Journal of Mathematical Extension
Vol. 15, No. 2, (2021) (17)1-15
URL: https://doi.org/10.30495/JME.2021.1356
ISSN: 1735-8299
Original Research Paper

Primary Submodules over a Multiplicatively Closed Subset of a Commutative Ring

N. Ilaghi
Payame Noor University

M. Maani-Shirazi
Payame Noor University

Sh. Khoshdel *
Payame Noor University

Abstract. In this paper, we introduce the concept of primary submodules over S which is a generalization of the concept of S-prime submodules. Suppose S is a multiplicatively closed subset of a commutative ring R and let M be a unital R-module. A proper submodule Q of M with $(Q :_R M) \cap S = \emptyset$ is called primary over S if there is an $s \in S$ such that, for all $a \in R$, $m \in M$, $am \in Q$ implies that $sm \in Q$ or $sa^n \in (Q :_R M)$, for some positive integer n. We get some new results on primary submodules over S. Furthermore, we compare the concept of primary submodules over S with primary ones. In particular, we show that a submodule Q is primary over S if and only if $(Q :_M s)$ is primary, for some $s \in S$.

AMS Subject Classification: 16p40; 13A15; 16D60
Keywords and Phrases: Multiplicatively closed subset, Multiplication module, Primary module, primary module over S

Received: July 2019; Accepted: July 2020
*Corresponding Author

1 Introduction

Throughout this paper, R will be a non-zero commutative ring with identity and all modules are unital. Let K and L be two submodules of an R-module M and I an ideal of R. We denote the ideal $\{a \in R \mid aL \subseteq K\}$ by $(K :_R L)$ and the submodule $\{m \in M \mid Im \subseteq K\}$ by $(K :_M I)$. In particular, we use $ann(M)$ instead of $(0 :_R M)$ and $(K :_M s)$ instead of $(K :_M Rs)$, where Rs is the principal ideal generated by an element $s \in R$. Also, the Jacobson radical of R is denoted by $J(R)$ and we use $U(R)$ for the set of all unit elements of R. By $Max(R)$ we mean the set of all maximal ideals of R.

Recall that a proper submodule Q of an R-module M is primary if, for all $a \in R$ and $m \in M$, $am \in Q$ implies that $m \in Q$ or $a^n \in (Q :_R M)$, for some positive integer n, see, for example, [3], [4], [8], [10] and [11]. An R-module M is called a multiplication module if $N = (N :_R M)M$, for every submodule N of M ([5] and [12]). A nonempty subset S of R is called a multiplicatively closed subset (briefly, m.c.s.) of R if $0 \notin S$, $1 \in S$ and $ss' \in S$, for all $s, s' \in S$ ([13]). Let S be a m.c.s. of R and P a submodule of M with $(P :_R M) \cap S = \emptyset$. Then P is called an S-prime submodule of M if there exists $s \in S$ such that $am \in P$ implies that $sm \in P$ or $sa \in (P :_R M)$, for each $a \in R$ and $m \in M$. Note that by taking $s = 1$, every prime submodule is an S-prime submodule. In [9], the concept of S-prime submodules was defined. We generalize this concept to primary submodules over S. A submodule Q of M with $(Q :_R M) \cap S = \emptyset$ is called primary over S if there exists $s \in S$ such that, for all $a \in R$ and $m \in M$, $am \in Q$ implies that $sm \in Q$ or $sa^n \in (Q :_R M)$, for some positive integer n. Since $1 \in S$, all primary submodules Q with $(Q :_R M) \cap S = \emptyset$ are primary over S . With an additional assumption, we show that the converse is true. Being Q a primary submodule over S is related to being $(Q :_R M)$ is so as an ideal. Also, If M is a finitely generated R-module, then we find an equivalent condition for a proper submodule Q to be primary over S in M.

2 Main Results

Definition 2.1. *Let S be a m.c.s. of the ring R and Q a submodule of*

M as an R-module with $(Q :_R M) \cap S = \emptyset$. Then Q is called a primary submodule over S if there exists $s \in S$ such that, for all $a \in R$ and $m \in M$, $am \in Q$ implies that $sm \in Q$ or $sa^n \in (Q :_R M)$, for some positive integer n.

Clearly every S-prime submodule is primary over S. For instance, in a vector space V over a field F, every proper submodule W of V is S-prime and so primary over S, where S is an arbitrary m.c.s. of F.

Example 2.2. Let p be a fixed prime number. Each proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is of the form $G_k = (\frac{1}{p^k} + \mathbb{Z})$, for some integer $k \geq 0$ and $(G_k :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$. Take the m.c.s. $S = \{1, q, q^2, ...\}$, for some prime number $q \neq p$. Note that $p(\frac{1}{p^{k+1}} + \mathbb{Z}) \in G_k$ but, for each $s \in S$, $s(\frac{1}{p^{k+1}} + \mathbb{Z}) \notin G_k$ and $sp^n \notin (G_k :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$, for all positive integer n. Hence G_k is not primary over S, for all non-negative integers k and so $\mathbb{Z}_{p^\infty}$ does not have any primary submodule over S.

Proposition 2.3. *Suppose S is a m.c.s. of the ring R and M is an R-module. If $S \subseteq U(R)$ and Q is primary over S then Q is primary.*

Proof. Since Q is primary over S, so there exists $s \in S$ satisfying the definition. Let $a \in R$, $m \in M$ and $am \in Q$. By assumption, as $s \in S \subseteq U(R)$, $m \in Q$ or $a^n \in (Q :_R M)$, for some positive integer n. Therefore Q is primary. □

Proposition 2.4. *Let S_1 and S_2 be multiplicatively closed subsets of the ring R such that $S_1 \subseteq S_2$, M an R-module and Q a primary submodule over S_1 of M with $(Q :_R M) \cap S_1 = \emptyset$. Then Q is primary over S_2 in case $(Q :_R M) \cap S_2 = \emptyset$.*

Proof. Since Q is primary over S_1 so there exists $s_1 \in S_1$ satisfying the definition. Let $a \in R$, $m \in M$ and $am \in Q$. By hypothesis, $s_1 m \in Q$ or $s_1 a^n \in (Q :_R M)$, for some positive integer n. But $S_1 \subseteq S_2$. Thus $s_1 \in S_2$ and we get the result. □

Recall that, for the m.c.s S of the ring R, the saturation S^* of S is defined as

$$S^* = \{x \in R \mid \frac{x}{1} \in U(S^{-1}R)\}.$$

Clearly, S^* is a m.c.s. of R containing S ([6]).

Proposition 2.5. *Let S be a m.c.s. of the ring R, M an R-module and Q a submodule of M. Then Q is primary over S if and only if Q is primary over S^*.*

Proof. Suppose that Q is primary over S. We show that $(Q :_R M) \cap S^* = \emptyset$. For this, let $x \in (Q :_R M) \cap S^*$. So $\frac{x}{1} \in U(S^{-1}R)$ and there exist $a \in R$, $s \in S$ such that $\frac{xa}{s} = 1$ which implies that $uxa = us$, for some $u \in S$. Put $us = s' \in S$. Then $s' = us = uxa \in (Q :_R M) \cap S$, a contradiction. So $(Q :_R M) \cap S^* = \emptyset$. Since Q is primary over S and $S \subseteq S^*$, by the above proposition, Q is primary over S^*.

Conversely, suppose that Q is primary over S^*. So there exists $s^* \in S^*$ satisfying the definition. Let $a \in R$, $m \in M$ and $am \in Q$. By hypothesis, $s^*m \in Q$ or $s^*a^n \in (Q :_R M)$, for some positive integer n. Also, $s^* \in S^*$. Thus there exist $s \in S$ and $b \in R$ such that $\frac{s^*b}{s} = 1$ and so $us = us^*b$, for some $u \in S$. By taking $us = s' \in S$, $s'm = usm = us^*bm \in Q$ or $s'a^n = usa^n = us^*ba^n \in (Q :_R M)$. Therefore Q is primary over S. □

Proposition 2.6. *Let S be a m.c.s. of the ring R, M an R-module and Q a submodule of M. If Q is a primary submodule over S then $S^{-1}Q$ is a primary submodule of $S^{-1}M$ as an $S^{-1}R$-module.*

Proof. Assume that Q is a primary submodule over S. Let $\frac{r}{s}\frac{m}{t} \in S^{-1}Q$, where $\frac{r}{s} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$. There exist $q \in Q$ and $v \in S$ such that $\frac{rm}{st} = \frac{q}{v}$ and so $uvrm = ustq \in Q$, for some $u \in S$. Since Q is primary over S, there exists $s' \in S$ so that $s'm \in Q$ or $s'(uvr)^n \in (Q :_R M)$, for some positive integer n. Thus $\frac{m}{t} = \frac{s'm}{s't} \in S^{-1}Q$ or $\frac{r^n}{s^n} = \frac{s'(uvr)^n}{s'(uvs)^n} \in S^{-1}(Q :_R M) \subseteq (S^{-1}Q :_{S^{-1}R} S^{-1}M)$. Therefore $S^{-1}Q$ is a primary submodule of $S^{-1}M$ as an $S^{-1}R$-module. □

The converse of the above proposition is not true in general.

Example 2.7. Consider the $\mathbb{Z}$-module $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take $N = \mathbb{Z} \times 0$ and $S = \mathbb{Z} - \{0\}$. Then S is a m.c.s. of $\mathbb{Z}$ and $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field. So $S^{-1}(\mathbb{Q} \times \mathbb{Q})$ is a vector space over $S^{-1}\mathbb{Z} = \mathbb{Q}$ and the proper submodule $S^{-1}N$ is a primary submodule of $S^{-1}(\mathbb{Q} \times \mathbb{Q})$. Obviously, $(N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q}) = 0$. Let s be an arbitrary element of S and choose a prime number p with $(p, s) = 1$.

Note that $p(\frac{1}{p}, 0) = (1, 0) \in N$, $s(\frac{1}{p}, 0) = (\frac{s}{p}, 0) \notin N = \mathbb{Z} \times 0$ and $sp^n \notin (N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q})$, for each positive integer n, which shows that N is not primary over S.

Now, we characterize primary submodules over S of modules over the ring R in case R is Noetherian.

Lemma 2.8. *Let R be a Noetherian ring, M an R-module and Q a submodule of M. Suppose that S is a m.c.s. of R such that $(Q :_R M) \cap S = \emptyset$. Then the following are equivalent.*

(i) Q is primary over S;

(ii) There exists $s \in S$ such that, for each ideal J of R and submodule N of M, $JN \subseteq Q$ implies that $sN \subseteq Q$ or $sJ^n \subseteq (Q :_R M)$, for some positive integer n.

Proof. (i) $\Longrightarrow$ (ii) Assume that Q is a primary submodule over S. Thus there exists $s \in S$ satisfying the definition. Let J be an ideal of R, N a submodule of M and $JN \subseteq Q$. If $sN \subseteq Q$ we are done. Otherwise, there exists $x \in N$ such that $sx \notin Q$. But R is a Noetherian ring and J is an ideal of R. Therefore $J = (a_1, a_2, ..., a_k)$. We have $a_i x \in Q$, for each $i = 1, ..., k$. Since Q is primary over S and $sx \notin Q$ so, for each $i = 1, ..., k$, $sa_i^{k_i} \in (Q :_R M)$, for some positive integer k_i. Put $n = \sum_{i=1}^{k}(k_i - 1) + 1$. In this case, $sJ^n \subseteq (Q :_R M)$.

(ii) $\Longrightarrow$ (i) Let $a \in R$, $m \in M$ and $am \in Q$. Put $J = Ra$ and $N = Rm$. Then $JN = Ram \subseteq Q$. By assumption, $sN = Rsm \subseteq Q$ or $sJ^n = Rsa^n \subseteq (Q :_R M)$, for some positive integer n and so either $sm \in Q$ or $sa^n \in (Q :_R M)$. Therefore Q is a primary submodule over S. $\square$

Corollary 2.9. *Suppose that S is a m.c.s. of a Noetherian ring R and Q an ideal of R such that $(Q :_R M) \cap S = \emptyset$. Then the following are equivalent.*

(i) Q is primary over S in R;

(ii) There exists $s \in S$ such that, for every ideals I and J of R, if $JI \subseteq Q$ then $sI \subseteq Q$ or $sJ^n \subseteq Q$, for some positive integer n.

Proposition 2.10. *Let S be a m.c.s. of the ring R and $f : M \longrightarrow M'$ an R-homomorphism. Then* *(i) If Q' is primary over S in M' provided that $(f^{-1}(Q') :_R M) \cap S = \emptyset$, then $f^{-1}(Q')$ is so in M.*

(ii) If f is an epimorphism and Q is primary over S in M such that $Kerf \subseteq Q$, then $f(Q)$ is so in M'.

Proof. (i) Since Q' is primary over S in M', there exists $s \in S$ satisfying the definition. Let $am \in f^{-1}(Q')$, for some $a \in R$ and $m \in M$. Then $f(am) = af(m) \in Q'$. By assumption, $sf(m) = f(sm) \in Q'$ or $sa^n \in (Q' :_R M')$, for some positive integer n. Now we show that $(Q' :_R M') \subseteq (f^{-1}(Q') :_R M)$. Take $x \in (Q' :_R M')$, we have $xM' \subseteq Q'$. Since $f(M) \subseteq M'$, $f(xM) = xf(M) \subseteq xM' \subseteq Q'$ so $xM \subseteq xM + Kerf = f^{-1}(f(xM)) \subseteq f^{-1}(Q')$ and thus $x \in (f^{-1}(Q') :_R M)$. As $(Q' :_R M') \subseteq (f^{-1}(Q') :_R M)$, we can conclude either $sm \in f^{-1}(Q')$ or $sa^n \in (f^{-1}(Q') :_R M)$. Hence $f^{-1}(Q')$ is primary over S in M.

(ii) First we claim that $(f(Q) :_R M') \cap S = \emptyset$. Otherwise, there exists an element $s \in (f(Q) :_R M') \cap S$ and so $f(sM) = sf(M) \subseteq sM' \subseteq f(Q)$. By taking their inverse images under f, we have $sM \subseteq sM + Kerf \subseteq Q + Kerf = Q$, which means $sM \subseteq Q$. Thus $s \in (Q :_R M) \cap S$, a contradiction. By assumption, Q is primary over S in M. Then there exists an element $s \in S$ satisfying the definition. Now take $a \in R$, $m' \in M'$ such that $am' \in f(Q)$. As f is an epimorphism, there exists $m \in M$ such that $m' = f(m)$. Hence $am' = af(m) = f(am) \in f(Q)$. Since $Kerf \subseteq Q$, $am \in Q$. But Q is primary over S in M. Hence we have $sm \in Q$ or $sa^n \in (Q :_R M)$, for some positive integer n. But $(Q :_R M) \subseteq (f(Q) :_R M')$. Therefore $f(sm) = sf(m) = sm' \in f(Q)$ or $sa^n \in (f(Q) :_R M')$. Consequently, $f(Q)$ is primary over S in M'. □

Being f an epimorphism in part (ii) is essential. Let us give an example.

Example 2.11. Let $R = \mathbb{Z}$, $M = \mathbb{Z}$, $S = \{-1, 1\}$ and $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $f(x) = 6x$. Then f is a $\mathbb{Z}$-homomorphism which is not onto. By Proposition 2.3, primary and primary submodules over S in $\mathbb{Z}$ are the same. We know that 0 and $(p)^n$, where p is an arbitrary prime number and n a positive integer, are all primary submodules in $\mathbb{Z}$ ([3]). Let $Q = 3\mathbb{Z}$. Then Q is primary and $f(Q) = 18\mathbb{Z}$ is not primary.

Corollary 2.12. *Let S be a m.c.s. of the ring R and L a submodule of* *the R-module M. Then*

(i) If Q' is primary over S in M with $(Q' :_R L) \cap S = \emptyset$ then $L \cap Q'$ is primary over S in L.

(ii) Suppose that Q is a submodule of M with $L \subseteq Q$. Then Q is primary over S in M if and only if $\frac{Q}{L}$ is so in $\frac{M}{L}$.

Proof. (i) Consider the injection $i : L \longrightarrow M$ defined by $i(m) = m$, for all $m \in L$. Then $i^{-1}(Q') = L \cap Q'$. Now we claim that $(i^{-1}(Q') :_R L) \cap S = \emptyset$. For it, let $s \in (i^{-1}(Q') :_R L) \cap S$. Then $sL \subseteq i^{-1}(Q') = L \cap Q' \subseteq Q'$ and thus $s \in (Q' :_R L) \cap S$, a contradiction. The result follows by Proposition 2.10.

(ii) Let Q be primary over S in M and $\pi : M \longrightarrow \frac{M}{L}$ be the canonical epimorphism defined by $\pi(m) = m + L$, for all $m \in M$. Since $L = Ker\pi \subseteq Q$ so $\frac{Q}{L}$ is primary over S in $\frac{M}{L}$, by Proposition 2.10 part (ii).

Conversely, assume that $\frac{Q}{L}$ is primary over S in $\frac{M}{L}$. There exists $s \in S$ satisfying the definition. Let $am \in Q$, for some $a \in R$ and $m \in M$. This implies that $a(m + L) = am + L \in \frac{Q}{L}$. By assumption, $s(m+L) = sm+L \in \frac{Q}{L}$ or $sa^n \in (\frac{Q}{L} :_R \frac{M}{L}) = (Q :_R M)$, for some positive integer n. Therefore, $sm \in Q$ or $sa^n \in (Q :_R M)$. Consequently, Q is primary over S in M. □

Proposition 2.13. *Let S be a m.c.s. of the ring R and M an R-module. The following statements hold.*

(i) If Q is primary over S in M then $(Q :_R M)$ is so in R.

(ii) If R is Noetherian, M a multiplication module over R and $(Q :_R M)$ is primary over S in R, then Q is so in M.

Proof. (i) Let Q be primary over S in M. There exists $s \in S$ satisfying the definition. Let $xy \in (Q :_R M)$, for some $x, y \in R$. Then $xym \in Q$, for all $m \in M$. If $sx^n \in (Q :_R M)$, for some positive integer n, we are done. Otherwise, $sym \in Q$, for all $m \in M$ which means that $sy \in (Q :_R M)$. Therefore $(Q :_R M)$ is primary over S in M.

(ii) Assume that M is a multiplication module over a Noetherian ring R and $(Q :_R M)$ is primary over S in R. There exists $s \in S$ satisfying in Corollary 2.9 part (ii). Let J be an ideal of R and N a submodule of M with $JN \subseteq Q$. Then $J(N :_R M) \subseteq (JN :_R M) \subseteq (Q :_R M)$. As $(Q :_R M)$ is primary over S in R, $s(N :_R M) \subseteq (Q :_R M)$ or $sJ^n \subseteq (Q :_R M)$, for some positive integer n. Thus $sN = s(N :_R$

$M)M \subseteq (Q :_R M)M = Q$ or $sJ^n \subseteq (Q :_R M)$. By Lemma 2.8, Q is primary over S in M. □

Let K and L be submodules of the multiplication R-module M. Recall that the product of K and L is defined as $KL = (K :_R M)(L :_R M)M$ ([1]).

Corollary 2.14. *Let M be a multiplication module over a Noetherian ring R and Q a submodule of M with $(Q :_R M) \cap S = \emptyset$, where S is a m.c.s. of R. The following are equivalent.*

(i) Q is primary over S in M;

(ii) There exists $s \in S$ such that, for every two submodules K and L of M with $KL \subseteq Q$, $sL \subseteq Q$ or $sK^n \subseteq Q$, for some positive integer n.

Proof. (i) $\Longrightarrow$ (ii) There exists $s \in S$ satisfying in Lemma 2.8 part (ii). Let K and L be submodules of M with $KL \subseteq Q$. Then $(K :_R M)(L :_R M)M \subseteq Q$ and so $s(L :_R M)M \subseteq Q$ or $s(K :_R M)^n \subseteq (Q :_R M)$, for some positive integer n. Hence $sL \subseteq Q$ or $s(K :_R M)^n \subseteq (Q :_R M)$. If $sL \subseteq Q$ we are done. Otherwise, $s(K :_R M)^n \subseteq (Q :_R M)$ and so $s(K :_R M)^n M \subseteq (Q :_R M)M = Q$. Therefore $sK^n \subseteq Q$.

(ii) $\Longrightarrow$(i) Let $am \in Q$, for some $a \in R$ and $m \in M$. Put $L = Rm$ and $J = Ra$. Then $JL \subseteq Q$ and $JLM \subseteq QM = Q$. Take $K = JM$. In this case, $KL \subseteq Q$. By hypothesis, $sL \subseteq Q$ or $sK^n \subseteq Q$, for some positive integer n. Then we have $sL \subseteq Q$ or $s(K :_R M)^n M \subseteq Q$. If $sL \subseteq Q$, since $L = Rm$ so $sm \in Q$ and we are done. Now suppose that $s(K :_R M)^n M \subseteq Q$. Since $K = JM$ and $J = Ra$, $sa^n \in (Q :_R M)$. Therefore Q is primary over S. □

Theorem 2.15. *Let S be a m.c.s. of the Noetherian ring R and M a finitely generated multiplication R-module. For a submodule Q of M with $(Q :_R M) \cap S = \emptyset$, the following are equivalent.*

(i) Q is primary over S;

(ii) $(Q :_R M)$ is primary over S in R;

(iii) $Q = IM$, for some primary ideal I over S in R with $ann(M) \subseteq I$.

Proof. (i) $\Longrightarrow$ (ii) It is clear by Proposition 2.13.

(ii) $\Longrightarrow$ (iii) By taking $I = (Q :_R M)$, we get the result.

(iii) $\Longrightarrow$ (i) Suppose that $Q = IM$, for some primary ideal I over S in R with $ann(M) \subseteq I$. There exists $s \in S$ satisfying in Corollary 2.9 part (ii). Assume that $JN \subseteq Q$, for some ideal J of R and submodule N of M. Then $J(N :_R M)M \subseteq IM$. By [12], $J(N :_R M) \subseteq I + annM = I$. By hypothesis, $s(N :_R M) \subseteq I \subseteq (Q :_R M)$ or $sJ^n \subseteq I \subseteq (Q :_R M)$, for some positive integer n. So $sN \subseteq Q$ or $sJ^n \subseteq (Q :_R M)$. Hence Q is primary over S in M. $\square$

Lemma 2.16. *Let S_i be a m.c.s of the ring R_i, for $i = 1, 2$. Put $S = S_1 \times S_2$ as a m.c.s. of the ring $R = R_1 \times R_2$. For each ideal $Q = Q_1 \times Q_2$ of R, the following are equivalent.*

(i) Q is primary over S in R;

(ii) Q_1 is primary over S_1 in R_1 and $Q_2 \cap S_2 \neq \emptyset$ or Q_2 is primary over S_2 in R_2 and $Q_1 \cap S_1 \neq \emptyset$.

Proof. (i) $\Longrightarrow$ (ii) Suppose that Q is primary over S in R. There exists $s = (s_1, s_2) \in S$ satisfying the definition. Since $(1, 0)(0, 1) = (0, 0) \in Q$, so $s(0, 1) = (0, s_2) \in Q$ or $s(1, 0)^n = (s_1, 0) \in Q$, for some positive integer n. Thus $s_1 \in Q_1 \cap S_1$ or $s_2 \in Q_2 \cap S_2$. Therefore $Q_1 \cap S_1 \neq \emptyset$ or $Q_2 \cap S_2 \neq \emptyset$. We may assume that $Q_1 \cap S_1 \neq \emptyset$ and show Q_2 is primary over S_2 in M_2. Since $Q \cap S = \emptyset$ so $Q_2 \cap S_2 = \emptyset$. Let $xy \in Q_2$, for some $x, y \in R_2$. Since $(0, x)(0, y) \in Q$ and Q is primary over S in R, either $s(0, y) = (0, s_2 y) \in Q$ or $s(0, x)^n = (0, s_2 x^n) \in Q$. This means that $s_2 y \in Q_2$ or $s_2 x^n \in Q_2$. Therefore Q_2 is primary over S_2 in R_2. In other case, one can easily show that Q_1 is primary over S_1 in R_1.

(ii) $\Longrightarrow$ (i) Assume that $Q_1 \cap S_1 \neq \emptyset$ and Q_2 is primary over S_2 in R_2. We show that Q is primary over S. Since $Q_1 \cap S_1 \neq \emptyset$, there exists $s_1 \in Q_1 \cap S_1$. Moreover, we have $s_2 \in S_2$ satisfying the definition of being primary over S_2. Let $(a, b)(c, d) = (ac, bd) \in Q$, for some $a, c \in R_1$ and $b, d \in R_2$. This implies that $bd \in Q_2$ and thus $s_2 d \in Q_2$ or $s_2 b^n \in Q_2$, for some positive integer n. Put $s = (s_1, s_2) \in S$. Then $s(c, d) = (s_1 c, s_2 d) \in Q$ or $s(a, b)^n = (s_1 a^n, s_2 b^n) \in Q$. Therefore Q is primary over S in R. In the other case, one can similarly prove that Q is primary over S in R. $\square$

Theorem 2.17. *Let S_i be a m.c.s of the ring R_i and M_i an R_i-module, for $i = 1, 2$. Suppose that $Q = Q_1 \times Q_2$ is a submodule of $M = M_1 \times M_2$ as an $R = R_1 \times R_2$-module. The following are equivalent.*

(i) Q is primary over S in M;

(ii) Q_1 is primary over S_1 in M_1 and $(Q_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or $(Q_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and Q_2 is primary over S_2 in M_2.

Proof. (i) $\Longrightarrow$ (ii) Assume that Q is primary over S in M. We have $s = (s_1, s_2) \in S$ satisfying the definition. By Proposition 2.13, $(Q :_R M) = (Q_1 :_{R_1} M_1) \times (Q_2 :_{R_2} M_2)$ is primary over S in R, and so by Lemma 2.16, either $(Q_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ or $(Q_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$. We may assume that $(Q_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$. Let $am \in Q_2$, for some $a \in R_2$, $m \in M_2$. Then $(1, a)(0, m) = (0, am) \in Q$. Since Q is primary over S so $s(0, m) = (0, s_2m) \in Q$ or $s(1, a)^n = (s_1, s_2a^n) \in (Q :_R M)$, for some positive integer n. This implies that $s_2m \in Q_2$ or $s_2a^n \in (Q_2 :_{R_2} M_2)$. Therefore Q_2 is primary over S_2. In the other case, one can similarly show that Q_1 is primary over S_1 in M_1.

(ii) $\Longrightarrow$ (i) Assume that $(Q_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and Q_2 is primary over S_2 in M_2. Thus there exists $s_1 \in (Q_1 :_{R_1} M_1) \cap S_1$ and we have $s_2 \in S_2$ satisfying the definition of primary over S_2. Now, let $(a_1, a_2)(m_1, m_2) = (a_1m_1, a_2m_2) \in Q$, for some $a_i \in R_i$ and $m_i \in M_i$, $i = 1, 2$. Then $a_2m_2 \in Q_2$. Since Q_2 is primary over S_2, so $s_2m_2 \in Q_2$ or $s_2a_2^n \in (Q_2 :_{R_2} M_2)$, for some positive integer n. Put $s = (s_1, s_2) \in S$. Then $s(m_1, m_2) = (s_1m_1, s_2m_2) \in Q = Q_1 \times Q_2$ or $s(a_1, a_2)^n = (s_1a_1^n, s_2a_2^n) \in (Q :_R M)$. Therefore Q is primary over S. Similarly, one can show that if Q_1 is primary over S_1 in M_1 and $(Q_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$, then Q is primary over S in M. □

Theorem 2.18. *Let S_i be a m.c.s of the ring R_i and M_i an R_i-module, for $i = 1, 2, ..., n$. Take $M = M_1 \times M_2 \times \cdots \times M_n$ as an $R = R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n$ as a m.c.s. of R. Assume that $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ is a submodule of M. The following statements are equivalent.*

(i) Q is primary over S in M;

(ii) Q_i is primary over S_i in M_i, for some $i \in \{1, 2, ..., n\}$ and $(Q_j :_{R_j} M_j) \cap S_j \neq \emptyset$, for all $j \in \{1, 2, ..., n\} - \{i\}$.

Proof. We prove it by induction on n. For $n = 1$, the result is true. If $n = 2$ then (i) $\Longleftrightarrow$ (ii) follows from Theorem 2.17. Assume that (i) and (ii) are equivalent for every positive integer $k < n$. Now, we shall prove that (i) $\Longleftrightarrow$ (ii) when $k = n$. Put $M' = M_1 \times M_2 \times \cdots \times M_{n-1}$,

$R' = R_1 \times R_2 \times \cdots \times R_{n-1}$, $Q' = Q_1 \times Q_2 \times \cdots \times Q_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$. By Theorem 2.17, $Q = Q' \times Q_n$ is primary over S in M if and only if Q' is primary over S' in M' and $(Q_n :_{R_n} M_n) \cap S_n \neq \emptyset$ or $(Q' :_{R'} M') \cap S' \neq \emptyset$ and Q_n is primary over S_n in M_n. The rest follows from the induction hypothesis. □

Lemma 2.19. *Let S be a m.c.s of the ring R and Q a primary submosule over S of an R-module M. The following statements hold, for some $s \in S$.*

(i) $(Q :_M s') \subseteq (Q :_M s)$, for all $s' \in S$.

(ii) $((Q :_R M) :_R s') \subseteq ((Q :_R M) :_R s)$, for all $s' \in S$.

Proof. (i) Since Q is primary over S in R so there exists $s \in S$ satisfying the definition. By taking $s' \in S$ and $m' \in (Q :_M s')$, $s'm' \in Q$. Thus $sm' \in Q$ or $ss'^n \in (Q :_R M)$, for some positive integer n. Since $(Q :_R M) \cap S = \emptyset$ so $sm' \in Q$, which means that $m' \in (Q :_M s)$.

(ii) We know that $(Q :_R M)$ is primary over S in R, so it is enough to replace $(Q :_R M)$ instead of Q in part (i). □

Proposition 2.20. *Let S be a m.c.s. of the ring R, M a finitely generated R-module and Q a submodule of M satisfying $(Q :_R M) \cap S = \emptyset$. The following are equivalent.*

(i) Q is primary over S in R;

(ii) $S^{-1}Q$ is a primary submodule of $S^{-1}M$ and there exists $s \in S$ satisfying $(Q :_M s') \subseteq (Q :_M s)$, for all $s' \in S$.

Proof. (i) $\Longrightarrow$ (ii) It is clear from Proposition 2.6 and Lemma 2.19.

(ii) $\Longrightarrow$ (i) Let $a \in R$, $m \in M$ and $am \in Q$. Then $\frac{am}{1} \in S^{-1}Q$. Since $S^{-1}Q$ is a primary submodule of $S^{-1}M$ and M is finitely generated so $\frac{m}{1} \in S^{-1}Q$ or $\frac{a^n}{1} \in (S^{-1}Q :_{S^{-1}R} S^{-1}M) = S^{-1}(Q :_R M)$, for some positive integer n. Thus $u'm \in Q$ or $ua^n \in (Q :_R M)$, for some $u, u' \in S$. By assumption, there exists $s \in S$ so that $(Q :_M s') \subseteq (Q :_M s)$, for all $s' \in S$. If $ua^n \in (Q :_R M)$ then $a^n M \subseteq (Q :_M u) \subseteq (Q :_M s)$ and thus $sa^n \in (Q :_R M)$. If $u'm \in Q$ a similar argument shows that $sm \in Q$. Therefore Q is primary over S in M. □

Theorem 2.21. *Let S be a m.c.s. of the ring R, M an R-module and Q a submodule of M such that $(Q :_R M) \cap S = \emptyset$. Then Q is primary over S in M if and only if $(Q :_M s)$ is primary in M, for some $s \in S$.*

Proof. Suppose that $(Q :_M s)$ is primary in M, for some $s \in S$. Let $am \in Q$, for some $a \in R$ and $m \in M$. Since $am \in Q \subseteq (Q :_M s)$ and $(Q :_M s)$ is primary, so $m \in (Q :_M s)$ or $a^n \in ((Q :_M s) :_R M)$, for some positive integer n. This implies that $sm \in Q$ or $sa^n \in (Q :_R M)$.

Conversely, suppose that Q is primary over S in M. Then there exists $s \in S$ satisfying the definition. Now we prove that $(Q :_M s)$ is primary in M. For it, let $a \in R$, $m \in M$ and $am \in (Q :_M s)$. Then $(sa)m \in Q$. Since Q is primary over S so $sm \in Q$ or $s^{n+1}a^n \in (Q :_R M)$, for some positive integer n. If $sm \in Q$ we are done. Otherwise $s^{n+1}a^n \in (Q :_R M)$ and so, by Lemma 2.19, $a^n \in ((Q :_R M) :_R s^{n+1}) \subseteq ((Q :_R M) :_R s)$. Thus we can conclude that $a^n \in ((Q :_M s) :_R M)$ and hence $(Q :_M s)$ is primary. □

Theorem 2.22. *Let M be a module over the ring R and Q a submodule of M such that $(Q :_R M) \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R. The following statements are equivalent.*

(i) Q is primary in M;

(ii) $(Q :_R M)$ is primary and Q is primary over $R - \mathbf{m}$, for all maximal ideals $\mathbf{m}$ of R.

Proof. (i) $\Longrightarrow$ (ii) Suppose that Q is primary. Since $(Q :_R M) \subseteq J(R)$, so $(Q :_R M) \subseteq \mathbf{m}$, for all maximal ideals $\mathbf{m}$. Hence $(Q :_R M) \cap (R - \mathbf{m}) = \emptyset$, for all maximal ideals $\mathbf{m}$. The rest follows clearly.

(ii) $\Longrightarrow$(i) Suppose $(Q :_R M)$ is primary and Q is primary over $R - \mathbf{m}$, for all maximal ideals $\mathbf{m}$. Let $a \in R$, $m \in M$ and $am \in Q$ with $a^n \notin (Q :_R M)$, for each positive integer n. Let $\mathbf{m}$ be a maximal ideal. Since Q is primary over $R - \mathbf{m}$ so there exists $s_{\mathbf{m}} \in R - \mathbf{m}$ such that $s_{\mathbf{m}}m \in Q$ or $s_{\mathbf{m}}a^k \in (Q :_R M)$, for some positive integer k. But $(Q :_R M)$ is primary, $s_{\mathbf{m}} \notin (Q :_R M)$ and $a^n \notin (Q :_R M)$, for each positive integer n. Thus $s_{\mathbf{m}}m \in Q$. Now define the set $\Omega = \{s_{\mathbf{m}} \mid \exists \mathbf{m} \in Max(R) \ni: s_{\mathbf{m}} \notin \mathbf{m}, s_{\mathbf{m}}m \in Q\}$. Now we show that (Ω), the ideal generated by Ω, is equal to R. For it, let $\mathbf{m'}$ be a maximal ideal such that $\Omega \subseteq \mathbf{m'}$. The definition of Ω requires that there exists $s_{\mathbf{m'}} \in \Omega$ and $s_{\mathbf{m'}} \notin \mathbf{m'}$. Since $\Omega \subseteq \mathbf{m'}$ so $s_{\mathbf{m'}} \in \Omega \subseteq \mathbf{m'}$, a contradiction. Thus $(\Omega) = R$ which implies that $1 = r_1 s_{\mathbf{m}_1} + r_2 s_{\mathbf{m}_2} + \cdots + r_n s_{\mathbf{m}_n}$, for some $r_i \in R$ and $s_{\mathbf{m}_i} \notin \mathbf{m}_i$ with $s_{\mathbf{m}_i}m \in Q$ and $\mathbf{m}_i \in Max(R)$, for each $i = 1, 2, ..., n$. Thus $m = r_1 s_{\mathbf{m}_1} m + r_2 s_{\mathbf{m}_2} m + \cdots + r_n s_{\mathbf{m}_n} m \in Q$ and so Q is primary. □

Corollary 2.23. *Suppose that M is a module over a ring R with unique maximal ideal $\boldsymbol{m}$. The following are equivalent.*

(i) Q is primary in M.

(ii) $(Q :_R M)$ is primary in R and Q is primary over $R - \boldsymbol{m}$ in M.

Proof. It is clear by Theorem 2.22. □

Suppose that M is a module over the ring R. Nagata introduced the idealization $R(+)M$ of M. Here $R(+)M = R \oplus M$ is a commutative ring whose addition is componentwise and multiplication is defined as $(a, m)(b, m^{'}) = (ab, am^{'} + bm)$, for each $a, b \in R$ and $m, m' \in M$ ([7]). The name comes from the fact that if N is a submodule of M, then $0(+)N$ is an ideal of $R(+)M$. If S is a m.c.s. of the ring R and N a submodule of an R-module M, then $S(+)N = \{(s, n) \mid s \in S, n \in N\}$ is a m.c.s. of $R(+)M$ ([2]).

Proposition 2.24. *Let S be a m.c.s. of the ring R and Q an ideal of R such that $Q \cap S = \emptyset$. The following are equivalent.*

(i) Q is a primary ideal over S in R;

(ii) $Q(+)M$ is a primary ideal over $S(+)0$ in $R(+)M$;

(iii) $Q(+)M$ is a primary ideal over $S(+)M$ in $R(+)M$.

Proof. (i) $\Longrightarrow$ (ii) Suppose Q is primary over S in R. There exists $s \in S$ satisfying the definition. Let $(x, m)(y, m') = (xy, xm' + ym) \in Q(+)M$, for some $x, y \in R$ and $m, m' \in M$. Then we have $xy \in Q$. Since Q is primary over S in R so $sy \in Q$ or $sx^n \in Q$, for some positive integer n. By putting $s' = (s, 0) \in S(+)0$, we have $s'(y, m') = (s, 0)(y, m') \in Q(+)M$ or $s'(x, m)^n = (s, 0)(x^n, nx^{n-1}m) = (sx^n, nsx^{n-1}m) \in Q(+)M$. Therefore $Q(+)M$ is primary over $S(+)0$ in $R(+)M$.

(ii) $\Longrightarrow$ (iii) It is clear by Proposition 2.4.

(iii) $\Longrightarrow$ (i) Suppose that $Q(+)M$ is primary over $S(+)M$ in $R(+)M$. There exists $s = (s_1, m_1) \in S(+)M$ satisfying the definition. Let $xy \in Q$, for some $x, y \in R$. Then $(x, 0)(y, 0) = (xy, 0) \in Q(+)M$. By hypothesis, $s(y, 0) = (s_1, m_1)(y, 0) = (s_1y, ym_1) \in Q(+)M$ or $s(x, 0)^n = (s_1, m_1)(x^n, 0) = (s_1x^n, x^nm_1) \in Q(+)M$, for some positive integer n. Hence, $s_1y \in Q$ or $s_1x^n \in Q$ and so Q is primary over S. □

Acknowledgements
The authors would like to thank the referees for their valuable comments and suggestions.

References

[1] R. Ameri, On the prime submodules of multiplication modules, *International Journal of Mathematics and Mathematical Sciences*, 27 (2003), 1715 – 1724.

[2] D. D. Anderson and M. Winders, Idealization of a module, *Journal of Commutative Algebra*, 1(1) (2009), 3 – 56.

[3] M. Atiyah, *Introduction to Commutative Algebra*, Boca Raton, FL., USA: CRC Press, 2018.

[4] S. Ebrahimi Atani, F. Callalpm and U. Tekir, A short note on the primary submodules of multiplication modules, *International Journal of Algebra*, 1(8) (2007), 381 – 384.

[5] Z. A. El-Bast and P. F. Smith, Multiplication modules, *Communications in Algebra*, 16(4) (1988), 755 – 779.

[6] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, No. 90. Kingston, Canada: Queens University, 1992.

[7] M. Nagata, *Local Rings*, New York, NY, USA: Interscience, 1962.

[8] E. S. Sevim, T. Arabaci, U. Tekir and S. Koc, A note on finite union of primary submodules, *Eurasian bulletin of mathematics ebm*, 2(1) (2019), 32 – 35.

[9] E. S. Sevim, T. Arabaci, U. Tekir and S. Koc, On S-prime submodules, *Turkish J. Math*, 43 (2019), 1036 – 1046.

[10] R. Y. Sharp, *Steps in Commutative Algebra*, Cambridge, UK: Cambridge University Press, 2000.

[11] P. F. Smith, Primary modules over commutative rings, *Glasgow Mathematical Journal*, 43(1) (2001), 103 − 111.

[12] P. F. Smith, Some remarks on multiplication modules, *Archiv der Mathematik*, 50(3) (1988), 223 − 235.

[13] F. Wang and H. Kim, *Fundations of Commutative Rings and Their Modules*, Singapore: Springer, 2016.

Nahid Ilaghi
Ph.D Student of Mathematics
Department of Mathematics
Payame Noor University (PNU), P. O. Box 19395-4697
Tehran, Iran
E-mail: ilaghi-nahid@yahoo.com

Mansooreh Maani-Shirazi
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University (PNU), P. O. Box 19395-4697
Tehran, Iran
E-mail: maani@pnu.ac.ir

Shamsolmolouk Khoshdel
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University (PNU), P. O. Box 19395-4697
Tehran, Iran
E-mail: khoshdel@pnu.ac.ir