x, y, z \in K$. Then by assumption, either $xy \in I$ or $z \in I$. If $xy \in I$, then we are done. If $z \in I$, then $zxyz \in I$. Thus again by assumption, either $zx \in I$ or $yz \in I$ as desired. $\square$

The following theorem is a characterization for a strongly 2-absorbing ideal of R.

Theorem 3.3. Let R be an integral domain with quotient field K and let I be an ideal of R. Then the following statements are equivalent:

(a) I is a strongly 2-absorbing ideal of R;

(b) For each $x, y \in K$ with $xy \notin I$ we have either $x^{-1}I \subseteq I$ or $y^{-1}I \subseteq I$.

Proof. $(a) \Rightarrow (b)$ Assume on the contrary that $x, y \in K$ with $xy \notin I$ and neither $x^{-1}I \not\subseteq I$ nor $y^{-1}I \not\subseteq I$. Then there exist $a, b \in I$ such that $x^{-1}a \notin I$ and $y^{-1}b \notin I$. Now as I is a strongly 2-absorbing ideal of R, we have $(x)(y)(x^{-1}y^{-1}a) = a \in I$ implies that $y^{-1}a \in I$. In the similar way we have $x^{-1}b \in I$. On the other hand,

$$a + b = (x)(y)(x^{-1}y^{-1}(a + b)) \in I$$

implies that either $xy \in I$ or $x^{-1}(a+b) \in I$ or $y^{-1}(a+b) \in I$. Therefore, either $xy \in I$ or $x^{-1}a \in I$ or $y^{-1}b \in I$, a contradiction.

$(b) \Rightarrow (a)$ Let $xyz \in I$ for some $x, y, z \in K$. If $xy \in I$, $xz \in I$, and $yz \in I$, then we are done. So suppose without loss of generality that $xy \notin I$. Then by part (b), either $x^{-1}I \subseteq I$ or $y^{-1}I \subseteq I$. If $x^{-1}I \subseteq I$, then $yz = yzxx^{-1} = (yzx)x^{-1} \in x^{-1}I \subseteq I$. Similarly, if $y^{-1}I \subseteq I$, then we have $xz \in I$, as desired. □

Corollary 3.4. Let R be an integral domain with quotient field K and let I be a strongly 2-absorbing ideal of R. Then for each $x, y \in K$ with $xy \notin I$ we have either $I \subseteq Rx$ or $I \subseteq Ry$.

Proof. Let $x, y \in K$ with $xy \notin I$. Then by Theorem 3.3 $(a) \Rightarrow (b)$, we have either $x^{-1}I \subseteq I$ or $y^{-1}I \subseteq I$. Thus either $I \subseteq Ix \subseteq Rx$ or $I \subseteq Ix \subseteq Ry$. □

Proposition 3.5, Example 3.6, and Example 3.7 show that the converse of Proposition 3.2 is not true in general.

Proposition 3.5. Let R be an integral domain with a prime ideal P such that there exists a discrete valuation overring (V, Q) of R centered at P (that is, $Q \cap R = P$), where $Q = xV$. Suppose that $ux^k \in P$ for all units u of V and natural numbers $k \geq 2$, but there is no unit u of V for which $ux \in P$. Then P is a strongly 2-absorbing ideal of R that is not a strongly prime ideal.

Proof. The fact that P is not a strongly prime ideal of R is immediate from the fact that $x^2 \in P$, but $x \notin P$, by assumption. Now, since P is a prime ideal of R, it is necessarily a 2-absorbing ideal of R. Let y and z be elements of the quotient field of R for which $yz \notin P$. By Theorem 3.3, it suffices to show that either $y^{-1}P \subseteq P$ or $z^{-1}P \subseteq P$. Observe that there exist units u and v of V and integers α and β for which $y = ux^{\alpha}$ and $z = vx^{\beta}$. Since $yz \notin P$, it must be the case that $\alpha + \beta \leq 1$. However, this means that either $\alpha \leq 0$ or $\beta \leq 0$. As such, either $-\alpha + \gamma \geq 2$ or $-\beta + \gamma \geq 2$ for all integers $\gamma \geq 2$, from which it follows that either $y^{-1}P \subseteq P$ or $z^{-1}P \subseteq P$ as needed. □

Example 3.6. If K is a field, then the ideal (X^2, X^3) in $K[[X^2, X^3]]$ the ring of formal power series in the indeterminates X^2 and X^3 over K

is an example of a strongly 2-absorbing prime ideal that is not strongly prime.

Example 3.7. If Q is the maximal ideal of a non-trivial DVR, V, then Q^2 is a strongly 2-absorbing ideal of V that is not a strongly prime ideal, since Q^2 is not even a prime ideal of V.

Proposition 3.8. Let R be an integral domain with quotient field K, I be a strongly 2-absorbing ideal of R, and Q be a prime ideal of R which is properly contained in I. Then I/Q is a strongly 2-absorbing ideal of R/Q.

Proof. Clearly, I/Q is a 2-absorbing ideal of R/Q. Now let $\phi : R \to R/Q$ denote the canonical homomorphism. Suppose that $x_1 = \phi(y_1)/\phi(z_1)$ and $x_2 = \phi(y_2)/\phi(z_2)$ are elements of the quotient field of R/Q such that $x_1x_2 \notin I/Q$. Then $(y_1/z_1)(y_2/z_2) \notin I$. Hence if $a \in I$, we have $(z_1/y_1)a \in I$ or $(z_2/y_2)a \in I$ by using Theorem 3.3. We can assume without loss of generality that $(z_1/y_1)a \in I$. It follows that $(\phi(z_1)/\phi(y_1))\phi(a) \in I/Q$. Thus $x^{-1}(I/Q) \subseteq I/Q$, as needed. □

Remark 3.9. Clearly, every strongly 2-absorbing ideal of R is a 2-absorbing ideal of R. But the converse is not true in general. Because for example, if we consider the integral domain $\mathbb{Z}$, then $K = \mathbb{Q}$ and $(8/15)(3/2)(5/2) = 2 \in 2\mathbb{Z}$ implies that $2\mathbb{Z}$ is not a strongly 2-absorbing ideal of $\mathbb{Z}$. But $2\mathbb{Z}$ is a 2-absorbing ideal of $\mathbb{Z}$.

Definition 3.10. We say that an integral domain R is a *2-absorbing pseudo-valuation domain* if every 2-absorbing ideal of R is a strongly 2-absorbing ideal of R.

Proposition 3.11. Every valuation domain is a 2-absorbing pseudo-valuation domain.

Proof. Let V be a valuation domain, and let I be a 2-absorbing ideal of V. Suppose $xyz \in I$, where $x, y, z \in K$, the quotient field of V. If x, y, and z are in V, we are done. Suppose without loss of generality that $x \notin V$. Since V is a valuation domain, we have $x^{-1} \in V$. Hence $yz = (x^{-1})(xyz) \in I$, as needed. □

Definition 3.12. Let R be an integral domain with quotient field K. We say that a non-zero prime ideal P of R is a *strongly semiprime* if whenever $x^2 \in P$ for element $x \in K$, we have $x \in P$.

Remark 3.13. Let R be an integral domain with quotient field K. Clearly every non-zero strongly prime ideal of R is a strongly semiprime ideal of R. But as we see in the following example the converse is not true in general.

Example 3.14. Consider an integral domain $\mathbb{Z}$. Then $K = \mathbb{Q}$ and $(4/3)(3/2) = 2 \in 2\mathbb{Z}$ implies that $2\mathbb{Z}$ is not a strongly prime ideal of $\mathbb{Z}$. But $2\mathbb{Z}$ is a strongly semiprime ideal of $\mathbb{Z}$.

Proposition 3.15. Let R be an integral domain with quotient field K.

(a) If P is a strongly semiprime and strongly 2-absorbing ideal of R, then P is a strongly prime ideal of R.

(b) If P_1 and P_2 are strongly semiprime ideals of R, then $P_1 \cap P_2$ is a strongly semiprime ideal of R.

Proof. (a) Let P be a strongly semiprime and strongly 2-absorbing ideal of R and let $x \in K \setminus R$. Then as P is strongly semiprime $x^2 \notin P$. Since P is strongly 2-absorbing, this implies that $x^{-1}P \subseteq P$ by Theorem 3.3. Now the result follows from [8, Proposition 1.2].

(b) This is clear. □

Theorem 3.16. Let R be an integral domain with quotient field K and let P and Q be non-zero strongly prime ideals of R. Then $P \cap Q$ is a strongly 2-absorbing ideal of R.

Proof. The proof is similar to that of Theorem 2.10. □

Proposition 3.17. Let R be an integral domain with quotient field K and let I be a strongly 2-absorbing ideal of R. Then we have the following:

(a) $\sqrt{I}$ is a strongly 2-absorbing ideal of R and $x^2 \in I$ for every $x \in \sqrt{I}$.

(b) If S is a multiplicatively closed subset of R such that $S \cap I = \emptyset$, then $S^{-1}I$ is a strongly 2-absorbing ideal of $S^{-1}R$.

Proof. (a) Since I is a strongly 2-absorbing ideal of R, observe that $x^2 \in I$ for every $x \in \sqrt{I}$. Let $x, y, z \in K$ such that $xyz \in \sqrt{I}$. Then $(xyz)^2 = x^2y^2z^2 \in I$. Since I is a strongly 2-absorbing ideal of R, we may assume without loss of generality that $x^2y^2 \in I$. Now since $(xy)^2 = x^2y^2 \in I$, we have $xy \in \sqrt{I}$ as desired.

(b) The proof is similar to that of Proposition 2.11. □

Corollary 3.18. Let R be an integral domain with quotient field K, N be a submodule of a finitely generated R-module M, and let S be a multiplicatively closed subset of R. If N is a strongly 2-absorbing submodule and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a strongly 2-absorbing $S^{-1}R$-submodule of $S^{-1}M$.

Proof. As M is finitely generated, $(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ by [10, Lemma 9.12]. Now the result follows from Proposition 3.17. □

Theorem 3.19. Let R be an integral domain with quotient field K and let I be a strongly 2-absorbing ideal of R. Then we have the following.

(a) If J and H are ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$.

(b) If J and I are prime ideals of R, then J and I are comparable.

Proof. The proof is similar to that of Theorem 2.8. □

Corollary 3.20. Let R be an integral domain with quotient field K and Q be a maximal ideal of R. If Q is a strongly 2-absorbing ideal of R, then R is a local ring with maximal ideal Q.

Proof. This follows from Theorem 3.19 (b). □

Recall that if K is the field of fractions of R, then an intermediate ring in the extension $R \subseteq K$ is called an *overring* of R.

Proposition 3.21. Let R be an integral domain with quotient field K, I be a strongly 2-absorbing ideal of R, and let T be an overring of R. Then IT is a strongly 2-absorbing ideal of T.

Proof. Let $x, y \in K$ and $xy \notin IT$. Then $xy \notin I$. Thus by Theorem 3.3, either $x^{-1}I \subseteq I$ or $y^{-1}I \subseteq I$. Therefore, either $x^{-1}IT \subseteq IT$ or $y^{-1}IT \subseteq IT$. Hence IT is a strongly 2-absorbing ideal of T, again by Theorem 3.3. $\square$

Proposition 3.22. Let R be an integral domain with quotient field K and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a chain of strongly 2-absorbing ideals of R. Then $\sum_{\lambda\in\Lambda} I_\lambda$ is a strongly 2-absorbing ideal of R.

Proof. Suppose that $x, y \in K$ with $xy \notin \sum_{\lambda\in\Lambda} I_\lambda$ and we have $x^{-1}\sum_{\lambda\in\Lambda} I_\lambda \nsubseteq \sum_{\lambda\in\Lambda} I_\lambda$ and $y^{-1}\sum_{\lambda\in\Lambda} I_\lambda \nsubseteq \sum_{\lambda\in\Lambda} I_\lambda$. Then there exist $\alpha, \beta \in \Lambda$ such that $x^{-1}I_\alpha \nsubseteq \sum_{\lambda\in\Lambda} I_\lambda$ and $y^{-1}I_\beta \nsubseteq \sum_{\lambda\in\Lambda} I_\lambda$. Hence, $x^{-1}I_\alpha \nsubseteq I_\alpha$ and $y^{-1}I_\beta \nsubseteq I_\beta$. Thus $y^{-1}I_\alpha \subseteq I_\alpha$ and $x^{-1}I_\beta \subseteq I_\beta$. By assumption, $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$. This implies that $x^{-1}I_\alpha \subseteq x^{-1}I_\beta \subseteq I_\beta \subseteq \sum_{\lambda\in\Lambda} I_\lambda$ or $y^{-1}I_\beta \subseteq y^{-1}I_\alpha \subseteq I_\alpha \subseteq \sum_{\lambda\in\Lambda} I_\lambda$. This is a contradiction. Thus by Theorem 3.3, $\sum_{\lambda\in\Lambda} I_\lambda$ is a strongly 2-absorbing ideal of R. $\square$

Recall that a *chained ring* is any ring whose set of ideals is totally ordered by inclusion.

Corollary 3.23. If R is a chained ring and contains a strongly 2-absorbing ideal, then R contains a unique largest strongly 2-absorbing ideal.

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.22. $\square$

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [6].

Corollary 3.24. Let R be an integral domain which is a chained ring with quotient field K and M be a faithful finitely generated multiplication R-module. If $\{N_i\}_{i\in I}$ is a family of strongly 2-absorbing submodules of M, then $\sum_{i\in I} N_i$ is a strongly 2-absorbing submodule of M.

Proof. This follows from Proposition 3.22 and the fact that

$$(\sum_{i\in I}(N_i :_R M)M :_R M) = \sum_{i\in I}(N_i :_R M)$$

by [7, Theorem 3.1]. $\square$

Proposition 3.25. Let R be an integral domain with quotient field K and M be an R-module. Then we have the following:

(a) If N is a strongly 2-absorbing submodule of M and $r \in K$ such that $r^{-1} \in R$, then $(N :_M r)$ is a strongly 2-absorbing submodule of M.

(b) If $f : M \to \acute{M}$ is a monomorphism of R-modules, then N is a strongly 2-absorbing submodule of M if and only if $f(N)$ is a strongly 2-absorbing submodule of $f(M)$.

(c) If N_1, N_2 are two submodules of M with $(N_1 :_R M)$ and $(N_2 :_R M)$ strongly prime ideals of R, then $N_1 \cap N_2$ is a strongly 2-absorbing submodule of M.

Proof. (a) The proof is similar to that of Proposition 2.13 (a).

(b) The proof is similar to that of Proposition 2.13 (b).

(c) Since $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$, the result follows from Proposition 3.16. □

Proposition 3.26. Let R be an integral domain with quotient field K, M be an R-module, and let $\{K_i\}_{i\in I}$ be a chain of strongly 2-absorbing submodules of M. Then $\cap_{i\in I} K_i$ is a strongly 2-absorbing submodule of M.

Proof. Let $a, b, c \in K$ and $abc \in (\cap_{i\in I} K_i :_R M) = \cap_{i\in I}(K_i :_R M)$. Assume to the contrary that $ab \notin \cap_{i\in I}(K_i :_R M)$, $bc \notin \cap_{i\in I}(K_i :_R M)$, and $ac \notin \cap_{i\in I}(K_i :_R M)$. Then there are $m, n, t \in I$ where $ab \notin (K_n :_R M)$, $bc \notin (K_m :_R M)$, and $ac \notin (K_t :_R M)$. Since $\{K_i\}_{i\in I}$ is a chain, we can assume without loss of generality that $K_m \subseteq K_n \subseteq K_t$. Then

$$(K_m :_R M) \subseteq (K_n :_R M) \subseteq (K_t :_R M).$$

As $abc \in (K_m :_R M)$, we have $ab \in (K_m :_R M)$ or $ac \in (K_m :_R M)$ or $bc \in (K_m :_R M)$. In any case, we have a contradiction. □

Definition 3.27. Let R be an integral domain with quotient field K. We say that a strongly 2-absorbing submodule N of an R-module M is a *minimal strongly 2-absorbing submodule* of a submodule H of M, if $H \subseteq N$ and there does not exist a strongly 2-absorbing submodule T of M such that $H \subset T \subset N$.

It should be noted that a minimal strongly 2-absorbing submodule of M means that a minimal strongly 2-absorbing submodule of the submodule 0 of M.

Lemma 3.28. Let R be an integral domain with quotient field K and let M be an R-module. Then every strongly 2-absorbing submodule of M contains a minimal strongly 2-absorbing submodule of M.

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.26. □

Theorem 3.29. Let R be an integral domain with quotient field K and let M be a Noetherian R-module. Then M contains a finite number of minimal strongly 2-absorbing submodules.

Proof. Suppose that the result is false. Let Σ denote the collection of all proper submodules N of M such that the module M/N has an infinite number of minimal strongly 2-absorbing submodules. Since $0 \in \Sigma$, we have $\Sigma \neq \emptyset$. Therefore Σ has a maximal member T, since M is a Noetherian R-module. Clearly, T is not a strongly 2-absorbing submodule. Therefore, there exist $a, b, c \in K$ such that $abc(M/T) = 0$ but $ab(M/T) \neq 0$, $ac(M/T) \neq 0$, and $bc(M/T) \neq 0$. The maximality of T implies that $M/(T + abM)$, $M/(T + acM)$, and $M/(T + bcM)$ have only finitely many minimal strongly 2-absorbing submodules. Suppose P/T is a minimal strongly 2-absorbing submodule of M/T. So $abcM \subseteq T \subseteq P$, which implies that $abM \subseteq P$ or $acM \subseteq P$ or $bcM \subseteq P$. Thus $P/(T + abM)$ is a minimal strongly 2-absorbing submodule of $M/(T + abM)$ or $P/(T + bcM)$ is a minimal strongly 2-absorbing submodule of $M/(T + bcM)$ or $P/(T + acM)$ is a minimal strongly 2-absorbing submodule of $M/(T + acM)$. Thus, there are only a finite number of possibilities for the submodule M/T. This is a contradiction. □

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Habibollah Ansari-Toroghy
Professor of Mathematics
Department of Pure Mathematics, Faculty of Mathematical Sciences
University of Guilan
Rasht, Iran
E-mail: ansari@guilan.ac.ir

Faranak Farshadifar
Assistant Professor of Mathematics
Department of Mathematics
Farhangian University
Tehran, Iran
E-mail: f.farshadifar@cfu.ac.ir

Sepideh Maleki-Roudposhti
Ph.D student of Mathematics
Department of Pure Mathematics, Faculty of Mathematical Sciences
University of Guilan
Rasht, Iran
E-mail: Sepidehmaleki.r@gmail.com