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Original Research Paper

Caputo Fractional Derivative Inequalities via $(h-m)$-Convexity

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Abstract. The aim of this study is to establish some new Caputo fractional integral inequalities. By applying definition of $(h-m)$-convexity and some straightforward inequalities an upper bound of the sum of left and right sided Caputo fractional derivatives has been established. Furthermore, a modulus inequality and a Hadamard type inequality have been analyzed. These results provide various fractional inequalities for all particular functions deducible from $(h-m)$-convexity, see Remark 1.3.

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1 Introduction

Nobody can deny the importance of fractional calculus in the field of engineering, fluid mechanics, mathematical analysis etc. Many mathematicians have been introduced many articles using fractional calculus

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(see [3, 4, 5, 6] and references therein).
The study on the fractional calculus continued with the contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov for detail (see, [7, 8]). In the realm of the fractional differential equations, Caputo fractional derivative and Riemann-Liouville ones are mostly used. They generalize the ordinary integral and differential operators. However, the fractional derivatives have fewer properties than the corresponding classical ones. On the other hand, besides the smooth requirement, Caputo derivative does not coincide with the classical derivative [9]. For detail of fractional derivatives readers are suggested [1, 2, 7].

Definition 1.1. [7] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous. Then the left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, x > a \tag{1}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, x < b. \tag{2}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^{n}_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^{n}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^{0}_{a+} f)(x) = ({}^C D^{0}_{b-} f)(x) = f(x) \tag{3}$$

where $n = 1$ and $\alpha = 0$.

The aim of this paper is to find fractional inequalities for the Caputo fractional derivatives via $(h-m)$-convex functions. The $(h-m)$-convexity is the generalization of convexity on right half of the real line including zero (see, [10, 11] and references therein). Moreover, these fractional inequalities appear as a compact formulation which contain various induced results for all functions deducible from $(h-m)$-convex functions, see Remark 1.3.

Definition 1.2. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)$-convex function, if f is non-negative and for all $x, y \in [0,b], m \in [0,1]$ and $\alpha \in (0,1)$, one has

$$f(\alpha x + m(1-\alpha)y) \le h(\alpha)f(x) + mh(1-\alpha)f(y). \tag{4}$$

If reverse of the inequality holds, then f is called $(h-m)$-concave function.

For suitable choice of h and m, class of $(h-m)$-convex functions is reduced to the different known classes of functions defined on $[0,b]$.

Remark 1.3. (i) By setting $m = 1$ in (4), it reduces to the definition of h-convex function.
(ii) By setting $h(\alpha) = \alpha$ in (4), it reduces to the definition of m-convex function.
(iii) By setting $h(\alpha) = \alpha$ and $m = 1$ in (4), it reduces to the definition of convex function.
(iv) By setting $h(\alpha) = 1$ and $m = 1$ in (4), it reduces to the definition of p-function.
(v) By setting $h(\alpha) = \alpha^s$ and $m = 1$ in (4), it reduces to the definition of s-convex function of second sense.
(vi) By setting $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$ in (4), it reduces to the definition of Godunova-Levin function.
(vii) By setting $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$ in (4), it reduces to the definition of s-Godunova-Levin function of second kind.

2 Main Results

Firstly, an upper bound of the sum of left and right Caputo fractional derivatives (CFD) has been proved by applying inequalities in the result of the definition of $(h-m)$-convex function. The established upper bound of (CFD) via $(h-m)$-convex function contains upper bounds in particular for h-convex, m-convex and convex functions, also for s-convex function of second sense, Godunova-Levin function and p-function. Further a modulus inequality is established for (CFD) by using $(h-m)$-convexity of $f^{(n+1)}$. At the end an inequality of Hadamard type is obtained. Moreover, presented results have been further studied in particular points in the domain.

Theorem 2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $f \in AC^n[a, b]$, $0 \le a < b$. If $f^{(n)}$ is $(h - m)$-convex, then for $\alpha, \beta > 1$ the following inequality for the Caputo fractional derivatives holds:*

$$\begin{aligned}&\left({}^C D_{a^+}^{\alpha-1} f\right)(x) + \left({}^C D_{b^-}^{\beta-1} f\right)(x) \\ &\le \left(\frac{(x-a)^{n-\alpha+1} f^{(n)}(a)}{\Gamma(n-\alpha+1)} + \frac{(b-x)^{n-\beta+1} f^{(n)}(b)}{\Gamma(n-\beta+1)} \right. \\ &\left. + m f^{(n)}\left(\frac{x}{m}\right) \left(\frac{(b-x)^{n-\beta+1}}{\Gamma(n-\beta+1)} + \frac{(x-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \right) \right) \int_0^1 h(z)dz.\end{aligned} \tag{5}$$

Proof. Using definition of $(h - m)$-convex function for $f^{(n)}$, we have

$$f^{(n)}(t) \le h\left(\frac{x-t}{x-a}\right) f^{(n)}(a) + mh\left(\frac{t-a}{x-a}\right) f^{(n)}\left(\frac{x}{m}\right), \tag{6}$$

where we use the identity

$$t = \frac{x-t}{x-a} a + m \frac{t-a}{x-a} \frac{x}{m}.$$

Also for $\alpha > 0$ and $t \in [a, x]$, we have

$$(x-t)^{n-\alpha} \le (x-a)^{n-\alpha}. \tag{7}$$

Multiplying (6) and (7) and integrating over $[a, x]$, we get

$$\begin{aligned}&\int_a^x (x-t)^{n-\alpha} f^{(n)}(t)dt \le f^{(n)}(a)(x-a)^{n-\alpha} \int_a^x h\left(\frac{x-t}{x-a}\right) dt \\ &+ m f^{(n)}\left(\frac{x}{m}\right)(x-a)^{n-\alpha} \int_a^x h\left(\frac{t-a}{x-a}\right) dt.\end{aligned} \tag{8}$$

Now by using definition of Caputo fractional derivative on left hand side of (8) and by change of variables on its right hand side, we get

$$\left({}^C D_{a^+}^{\alpha-1} f\right)(x) \le \frac{(x-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \left(f^{(n)}(a) + m f^{(n)}\left(\frac{x}{m}\right)\right) \int_0^1 h(z)dz. \tag{9}$$

Similarly using $(t-x)^{n-\beta} \leq (x-b)^{n-\beta}$, $t \in [x,b]$ and definition of $(h-m)$-convexity of $f^{(n)}$ for the identity

$$t = \frac{t-x}{b-x}b + m\frac{b-t}{b-x}\frac{x}{m}$$

one can have

$$\left({}^C D_{b^-}^{\beta-1} f\right)(x) \leq \frac{(b-x)^{n-\beta+1}}{\Gamma(n-\beta+1)}\left(f^{(n)}(b) + m f^{(n)}\left(\frac{x}{m}\right)\right)\int_0^1 h(z)dz. \tag{10}$$

Adding (9) and (10), we get the inequality in (5). □

Corollary 2.2. *By taking $\alpha = \beta$ in (5), we get the following inequality for Caputo fractional derivatives:*

$$\begin{aligned}
&\left({}^C D_{a^+}^{\alpha-1} f\right)(x) + \left({}^C D_{b^-}^{\alpha-1} f\right)(x) \\
&\leq \frac{1}{\Gamma(n-\alpha+1)}\Big((x-a)^{n-\alpha+1} f^{(n)}(a) + (b-x)^{n-\alpha+1} f^{(n)}(b) \\
&+ m f^{(n)}\left(\frac{x}{m}\right)\left((b-x)^{n-\alpha+1} + (x-a)^{n-\alpha+1}\right)\Big)\int_0^1 h(z)dz.
\end{aligned}$$

Remark 2.3. (i) If we put $m = 1$ in (5), then bound of Caputo fractional derivative for h-convex function is established.
(ii) If we put $h(\alpha) = \alpha$ in (5), then bound of Caputo fractional derivative for m-convex function is established.
(iii) If we put $h(\alpha) = \alpha$ and $m = 1$ in (5), then bound of Caputo fractional derivative for convex function is established..
(iv) If we put $h(\alpha) = 1$ and $m = 1$ in (5), then bound of Caputo fractional derivative for p-function is established..
(v) If we put $h(\alpha) = \alpha^s$ and $m = 1$ in (5), then bound of Caputo fractional derivative for s-convex function of second sense is established.
(vi) If we put $h(\alpha) = \frac{1}{\alpha}$ and $m = 1$ in (5), then bound of Caputo fractional derivative for Godunova-Levin function is established.
(vii) If we put $h(\alpha) = \frac{1}{\alpha^s}$ and $m = 1$ in (5), then bound of Caputo fractional derivative for s-Godunova-Levin function of second kind is established.

Theorem 2.4. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^{n+1}[a,b]$, $0\le a<b$. If $|f^{(n+1)}|$ is $(h-m)$-convex, then for $\alpha,\beta>0$ the following inequality for the Caputo fractional derivatives holds:*

$$\left|({}^CD^{\alpha}_{a^+}f)(x)+({}^CD^{\beta}_{b^-}f)(x)-\left(\frac{(x-a)^{n-\alpha}f^{(n)}(a)}{\Gamma(n-\alpha+1)}+\frac{(b-x)^{n-\beta}f^{(n)}(b)}{\Gamma(n-\beta+1)}\right)\right| \tag{11}$$

$$\le\left(\frac{(x-a)^{n-\alpha+1}|f^{(n+1)}(a)|}{\Gamma(n-\alpha+1)}+\frac{(b-x)^{n-\beta+1}|f^{(n+1)}(b)|}{\Gamma(n-\beta+1)}\right.$$

$$\left.+m\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|\left(\frac{(x-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)}+\frac{(b-x)^{n-\beta+1}}{\Gamma(n-\beta+1)}\right)\right)\int_0^1 h(z)dz.$$

Proof. Since $|f^{(n+1)}|$ is $(h-m)$-convex, therefore for $t\in[a,x]$, we have

$$|f^{(n+1)}(t)|\le h\left(\frac{x-t}{x-a}\right)|f^{(n+1)}(a)|+mh\left(\frac{t-a}{x-a}\right)\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|,$$

from which we can write

$$-\left(h\left(\frac{x-t}{x-a}\right)|f^{(n+1)}(a)|+mh\left(\frac{t-a}{x-a}\right)\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|\right) \tag{12}$$

$$\le f^{(n+1)}(t)\le h\left(\frac{x-t}{x-a}\right)|f^{(n+1)}(a)|+mh\left(\frac{t-a}{x-a}\right)\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|.$$

We consider the second inequality of (12), that is

$$f^{(n+1)}(t)\le h\left(\frac{x-t}{x-a}\right)|f^{(n+1)}(a)|+mh\left(\frac{t-a}{x-a}\right)\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|. \tag{13}$$

Now for $\alpha>0$, we have the following inequality

$$(x-t)^{n-\alpha}\le(x-a)^{n-\alpha},\quad t\in[a,x]. \tag{14}$$

Multiplying the last two inequalities and integrating with respect to t over $[a,x]$, we have

$$\int_a^x(x-t)^{n-\alpha}f^{(n+1)}(t)dt \tag{15}$$

$$\le(x-a)^{n-\alpha}\left[|f^{(n+1)}(a)|\int_a^x h\left(\frac{x-t}{x-a}\right)dt+m\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|\right.$$

$$\left.\int_a^x h\left(\frac{t-a}{x-a}\right)dt\right].$$

The left hand side of (15) is calculated as follows

$$\int_a^x (x-t)^{n-\alpha} f^{(n+1)}(t)dt = -f^{(n)}(a)(x-a)^{n-\alpha} + \Gamma(n-\alpha+1)({}^C D_{a^+}^{\alpha} f)(x),$$

while using change of variables in the right hand side of (15), the resulting inequality takes the form as follows

$$\begin{aligned} &({}^C D_{a^+}^{\alpha} f)(x) - \frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} \\ &\le \frac{(x-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \left(|f^{(n+1)}(a)| + m \left| f^{(n+1)}\left(\frac{x}{m}\right) \right| \right) \int_0^1 h(z)dz. \end{aligned} \tag{16}$$

If we consider from (12), the left hand side inequality and proceeding as we did for the right side inequality, we get

$$\begin{aligned} &\frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} - ({}^C D_{a^+}^{\alpha} f)(x) \\ &\le \frac{(x-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \left(|f^{(n+1)}(a)| + m \left| f^{(n+1)}\left(\frac{x}{m}\right) \right| \right) \int_0^1 h(z)dz. \end{aligned} \tag{17}$$

From (16) and (17), we get

$$\begin{aligned} &\left| ({}^C D_{a^+}^{\alpha} f)(x) - \frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} \right| \\ &\le \frac{(x-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \left(|f^{(n+1)}(a)| + m \left| f^{(n+1)}\left(\frac{x}{m}\right) \right| \right) \int_0^1 h(z)dz. \end{aligned} \tag{18}$$

On the other hand for $t \in [x,b]$ using $(h-m)$-convexity of $|f^{(n+1)}|$, we have

$$|f^{(n+1)}(t)| \le h\left(\frac{t-x}{b-x}\right) |f^{(n+1)}(b)| + mh\left(\frac{b-t}{b-x}\right) \left| f^{(n+1)}\left(\frac{x}{m}\right) \right|. \tag{19}$$

Also for $t \in [x,b]$ and $\beta > 0$, we have

$$(t-x)^{n-\beta} \le (b-x)^{n-\beta}. \tag{20}$$

By adopting the same treatment as we have done for (12) and (14), one can obtain from (19) and (20), the following inequality

$$\left|({}^{C}D^{\beta}_{b-}f)^{(n)}(x)-\frac{f^{(n)}(b)(b-x)^{n-\beta}}{\Gamma(n-\beta+1)}\right| \tag{21}$$
$$\leq\frac{(b-x)^{n-\beta+1}}{\Gamma(n-\beta+1)}\left(|f^{(n+1)}(b)|+m\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|\right)\int_0^1 h(z)dz.$$

By combining the inequalities (18) and (21) via triangular inequality, we get the required inequality. □

Corollary 2.5. *By taking $\alpha=\beta$ in (11), then we get the following inequality for Caputo fractional derivatives:*

$$\Big|({}^{C}D^{\alpha}_{a+}f)(x)+({}^{C}D^{\alpha}_{b-})f(x)$$
$$-\frac{1}{\Gamma(n-\alpha+1)}\left((x-a)^{n-\alpha}f^{(n)}(a)+(b-x)^{n-\alpha}f^{(n)}(b)\right)\Big|$$
$$\leq\frac{1}{\Gamma(n-\alpha+1)}\Big((x-a)^{n-\alpha+1}|f^{(n+1)}(a)|+(b-x)^{\alpha+1}|f^{(n+1)}(b)|$$
$$+m\left|f^{(n+1)}\left(\frac{x}{m}\right)\right|\left((x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}\right)\Big)\int_0^1 h(z)dz.$$

Remark 2.6. Axioms (i)-(vii) of Remark 2.3 for Theorem 2.1 are valid for Theorem 2.4.

First of all we prove the following lemma.

Lemma 2.7. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,b]$, $0\leq a<b$. If $f^{(n)}$ is $(h-m)$-concave and $f^{(n)}\left(\frac{a+b-x}{m}\right)=f^{(n)}(x)$, then the following inequality holds:*

$$f^{(n)}\left(\frac{a+b}{2}\right)\geq(m+1)h\left(\frac{1}{2}\right)f^{(n)}(x),\ x\in[a,b]. \tag{22}$$

Proof. We have

$$\frac{a+b}{2}=\frac{1}{2}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)+\frac{1}{2}\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right). \tag{23}$$

Since $f^{(n)}$ is $(h-m)$-concave, therefore we have

$$\begin{aligned}
&f^{(n)}\left(\frac{a+b}{2}\right) \qquad (24)\\
&\geq h\left(\frac{1}{2}\right)\left[f^{(n)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)+mf^{(n)}\left(\frac{x-a}{m(b-a)}a+\frac{b-x}{m(b-a)}b\right)\right]\\
&= h\left(\frac{1}{2}\right)\left(f^{(n)}(x)+mf^{(n)}\left(\frac{a+b-x}{m}\right)\right).
\end{aligned}$$

Now by using the condition $f^{(n)}\left(\frac{a+b-x}{m}\right)=f^{(n)}(x)$, inequality in (22) can be obtained. □

Theorem 2.8. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in AC^n[a,b]$, $0\leq a<b$. If $f^{(n)}$ is $(h-m)$-concave, then for $\alpha,\beta>0$ the following inequality for the Caputo fractional derivatives holds:*

$$\begin{aligned}
&\left(mf^{(n)}\left(\frac{b}{m}\right)+f^{(n)}(a)\right) \qquad (25)\\
&\leq \frac{\Gamma(n-\beta)({}^{C}D^{\beta}_{a+}f)(b)}{(b-a)^{n-\beta}}+\frac{\Gamma(n-\alpha)({}^{C}D^{\alpha}_{b-}f)(a)}{(b-a)^{n-\alpha}}\\
&\leq \frac{1}{(m+1)h\left(\frac{1}{2}\right)}\left(\frac{1}{n-\beta}+\frac{1}{n-\alpha}\right)f^{(n)}\left(\frac{a+b}{2}\right).
\end{aligned}$$

Proof. For $\alpha,\beta>0$, $n-\alpha-1\leq 0$ and

$$(b-a)^{n-\alpha-1}\leq (x-a)^{n-\alpha-1}. \qquad (26)$$

Since $f^{(n)}$ is $(h-m)$ concave, therefore we have

$$mh\left(\frac{x-a}{b-a}\right)f^{(n)}\left(\frac{b}{m}\right)+h\left(\frac{b-x}{b-a}\right)f^{(n)}(a)\leq f^{(n)}(x). \qquad (27)$$

Multiplying (26) with (27) and then integrating over $[a,b]$, we get

$$\begin{aligned}
&(b-a)^{n-\alpha-1}\left[mf^{(n)}\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)dx\right.\\
&\left.+f^{(n)}(a)\int_a^b h\left(\frac{b-x}{b-a}\right)dx\right]\leq \int_a^b (x-a)^{n-\alpha-1}f^{(n)}(x)dx
\end{aligned}$$

which implies

$$
\begin{aligned}
&(-1)^n\Gamma(n-\alpha)({}^CD^{\alpha}_{b-}f)(a) \\
&\geq (b-a)^{n-\alpha}\left(mf^{(n)}\left(\frac{b}{m}\right)+f^{(n)}(a)\right)\int_0^1 h(z)dz.
\end{aligned} \tag{28}
$$

Also

$$
(b-a)^{n-\beta-1}\leq (b-x)^{n-\beta-1}. \tag{29}
$$

Multiplying (29) with (27), and then integrating over $[a,b]$, we get

$$
\begin{aligned}
&(b-a)^{n-\beta-1}\left[mf^{(n)}\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)dx\right. \\
&\left.+f^{(n)}(a)\int_a^b h\left(\frac{b-x}{b-a}\right)dx\right]\leq \int_a^b (b-x)^{n-\beta-1}f^{(n)}dx
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(-1)^n\Gamma(n-\beta)({}^CD^{\beta}_{a+}f)(a) \\
&\geq (b-a)^{n-\beta}\left(mf^{(n)}\left(\frac{b}{m}\right)+f^{(n)}(a)\right)\int_0^1 h(z)dz.
\end{aligned} \tag{30}
$$

From (28) and (30), we get

$$
\begin{aligned}
&\frac{\Gamma(n-\beta)({}^CD^{\beta}_{a+}f)(b)}{(b-a)^{n-\beta}}+\frac{\Gamma(n-\alpha)({}^CD^{\alpha}_{b-}f)(a)}{(b-a)^{n-\alpha}} \\
&\geq \left(mf^{(n)}\left(\frac{b}{m}\right)+f^{(n)}(a)\right)\int_0^1 h(z)dz.
\end{aligned} \tag{31}
$$

Now from Lemma 1, we have

$$
(m+1)h\left(\frac{1}{2}\right)f^{(n)}(x)\leq f^{(n)}\left(\frac{a+b}{2}\right). \tag{32}
$$

Multiplying (32) by $(x-a)^{n-\alpha-1}$ and integrating over $[a,b]$, we have

$$
\int_a^b (x-a)^{n-\alpha-1}f^{(n)}(x)dx\leq \frac{f^{(n)}\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)}\int_a^b (x-a)^{n-\alpha-1}dx,
$$

$$\Gamma(n-\alpha)({}^{C}D^{\alpha}_{b^-}f)(a) \leq \frac{(b-a)^{n-\alpha}}{(n-\alpha)(m+1)h\left(\frac{1}{2}\right)} f^{(n)}\left(\frac{a+b}{2}\right). \tag{33}$$

Multiplying (32) by $(b-x)^{n-\beta-1}$ and integrating over $[a,b]$, we have

$$\int_a^b (b-x)^{n-\beta-1} f^{(n)}(x)dx \leq \frac{f^{(n)}\left(\frac{a+b}{2}\right)}{(m+1)h\left(\frac{1}{2}\right)} \int_a^b (b-x)^{n-\beta-1}dx,$$

$$\Gamma(n-\beta)({}^{C}D^{\beta}_{a^+}f)(x) \leq \frac{(b-a)^{n-\beta}}{(n-\beta)(m+1)h\left(\frac{1}{2}\right)} f^{(n)}\left(\frac{a+b}{2}\right). \tag{34}$$

Adding (33) and (34), we obtain

$$\begin{aligned} &\frac{\Gamma(n-\beta)({}^{C}D^{\beta}_{a^+}f)(b)}{(b-a)^{n-\beta}} + \frac{\Gamma(n-\alpha)({}^{C}D^{\alpha}_{b^-}f)(a)}{(b-a)^{n-\alpha}} \\ &\leq \frac{1}{(m+1)h\left(\frac{1}{2}\right)}\left(\frac{1}{n-\beta}+\frac{1}{n-\alpha}\right) f^{(n)}\left(\frac{a+b}{2}\right). \end{aligned} \tag{35}$$

From (31) and (35), we get required inequality. □

Corollary 2.9. *By taking $\alpha = \beta$ in (25), then we get the following inequality for Caputo fractional derivatives:*

$$\begin{aligned} \left(mf^{(n)}\left(\frac{b}{m}\right) + f^{(n)}(a)\right) &\leq \frac{\Gamma(n-\alpha)}{(b-a)^{n-\alpha}}\left(({}^{C}D^{\beta}_{a^+}f)(b) + ({}^{C}D^{\alpha}_{b^-}f)(a)\right) \\ &\leq \frac{2}{(m+1)(n-\alpha)h\left(\frac{1}{2}\right)} f^{(n)}\left(\frac{a+b}{2}\right). \end{aligned}$$

Remark 2.10. Axioms (i)-(vii) of Remark 2.3 for Theorem 2.1 are valid for Theorem 2.8.

Next we give the following results as an application of previous established results. First we apply Theorem 2.1, and get the following result.

Theorem 2.11. *Under the assumptions of Theorem 2.1, we have*

$$\begin{aligned} &\left({}^{C}D^{\alpha-1}_{a^+}f\right)(b) + \left({}^{C}D^{\beta-1}_{b^-}f\right)(a) \\ &\leq \left(\frac{(b-a)^{n-\alpha+1}f^{(n)}(a)}{\Gamma(n-\alpha+1)} + \frac{(b-a)^{n-\beta+1}f^{(n)}(b)}{\Gamma(n-\beta+1)}\right. \\ &\left.+m\left(f^{(n)}\left(\frac{a}{m}\right)\frac{(b-a)^{n-\beta+1}}{\Gamma(n-\beta+1)} + f^{(n)}\left(\frac{b}{m}\right)\frac{(b-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)}\right)\right)\int_0^1 h(z)dz. \end{aligned} \tag{36}$$

Proof. If we take $x = a$ in (5), then we get following inequality

$$\left({}^{C}D_{b^-}^{\beta-1}f\right)(a) \le \frac{(b-a)^{n-\beta+1}}{\Gamma(n-\beta+1)}\left(f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m}\right)\right)\int_0^1 h(z)dz. \tag{37}$$

If we take $x = b$ in (5), then we get following inequality

$$\left({}^{C}D_{a^+}^{\alpha-1}f\right)(b) \le \frac{(b-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)}\left(f^{(n)}(a) + mf^{(n)}\left(\frac{b}{m}\right)\right)\int_0^1 h(z)dz. \tag{38}$$

Adding (37) and (38), we get required inequality in (36). $\square$

Corollary 2.12. *By taking $\alpha = \beta$ in (36), then we get the following inequality for Caputo fractional derivatives:*

$$\begin{aligned}
&\left({}^{C}D_{a^+}^{\alpha-1}f\right)(b) + \left({}^{C}D_{b^-}^{\alpha-1}f\right)(a) \\
&\le \frac{(b-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)}\left(f^{(n)}(a) + f^{(n)}(b) + m\left(f^{(n)}\left(\frac{a}{m}\right) + f^{(n)}\left(\frac{b}{m}\right)\right)\right) \\
&\times \int_0^1 h(z)dz.
\end{aligned}$$

Next we apply Theorem 2.4, and get the following result.

Theorem 2.13. *Under the assumptions of Theorem 2.4, we have*

$$\left|({}^{C}D_{a^+}^{\alpha}f)(b) + ({}^{C}D_{b^-}^{\beta}f)(a) - \left(\frac{(b-a)^{n-\alpha}f^{(n)}(a)}{\Gamma(n-\alpha+1)} + \frac{(b-a)^{n-\beta}f^{(n)}(b)}{\Gamma(n-\beta+1)}\right)\right| \tag{39}$$

$$\begin{aligned}
&\le \left(\frac{(b-a)^{n-\alpha+1}|f^{(n+1)}(a)|}{\Gamma(n-\alpha+1)} + \frac{(b-a)^{n-\beta+1}|f^{(n+1)}(b)|}{\Gamma(n-\beta+1)}\right. \\
&\left. + m\left(\frac{(b-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)}\left|f^{(n+1)}\left(\frac{b}{m}\right)\right| + \frac{(b-a)^{n-\beta+1}}{\Gamma(n-\beta+1)}\left|f^{(n+1)}\left(\frac{a}{m}\right)\right|\right)\right) \\
&\times \int_0^1 h(z)dz.
\end{aligned}$$

Proof. If we take $x = a$ in (11), then we get following inequality

$$\left|({}^C D^{\beta}_{b^-} f)^{(n)}(a) - \frac{f^{(n)}(b)(b-a)^{n-\beta}}{\Gamma(n-\beta+1)}\right| \tag{40}$$
$$\leq \frac{(b-a)^{n-\beta+1}}{\Gamma(n-\beta+1)} \left(|f^{(n+1)}(b)| + m\left|f^{(n+1)}\left(\frac{a}{m}\right)\right|\right) \int_0^1 h(z)dz.$$

If we take $x = b$ in (11), then we get following inequality

$$\left|({}^C D^{\alpha}_{a^+} f)(b) - \frac{f^{(n)}(a)(b-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}\right| \tag{41}$$
$$\leq \frac{(b-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \left(|f^{(n+1)}(a)| + m\left|f^{(n+1)}\left(\frac{b}{m}\right)\right|\right) \int_0^1 h(z)dz.$$

Adding (40) and (41), we get required inequality in (39). □

Corollary 2.14. *By taking $\alpha = \beta$ in (39), then we get the following inequality for Caputo fractional derivatives:*

$$\left|({}^C D^{\alpha}_{a^+} f)(b) + ({}^C D^{\beta}_{b^-} f)(a) - \frac{(b-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(f^{(n)}(a) + f^{(n)}(b)\right)\right|$$
$$\leq \frac{(b-a)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \Bigg(|f^{(n+1)}(a)| + |f^{(n+1)}(b)|$$
$$+m\left(\left|f^{(n+1)}\left(\frac{b}{m}\right)\right| + \left|f^{(n+1)}\left(\frac{a}{m}\right)\right|\right)\Bigg) \int_0^1 h(z)dz.$$

By applying Theorem 2.8, similar results can be established, therefore we leave it for reader.

Concluding remarks

This paper have been prepared to address Caputo fractional integral inequalities via functions whose nth derivatives are $(h-m)$-convex. It is remarkable to mention that the presented results contain Caputo fractional inequalities for h-convex functions, m-convex functions, convex functions, Godunova-Levin functions, p-functions and s-convex functions in second sense on the domain of nonnegative real numbers. Further in application point of the results of this paper may be useful in studying uniqueness of solutions fractional differential equations, analyzing fractional models of different dynamic systems, complex systems etc. In future we will try to study existence of solutions of fractional systems under constraints of such fractional derivative inequalities.

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