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A Good Approximate Solution for Liénard Equation in a Large Interval Using Block Pulse Functions

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Abstract. In this paper, the Block pulse functions (BPFs) and their operational matrices of integration and differentiation are used to solve Liénard equation in a large interval. This method converts the equation to a system of nonlinear algebraic equations whose solution is the coefficients of Block pulse expansion of the solution of the Liénard equation. Moreover, this method is examined by comparing the results with the results obtained by the Adomian decomposition method (ADM) and the Variational iteration method (VIM).

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1. Introduction

A variety of problems in science and engineering can be modeled by nonlinear ordinary differential equations (ODEs). A special kind of the nonlinear ordinary differential equations is the Liénard equation:

$$y'' + f(y)y' + g(y) = h(t), \tag{1}$$

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which is not only regarded as a generalization of the damped pendulum equation or a damped spring-mass system (where $f(y)y'$ is the damping force, $g(y)$ is the restoring force, and $h(x)$ is the external force), but also is used as a nonlinear model in many physically significant fields by taking different choices for $f(y)$, $g(y)$ and $h(t)$. For example, the choices $f(y) = \epsilon(y^2 - 1)$, $g(y) = y$, and $h(t) = 0$ lead (1) to the Van der Pol equation served as a nonlinear model for electronic oscillation [4, 17, 7]. Therefore, studying (1) is of physical significance and in the general case, it is commonly believed that obtaining its analytical solution is very difficult [7]. In this study we consider the special form of (1) as:

$$y''(t) + \ell y(t) + \mu y^3(t) + \nu y^5(t) = 0, \tag{2}$$

where ℓ, μ and ν are real coefficients.

The Liénard equation (2), has been considered by many authors. Kong [8] discussed the explicit exact solutions for (2) and its applications, Feng [3] obtained an explicit exact solutions to this equation, and applied the results to find some explicit exact solitary wave solutions to the nonlinear Schrödinger equation and the Pochhammer−Chree equation, Jianhua Su et al. [16] studied explicit exact solutions for the Liénard equation and its applications and also obtained more explicit exact solitary wave solutions for the generalized Pochhammer−Chree equation given by seeking qualitatively the homoclinic and heteroclinic orbits for this class of Liénard equation, Dogan Kaya et al. [7] applied the Adomian decomposition method (ADM) to obtain explicit exact and numerical solutions to the this equation, Matinfar et al. implemented the variational iteration method (VIM) [11] and the variational homotopy perturbation method (VHPM) [12] to find approximate solutions to this equation. In all mentioned methods, solution of problem (2), have been studied in small interval [0,1]. In this study we will investigate solution of problem (2), in a large interval by Block pulse functions (BPFs).

Block pulse functions (BPFs) are easy to use and this simplicity allows one to use them for solving integral and differential equations [5, 1]. In this study, we will apply the BPFs to find approximate solutions to the Liénard equation for a given nonlinearity. In the proposed method both of the operational matrices of integration and differentiation of BPFs are

used to solve the Liénard equation. The method is based on reducing the equation to a system of nonlinear algebraic equation by expanding the solution as BPFs with unknown coefficients. The proposed method is simple to understand and easy to implement using suitable computer softwares. Moreover, this method is examined by comparing the results with the ADM and VIM which this equation have been solved with them. Numerical results show the efficiency of the proposed method of this paper.
The structure of this paper is as follows: In section 2 we introduce Block pulse functions and their properties [15, 6]. In section 3 the rate of convergence and error analysis of BPFs expansion are presented. In section 4 we implement the proposed method for the Liénard equation. In section 5 the error investigation of proposed method is investigated. In section 6 we apply the proposed method in some examples in order to show the accuracy and efficiency of the method. Finally a conclusion is draw in section 7.

2. Block Pulse Functions (BPFs)

BPFs have been studied by many authors and also were applied for solving different problms (for example see [1, 13, 9, 14] and references therein). Here, we present a brief review of BPFs and their properties:

Definition 2.1. *An m-set of BPFs is defined over the interval $[0, T)$ as:*

$$b_i(t) = \begin{cases} 1, & \frac{iT}{m} \leqslant t < \frac{(i+1)T}{m}, \\ 0, & otherwise, \end{cases} \tag{3}$$

for $i = 0, 1, 2, \ldots, (m-1)$.
The most important properties of BPFs are disjointness, orthogonality and completeness.

- *Disjointness: This property can be clearly obtained from the definition of BPFs:*

$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \tag{4}$$

- *Orthogonality: It is clear that*

$$\int_0^T b_i(\tau)b_j(\tau)d\tau = \frac{T}{m}\delta_{ij}. \tag{5}$$

where δ_{ij} is the Kroneker delta.

- *Completeness: For every function $f \in L^2[0,T)$, the sequence $\{b_i(t)\}$ is complete if $\int b_i f = 0$ results in $f = 0$ almost everywhere. Because of completeness of $\{b_i(t)\}$, Parseval's identity holds, i.e. we have $\int_0^T [f(t)]^2 dt = \sum_0^\infty f_i^2 \|b_i(t)\|^2$, for every real bounded function $f \in L^2[0,T)$ and:*

$$f_i = \frac{m}{T}\int_0^T b_i(\tau)f(\tau)d\tau = \frac{m}{T}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} b_i(\tau)f(\tau)d\tau. \tag{6}$$

2.1 Vector Forms

Consider the first m-terms of BPFs and for the sake of simplicity write them as m-vector:

$$B_m(t) = [b_0(t), b_1(t), \dots, b_i(t), \dots, b_{m-1}(t)]^T, \quad t \in [0,T). \tag{7}$$

The above representation and the disjointness property result [1]:

$$B_m(t)B_m^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{m-1}(t) \end{pmatrix}, \tag{8}$$

$$B_m^T(t)B_m(t) = 1, \tag{9}$$

$$B_m(t)B_m^T(t)V = \tilde{V}B_m(t), \tag{10}$$

where V is an m-vector and $\tilde{V} = diag(V)$.

2.2 Function Approximation

Any absolutely integrable function $f(t)$ defined over $[0, T)$ can be expanded in BPFs as:

$$f(t) = \sum_{i=0}^{\infty} f_i b_i(t), \tag{11}$$

where f_i is obtained in (6).
If the infinite series in (11) is truncated, then (11) can be written as

$$f(t) \simeq \sum_{i=0}^{m-1} f_i b_i(t) = F_m^T B_m(t), \tag{12}$$

where $F_m = [f_0, f_1, \ldots, f_{m-1}]^T$ and $B_m(t)$ is defined in (7).
Also the block pulse coefficients f_i are obtained as (6), such that the mean square error between $f(t)$ and its block pulse expansion (12) in the interval of $t \in [0, T)$ is minimal:

$$\epsilon = \frac{1}{T} \int_0^T \left(f(t) - \sum_{i=0}^{m-1} f_i b_i(t) \right)^2 dt.$$

2.3 The Operational Matrices

Chen and Hsiao [2] introduced the concept of the operational matrix in 1975, and Kilicman and Al Zhour [10] investigated the generalized integral operational matrix, that is, the integral of the matrix $B_m(t)$ defined in (7), which can be approximated by:

$$\int_0^t B_m(\tau) d\tau \simeq P B_m(t), \tag{13}$$

where P is the $m \times m$ operational matrix of one-time integration of $B_m(t)$. Moreover, we can compute the generalized operational matrices P^n of n-times integration of $B_m(t)$ as:

$$\underbrace{\int_0^t \cdots \int_0^t}_{n-times} B_m(\tau)(d\tau)^n \simeq P^n B_m(t). \tag{14}$$

In [10] it is shown that P^n has the following form:

$$P^n = \left(\frac{T}{m}\right)^n \frac{1}{(n+1)!} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \dots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

where $\xi_i = (i+1)^{n+1} - 2i^{n+1} + (i-1)^{n+1}$.

Also the generalized BPFs operational matrices D^n for differentiation can be derived by inverting the P^n matrices, which has the following form [10]:

$$D^n = (n+1)! \left(\frac{m}{T}\right)^n \begin{pmatrix} 1 & d_1 & d_2 & \dots & d_{m-1} \\ 0 & 1 & d_1 & \dots & d_{m-2} \\ 0 & 0 & 1 & \dots & d_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

where $d_i = -\sum_{j=1}^{i} \xi_j d_{i-j}$ for all $i = 1,\ 2, \dots,\ m-1$, and $d_0 = 1$.

3. The Rate of Convergence and Error Analysis of BPFs Expansion

In this section we review some materials about the rate of convergence and error analysis of the BPFs expansion of a continuous function.

Definition 3.1. *Suppose $f_m = F_m^T B_m(t)$ be the expansion of $f(t)$ by BPFs, then the corresponding error is defined as follows:*

$$e_m(t) = f_m(t) - f(t). \tag{17}$$

Now we will prove the following convergence theorem.

Theorem 3.2. *Suppose that $f(t)$ satisfies a Lipschitz condition on $[0, T]$, that is:*

$$\exists M > 0,\ \forall \zeta,\ \eta \in [o, T] : |f(\zeta) - f(\eta)| \leqslant M|\zeta - \eta|. \tag{18}$$

Then the BPFs expansion will be convergence in the sense that $e_m(t)$ *toward zero as* m *toward infinity. Moreover, the convergence is of order one, that is:*

$$\|e_m(t)\| = \mathcal{O}(\frac{1}{m}). \tag{19}$$

Proof. By defining the error between $f(t)$ and its BPFs expansion over every subinterval I_i as:

$$e_i(t) = f_i b_i(t) - f(t), \qquad t \in I_i \ (i = 0, 1, \dots, m-1),$$

where $I_i = [\frac{iT}{m}, \frac{(i+1)T}{m})$, we have :

$$\|e_i(t)\|^2 = \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} e_i(t)^2 dt = \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \left(f_i b_i(t) - f(t)\right)^2 dt$$

$$= \frac{T}{m} \left(f_i - f(\eta_i)\right)^2, \ \eta_i \in I_i, \tag{20}$$

where we used mean value theorem for integral.
From (6) and the mean value theorem, we have:

$$f_i = \frac{m}{T} \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} f(t)dt = \frac{m}{T}\frac{T}{m} f(\zeta_i) = f(\zeta_i), \qquad \zeta_i \in I_i. \tag{21}$$

By substituting (21) in (3.) we have:

$$\|e_i(t)\|^2 = \frac{T}{m} \left(f(\zeta_i) - f(\eta_i)\right)^2 \leqslant \frac{TM^2}{m} |\zeta_i - \eta_i|^2 \leqslant \frac{T^3 M^2}{m^3}. \tag{22}$$

This leads to:

$$\|e_m(t)\|^2 = \int_0^T e_m(t)^2 dt = \int_0^T \left(\sum_{i=0}^{m-1} e_i(t)\right)^2 dt$$

$$= \int_0^T \left(\sum_{i=0}^{m-1} e_i(t)^2\right) dt + 2 \sum_{i \leqslant j} \int_0^T e_i(t) e_j(t) dt. \tag{23}$$

Since for $i \neq j$, $I_i \cap I_j = \emptyset$, then

$$\|e_m(t)\|^2 = \int_0^T \left(\sum_{i=0}^{m-1} e_i(t)^2 \right) dt = \sum_{i=0}^{m-1} \|e_i\|^2. \tag{24}$$

Substituting (22) into (24), we have:

$$\|e_m(t)\|^2 \leqslant \frac{T^3 M^2}{m^2}, \tag{25}$$

or, in other words, $\|e_m(t)\| = \mathcal{O}(\frac{1}{m})$. This completes the proof. □

Corollary 3.3. *Let $f_m(t)$ be the expansion of $f(t)$ by BPFs and $e_m(t)$ be the corresponding error which is defined in (17), then we have:*

$$\|e_m(t)\| \leqslant \frac{MT\sqrt{T}}{m}. \tag{26}$$

proof. It is an immediate consequence of Theorem 3.2. □

4. Implementation of the Numerical Method

In this section, the BPFs expansion together with their operational matrices of integration and differentiation are used to obtain numerical solutions of equation(2). Let us consider the Liénard equation (2) with the initial conditions:

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{27}$$

where α and β are fixed constants.
For solving this equation we assume:

$$y''(t) = K_m^T B_m(t), \tag{28}$$

where K_m^T is an unknown vector and $B_m(t)$ is the vector defined in (6). By two-times integration of (28) with respect to t and using the initial conditions (27), we have:

$$y'(t) = K_m^T P B_m(t) + \beta = K_m^T P B_m(t) + [\beta, \beta, \ldots, \beta] B_m(t), \tag{29}$$

and

$$y(t) = K_m^T P^2 B_m(t) + [\beta, \beta, \ldots, \beta] P B_m(t) + [\alpha, \alpha, \ldots, \alpha] B_m(t). \quad (30)$$

Equation (30) can be written as:

$$\begin{aligned} y(t) &= K_m^T P^2 B_m(t) + [\beta, \ldots, \beta] P B_m(t) + [\alpha, \ldots, \alpha] B_m(t) \\ &= [a_1, a_2, \ldots, a_m] B_m(t). \end{aligned} \quad (31)$$

Now from (4) we obtain:

$$[y(t)]^3 = [a_1^3, a_2^3, \ldots, a_m^3] B_m(t), \quad (32)$$

and

$$[y(t)]^5 = [a_1^5, a_2^5, \ldots, a_m^5] B_m(t). \quad (33)$$

Finally substituting (28), (4.), (32) and (33) into (2) and considering (4.) we have:

$$\left[K_m^T + \ell[a_1, a_2, \ldots, a_m] + \mu[a_1^3, a_2^3, \ldots, a_m^3] + \nu[a_1^5, a_2^5, \ldots, a_m^5]\right] B_m(t) = 0, \quad (34)$$

where

$$K_m^T = [[a_1, a_2, \ldots, a_m] - [\beta, \ldots, \beta] P - [\alpha, \ldots, \alpha]] D^2.$$

Equation(34) is a nonlinear system of algebraic equations for the unknown vector $[a_1, a_2, \ldots, a_m]$, and can be solved by Newton's iteration method. Finally, $y(t)$, as the solution of (2), is $y(t) = [a_1, a_2, \ldots, a_m] B_m(t)$.

5. Error Investigation of the Proposed Method

In this section we investigate error of the proposed method.

Lemma 5.1. *Suppose $f(t)$ is approximated by BPFs on interval $[0, T)$ as:*

$$f_m(t) = \sum_{i=0}^{m-1} f_i b_i(t),$$

and moreover suppose by solving some problems we have found $\hat{f}_i$ *as an approximation of* f_i *and put:*

$$\hat{f}_m(t) = \sum_{i=0}^{m-1} \hat{f}_i b_i(t),$$

Then for each $t \in [0, T)$ we have:

$$\|\hat{f}_m(t) - f_m(t)\| \leqslant \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

Proof. We have:

$$\begin{aligned}
\|\hat{f}_m(t) - f_m(t)\| &= \left(\int_0^T \left(\sum_{i=0}^{m-1} \left(\hat{f}_i - f_i\right) b_i(t)\right)^2 dt\right)^{\frac{1}{2}} \\
&= \sum_{i=0}^{m-1} |\hat{f}_i - f_i| \left(\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} b_i(t)^2 dt\right)^{\frac{1}{2}} \\
&= \sum_{i=0}^{m-1} |\hat{f}_i - f_i| \sqrt{\frac{T}{m}} \leqslant \sqrt{\frac{T}{m}} \sum_{i=0}^{m-1} \|\hat{f}_m(t) - f(t)\|_\infty \\
&= \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.
\end{aligned}$$

This completes the proof. □

Corollary 5.2. *Let by solving some problems we obtain* $\hat{f}_m(t)$ *as the approximation of* $f(t)$*. Then we have:*

$$\|\hat{f}_m(t) - f(t)\| \leqslant \frac{MT\sqrt{T}}{m} + \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

Proof. For every $t \in [0, T)$ we have:

$$\begin{aligned}
\|\hat{f}_m(t) - f(t)\| &= \|\hat{f}_m(t) - f(t) - f_m(t) + f_m(t)\| \\
&\leqslant \|f_m(t) - f(t)\| + \|\hat{f}_m(t) - f_m(t)\|.
\end{aligned}$$

Now from (26) and Lemma 5.1, we have:

$$\|\hat{f}_m(t) - f(t)\| \leqslant \frac{MT\sqrt{T}}{m} + \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

This completes the proof. □

6. Test Problems

In this section we investigate two important cases of Liénard equation to show the reliability of the proposed method. Moreover, the obtained results are compared with the results that have been obtained by the ADM and VIM.

Example 6.1. In this example, we consider the Liénard equation (2), with the initial conditions [7, 11, 12]:

$$y(0) = \sqrt{\frac{K}{2+D}}, \qquad y'(0) = 0,$$

where

$$K = 4\sqrt{\frac{3\ell}{3\mu^2 - 16\nu\ell}}, \qquad D = -1 + \frac{\sqrt{3}\mu}{\sqrt{3\mu^2 - 16\nu\ell}}.$$

The exact solution of this problem is:

$$y(t) = \sqrt{\frac{K sech^2(t\sqrt{-\ell})}{2 + D sech^2(t\sqrt{-\ell})}}.$$

In order to show the efficiency of the proposed method and comparison with others numerical methods, this problem is now solved for $\ell = -1$, $\mu = 4$ and $\nu = 3$. For some selected values of m, numerical solutions are shown in Fig 1.

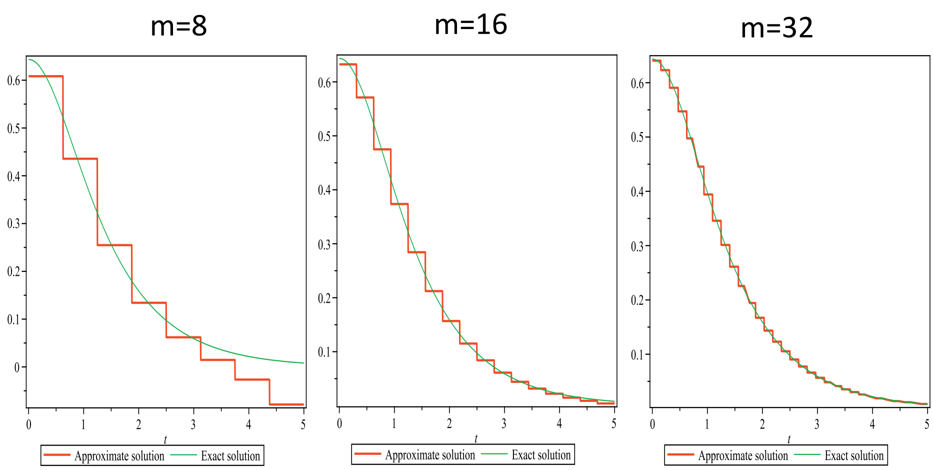

Figure 1: Approximate solutions of example 1, for some values of m.

A comparison between the proposed method for $m = 128$ with 3-terms approximate solution ($\phi_3(t)$) of the ADM [7], 3-iterations ($\psi_3(t)$) of the VIM and the exact solution are performed in Fig 2 (left hand side). Also, to see how much increasing number of terms and orders in the ADM and VIM can be improved the solution of this problem, a comparison between the proposed method for $m = 128$, with $\phi_6(t)$ of the ADM, $\psi_6(t)$ of the VIM and the exact solution are performed in Fig 2 (right hand side).

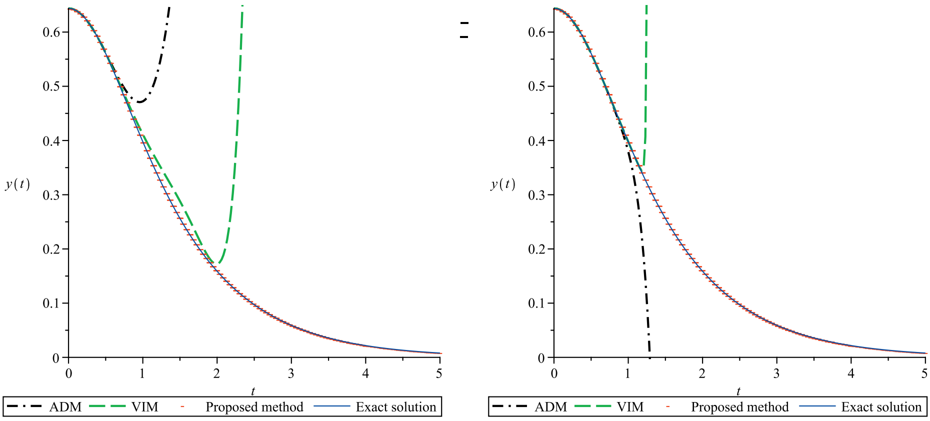

Figure 2: Numerical solutions of example 1.

From Fig 2, it is concluded that increasing number of terms and iterations for the ADM and VIM can not improve the solution of this problem in a large domain. In contrast by applying more numbers of basic block pulse functions (i.e. increasing m) a good approximate solution for this problem can be obtained in a large domain.

Example 6.2. Let us consider the equation (2) with the initial conditions [7, 11, 12]:

$$y(0) = \sqrt{\frac{-2\ell}{\mu}}, \qquad y'(0) = -\frac{-\ell\sqrt{-\ell}}{\mu\sqrt{\frac{-2\ell}{\mu}}}.$$

The exact solution of this problem is:

$$y(t) = \sqrt{\frac{-2\ell(1+\tanh(t\sqrt{-\ell}))}{\mu}}.$$

In order to verify numerically whether the proposed method lead to accurate solutions, we solve this problem by the proposed method for $\ell = -1$, $\mu = 4$ and $\nu = -3$. Numerical results for some values of m are shown in Fig 3.

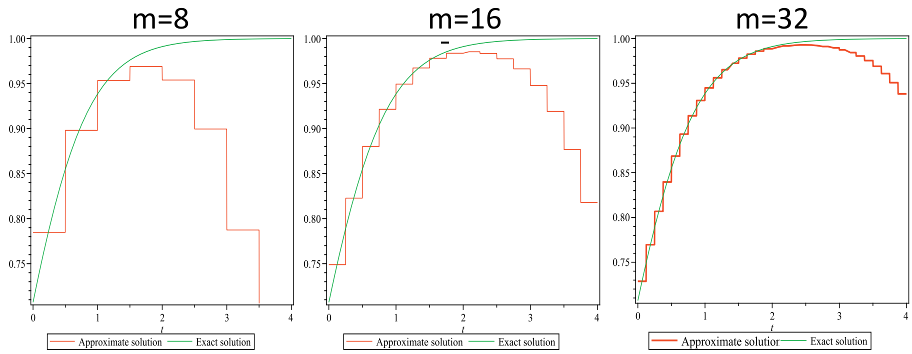

Figure 3: Approximate solutions of example 2, for some values of m.

A comparison between the proposed method for $m = 128$, with $\phi_3(t)$ of the ADM [7], $\psi_3(t)$ of the VIM [11] and the exact solution are performed in Fig 4 (left hand side). Moreover in order to see how much increasing number of terms and orders in the ADM and VIM can be improved the solution of this problem, a comparison between the proposed method for $m = 128$, with $\phi_6(t)$ of the ADM , $\psi_6(t)$ of the VIM and the exact solution are performed in Fig 4 (right hand side).

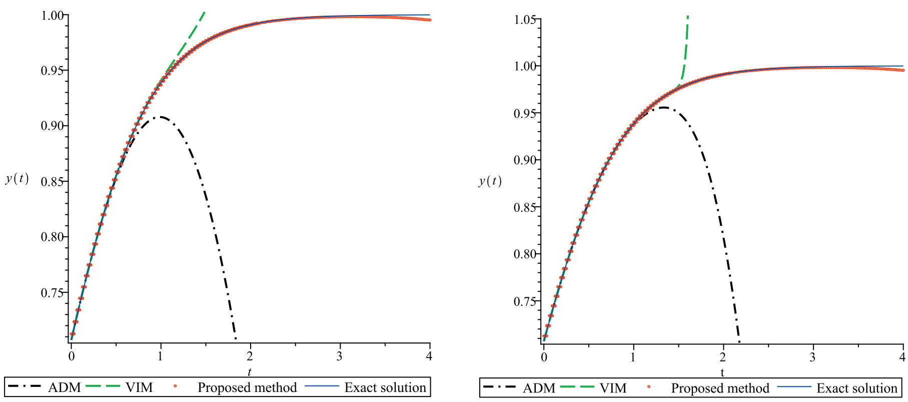

Figure 4: Numerical solutions of example 2.

From Fig 4, it is concluded that increasing number of terms and orders for the ADM and VIM can not improve the solution of this problem in a large domain. But by applying more number of basic block pulse functions a good approximate solution for this problem can be obtained in a large domain.

7. Conclusion

The Block pulse functions (BPFs) and their operational matrices are used to solve Liénard equation. The proposed method converts Liénard equation to a nonlinear system of algebraic equations whose solution vector gives the coefficients of the Block pulse functions of the solution of the Liénard equation. A comparison between the proposed method, ADM, VIM and the exact solution is performed. The obtained results show that the proposed method is very efficient and accurate in comparison with ADM and VIM. Also the proposed approach can provide a suitable approximate solution in a large interval by using only a few number of BPFs. Moreover, the proposed method doesn't require using Adomian polynomials, Lagrange multipliers, correction functionals, stationary conditions and the variational methods, etc., which eliminates the complications that exist in VIM. So in comparison with the ADM and VIM, the proposed method is simpler in principle and is more convenient for computer algorithms.

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