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Tuples of Operators with Hereditarily Transitivity Property

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Abstract. In this paper, we investigate the relation between hypercyclicity and d-dense orbits of a tuple of operators. Also, we characterize conditions under which a tuple of continuous operators is hereditarily transitive.

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1. Introduction

In the following, by an n-tuple of operators we mean a finite sequence of length n of commuting continuous operators acting on a Banach space X.

Definition 1.1. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let*

$$\mathcal{F}_{\mathcal{T}} = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geqslant 0, i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}_{\mathcal{T}}\}.$$

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If $T_1, T_2, ..., T_n$ are also linear, a vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic ([5]). By $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}_{\mathcal{T}}$, i.e.

$$\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}_{\mathcal{T}}\}.$$

Definition 1.2. *We say that a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is topologically transitive with respect to an n-tuple of nonnegative integer sequences*

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j),$$

if for every nonempty open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that

$$T_1^{k_{j_0(1)}} T_2^{k_{j_0(2)}} ... T_n^{k_{j_0(n)}}(U) \cap V \neq \emptyset.$$

Also, we say that an n-tuple $\mathcal{T}$ is topologically transitive if it is topologically transitive with respect an n-tuple of nonnegative integer sequences.

Definition 1.3. *A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called topologically mixing if for any given open sets U and V, there exist positive integers $M(1)$, ..., $M(n)$ such that*

$$T_1^{m(1)} ... T_n^{m(n)}(U) \cap V \neq \emptyset$$

for all $m(i) \geqslant M(i)$, $i = 1, ..., n$.

Definition 1.4. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of continuous operators acting on a separable infinite dimensional Banach space X. We say that $\mathcal{T}$ is hereditarily transitive with respect to a tuple*

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j)$$

of integers if it is topologically transitive with respect to any tuple of subsequence

$$(\{k_{j_i(1)}\}_i, \{k_{j_i(2)}\}_i, ..., \{k_{j_i(n)}\}_i)$$

of

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, ..., \{k_{j(n)}\}_j).$$

Also, we say that $\mathcal{T}$ is hereditarily transitive if it is hereditarily transitive with respect an n-tuple of nonnegative integers.

Here, we want to extend some properties of hereditarily transitivity and hypercyclic operators to a tuple of commuting operators. For some topics we refer to [1-20].

2. Main Results

In this section we characterize the equivalent conditions for a tuple of operators, being hereditarily transitive.

Theorem 2.1. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of continuous operators acting on a separable infinite dimensional Banach space X. Then the followings are equivalent:*
(i) *$\mathcal{T}$ is topologically mixing.*
(ii) *$\mathcal{T}$ is hereditarily transitive with respect to the n-tuple of full sequences.*

Proof. (i) $\rightarrow$ (ii): Suppose that $\mathcal{T}$ is topologically mixing and consider an arbitrary tuple of nonnegative sequences of integers

$$(\{m_k^{(1)}\}_k, \{m_k^{(2)}\}_k, ..., \{m_k^{(n)}\}_k).$$

Let U, V be two nonempty open subsets of X. Since $\mathcal{T}$ is topologically mixing, there exists a tuple $(M_1, M_2, ..., M_n)$ of integers such that

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(U) \cap V \neq \emptyset$$

for all $m_i > M_i$, i=1,...,n. This implies that there exists

$$(m_{k_0}^{(1)}, m_{k_0}^{(2)}, ..., m_{k_0}^{(n)}) \in (\{m_k^{(1)}\}_k, \{m_k^{(2)}\}_k, ..., \{m_k^{(n)}\}_k)$$

such that

$$T_1^{m_{k_0}^{(1)}} T_2^{m_{k_0}^{(2)}} ... T_n^{m_{k_0}^{(n)}}(U) \cap V \neq \emptyset.$$

Since $(\{m_k^{(1)}\}_k, \{m_k^{(2)}\}_k, ..., \{m_k^{(n)}\}_k)$ is arbitrary, thus $\mathcal{T}$ is hereditarily transitive with respect to all tuples of subsequences

$$(\{m_k^{(1)}\}_k, \{m_k^{(2)}\}_k, ..., \{m_k^{(n)}\}_k)$$

of the tuple of full integer sequences and so (ii) holds.
(ii) $\to$ (i): Suppose that $\mathcal{T}$ is hereditarily transitive with respect to the tuple of full integer sequences, and also $\mathcal{T}$ is not topologically mixing. Thus there exist two nonempty open sets U, V such that for all $M_i \in \mathbb{N}$, $i = 1, ..., n$, we can find $m_i > M_i$ for $i = 1, ..., n$ satisfying

$$T_1^{m_1} T_2^{m_2} ... T_n^{m_n}(U) \cap V = \emptyset.$$

This implies that there exists a tuple of sequences of integers

$$(\{m_k^{(1)}\}_k, \{m_k^{(2)}\}_k, ..., \{m_k^{(n)}\}_k)$$

such that

$$T_1^{m_k^{(1)}} T_2^{m_k^{(2)}} ... T_n^{m_k^{(n)}}(U) \cap V \neq \emptyset$$

for all $k \geqslant 0$, which contradicts the assertion (ii). This completes the proof. $\square$

Remember that if $d > 0$, we say a tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ of bounded linear operators acting on a separable infinite dimensional Banach space X has a d-dense orbit in a nonempty open subset U of X, if there exists $x \in X$ such that for any $y \in U$,

$$B(y, d) \cap Orb(\mathcal{T}, x) \neq \emptyset.$$

The following theorem extends theorem 3.7 of [9] for tuples.

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be a tuple of bounded linear operators acting on a separable infinite dimensional Banach space X such that $(T_1^{m_1} T_2^{m_2} ... T_n^{m_n})^*$ has no eigenvalue for all $m_i > 0$, $i = 1, ..., n$. Also, let W be a nonempty open subset of X and $E = \bigcup_{c \geqslant 0} cW$. If there exist $x \in X$ and $d > 0$ such that $Orb(\mathcal{T}, x)$ is d-dense in E, then $\mathcal{T}$ is hypercyclic.*

Proof. Let U, V be any nonempty open sets in E. Consider $y \in U$ and $z \in V$. Then there exists $\epsilon > 0$ such that $B(y, \epsilon) \subset U$ and $B(z, \epsilon) \subset V$. Let $\delta > 2d/\epsilon$ and choose the sequences $\{y_k\}$, $\{z_k\}$ in E such that $\|y_n - y_k\| \geqslant d$ and $\|z_n - z_k\| \geqslant d$ for all $n \neq k$, and also

$$\|y_k - \delta y\| = \|z_k - \delta z\| = \delta\epsilon/2.$$

Now, since $Orb(\mathcal{T},x)$ is d-dense in E, thus we have

$$B(y_k,d)\cap Orb(\mathcal{T},x)\neq\emptyset$$

and

$$B(z_k,d)\cap Orb(\mathcal{T},x)\neq\emptyset$$

for all k. Thus, there exist sequences of integers $\{m_k^{(i)}\}_k$, $\{p_k^{(i)}\}_k$ for $i=1,...,n$ such that

$$T_1^{m_k^{(1)}}T_2^{m_k^{(2)}}...T_n^{m_k^{(n)}}x\in B(\delta y,\delta\epsilon),$$

and

$$T_1^{p_k^{(1)}}T_2^{p_k^{(2)}}...T_n^{p_k^{(n)}}x\in B(\delta z,\delta\epsilon)$$

for all k. We can choose $m_{k_0}^{(i)}<p_{k_0}^{(i)}$ for $i=1,...,n$, such that

$$T_1^{m_{k_0}^{(1)}}T_2^{m_{k_0}^{(2)}}...T_n^{m_{k_0}^{(n)}}x\in B(\delta y,\delta\epsilon),$$

and

$$T_1^{p_{k_0}^{(1)}}T_2^{p_{k_0}^{(2)}}...T_n^{p_{k_0}^{(n)}}x\in B(\delta z,\delta\epsilon).$$

Hence we get

$$T_1^{p_{k_0}^{(1)}-m_{k_0}^{(1)}}T_2^{p_{k_0}^{(2)}-m_{k_0}^{(2)}}...T_n^{p_{k_0}^{(n)}-m_{k_0}^{(n)}}(B(\delta y,\delta\epsilon))\cap B(\delta z,\delta\epsilon)\neq\emptyset.$$

Therefore,

$$T_1^{p_{k_0}^{(1)}-m_{k_0}^{(1)}}T_2^{p_{k_0}^{(2)}-m_{k_0}^{(2)}}...T_n^{p_{k_0}^{(n)}m_{k_0}^{(n)}}(B(y,\epsilon))\cap B(z,\epsilon)\neq\emptyset$$

which implies that

$$T_1^{p_{k_0}^{(1)}-m_{k_0}^{(1)}}T_2^{p_{k_0}^{(2)}-m_{k_0}^{(2)}}...T_n^{p_{k_0}^{(n)}-m_{k_0}^{(n)}}(U)\cap V\neq\emptyset.$$

Thus there exists $w\in X$ such that $Orb(\mathcal{T},w)\cap E$ is dense in E. Now since $(T_1^{m_1}T_2^{m_2}...T_n^{m_n})^*$ has no eigenvalue for all $m_i>0$ $(i=1,...,n)$, by Corollary 5.6 in [5], $\mathcal{T}$ is hypercyclic and so the proof is complete. □

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