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Original Research Paper

The Application of Fuzzy Laplace Transform by Using Generalized Hukuhara Derivative

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Abstract. In this research an analytic method is used to solve fuzzy heat equation with fuzzy initial values, which is based on generalized Hukuhara differentiability. At first, we define fuzzy laplace transform in t considering x as a parameter on fuzzy functions $u(x,t)$ and their derivatives by using generalized Hukuhara differentiability. The fuzzy laplace transform is used in an analytic method for solving the fuzzy partial differential equation with generalized Hukuhara differantiation. Finally, the method is explained by presenting examples.

AMS Subject Classification: MSC code1; MSC code 2; more
Keywords and Phrases: Fuzzy-number, Fuzzy-valued function, Generalized differentiability, Fuzzy partial differential equation, Fuzzy Laplace transform.

1 Introduction

Often the engineering design problems deal with a set of partial differential equations (PDEs), such as heat transfer and solid and fluid

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mechanicals [10, 6]. When a physical problem is transformed into a deterministic parabolic partial differential equation, we cannot usually be sure that this modeling is perfect, and the initial and boundary value may not be known exactly. If the underlying structure be fuzzy numbers ,then instead of a deterministic problem, we get a fuzzy partial differential equation with fuzzy initial and boundary values. The concept of the fuzzy derivative have been discussed[8, 9, 10, 11, 12, 13, 15].
To overcome some shortcomings of previous methods, Bede and Gal introduced the weakly generalized differential of a fuzzy number valued function. Also, they presented the sense of strongly generalized differentiability for fuzzy-valued functions [4]. The strongly generalized differentiability defined by considering lateral H-derivatives. Clearly the disadvantage of strongly generalized differentiability of a function in comparison H- differentiability is that, a fuzzy differential equation has no unique solution. Thus, a generalization of the Hukuhara difference and derivative for interval valued functions presented by Stefanini and Bede [18]. They shown that, this concept of differentiability has relationships with weakly generalized differentiability and strongly generalized differentiability [17]. Recently, a general difference researched in[16]. Afterwards, a generalized differentiability ideas based on general difference for fuzzy valued function recently examined in[5]. Also for solving (FPDEs) ,J. Buckley and T. Feuring proposed a procedure to examine solutions of fuzzy partial differential equations [7]. Also numerical method to solve FPDE have been discussed [1, 2, 14, 18] . Rest of the paper is organized as follows: In section 2, some basic definitions and results are stated with will be used throughout this paper. In section 3, fuzzy Laplace transform is defined on to dimensional fuzzy functions and their derivatives by using generalized Hukuhara differentiation. In section 4, the fuzzy laplace transform is used in a analytic method for solving to fuzzy partial differential equation by use of generalized Hukuhara differentiation to illustrative examples, and the conclusion is drawn in section 5.

2 Preliminaries

We now recall some definitions needed through the paper.

In what follows, we briefly recall the basic definitions and properties of the generalized Hukuhara derivative. We denote by R_F, the set of fuzzy numbers that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which defined over the real line. For $0 < \alpha \leq 1,\ set\ [u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$,and $[u]^0 = cl\{x \in R^n | u(x) > 0\}$. We represent $[u]^\alpha = [\overset{-}{u}(\alpha), \overset{+}{u}(\alpha)]$, so if $u \in R_F$, For arbitrary $u, v \in R_F$ and $k \in R$,the addition and scalar multiplication are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha, [ku]^\alpha = k[u]^\alpha$ respectively.

A triangular fuzzy number defined as a fuzzy set in R_F , that is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a \leq b \leq c$ such that $\overset{-}{u}(\alpha) = a + (b-a)\alpha$ and $\overset{+}{u}(\alpha) = c - (c-b)\alpha$ are endpoints of α-level sets for all $\alpha \in [0, 1]$.

Also, $\ominus$ is the Hukuhara difference(H-difference), it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 2.1. *[6] The generalized Hukuhara difference of two fuzzy numbers $u, v \in R_F$ is defined as follows*

$$u \ominus_{gH} v = w \iff \begin{cases} (i).u = v + w; \\ or(ii).v = u + (-1)w. \end{cases}$$

Definition 2.2. *[3] In terms of α-levels we have*

$$\begin{aligned}[u \ominus_{gH} v]^\alpha =& [\min\{\overset{-}{u}(\alpha) - \overset{-}{v}(\alpha), \overset{+}{u}(\alpha) - \overset{+}{v}(\alpha)\}, \\ & max\{\overset{-}{u}(\alpha) - \overset{-}{v}(\alpha), \overset{+}{u}(\alpha) - \overset{+}{v}(\alpha)\}]\end{aligned}$$

and if the H-difference exists, then $u \ominus v = u \ominus_{gH} v$; the conditions for the existence of $w = u \ominus_{gH} v \in R_F$ are

$$case(i) \begin{cases} \overset{-}{w}(\alpha) = \overset{-}{u}(\alpha) - \overset{-}{v}(\alpha)\ and\ \overset{+}{w}(\alpha) = \overset{+}{u}(\alpha) - \overset{+}{v}(\alpha), \quad \forall \alpha \in [0,1], \\ with\ \overset{-}{w}(\alpha)\ increasing\ \ , \overset{+}{w}(\alpha)\ decreasing\ \ , \overset{-}{w}(\alpha) \leq \overset{+}{w}(\alpha), \end{cases}$$

$$case(ii) \begin{cases} \overset{-}{w}(\alpha) = \overset{+}{u}(\alpha) - \overset{+}{v}(\alpha)\ and\ \overset{+}{w}(\alpha) = \overset{-}{u}(\alpha) - \overset{-}{v}(\alpha), \forall \alpha \in [0,1], \\ with\ \overset{-}{w}(\alpha)\ increasing\ \ , \overset{+}{w}(\alpha)\ decreasing\ \ , \overset{-}{w}(\alpha) \leq \overset{+}{w}(\alpha), \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

Definition 2.3. *[5]The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \to R_F$ at $x_0 \epsilon (a, b)$ is defined as*

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h} \tag{1}$$

if $f'_{gH}(x_0) \in R_F$ satisfying (1) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition 2.4. *[5] Let $f : [a, b] \to R_F$ and $x_0 \in (a, b)$, with $f^-(x, \alpha)$ and $f^+(x, \alpha)$ both differentiable at x_0. Also, we say that*

- *f is $[(i) - gH]$- differentiable at x_0 if*

$$f'_{i.gH}(x_0; \alpha) = [(f^-)'(x_0 : \alpha), (f^+)'(x_0 : \alpha)], \qquad 0 \le \alpha \le 1, \tag{2}$$

- *f is $[(ii) - gH]$ differentiable at x_0 if*

$$f'_{ii,gH}(x_0; \alpha) = [(f^+)'(x_0 : \alpha), (f^-)'(x_0 : \alpha)], \qquad 0 \le \alpha \le 1, \tag{3}$$

Definition 2.5. *[17]*

We say that a point $x_0 \in (a, b)$, is a switching point for the differentiability of f , if in any neighborhood V of x_0 there exist points $x_1 < x_2 < x_3$ such that

type(I) *at x_1(2) holds while (3) does not hold and at x_2(3) holds and (2) does not hold, or*

type(II) *at x_1(3) holds while (2) does not hold and at x_2(2) holds and (3) does not hold.*

Definition 2.6. *[3] Let $f : (a, b) \to \mathbb{R}_F$ be a fuzzy-valued function and $c \in (a, b)$. The fuzzy-valued function f is gH-differentiable at c if $\lim_{x \to c} \frac{f(x) \ominus_{gH} f(c)}{x - c}$ exists in $\mathbb{R}_F$, it is called the gH-derivative of f at c, and it is denoted $f'_{gH}(c)$.*

Definition 2.7. *[3] Let $f:(a,b)\to\mathbb{R}_F$. We say that $f(x)$ is gH-differentiable of the 2^{th}–order at $x_0\in(a,b)$ whenever the function $f(x)$ is gH-differentiable of the order i, i = 0, 1, at x_0, $f^{(i)}_{gH}(x_0)\in\mathbb{R}_F$, moreover there is not any switching point on (a,b). Then there exist $f''_{gH}(x_0)\in\mathbb{R}_F$ such that $f''_{gH}(x_0)=\lim_{h\to 0}\frac{f'_{gH}(x_0+h)\ominus_{gH}f'_{gH}(x_0)}{h}$. if $f'_{gH}(x_0+h)\ominus_{gH} f'_{gH}(x_0)\in\mathbb{R}_F$.*

Definition 2.8. *[3]*

Let $f:[a,b]\to\mathbb{R}_F$ and $f'_{gH}(x)$, gH-differentiable at $x_0\in(a,b)$, moreover there isn't any switching point on (a,b) and $(f^-)'(x;\alpha)$ and $(f^+)'(x;\alpha)$ both differentiable at x_0. We say that

- *$f'_{gH}(x)$ is $[(i)-gH]$-differentiable whenever the type of gH-differentiability $f(x)$ and $f'_{gH}(x)$ are the same:*

$$f''_{i.gH}(x_0;\alpha)=[(f^-)''(x_0;\alpha)\ ,\ (f^+)''(x_0;\alpha)],\quad 0\leq\alpha\leq 1$$

- *$f'_{gH}(x)$ is $[(ii)-gH]$-differentiable if the type of gH-differentiability $f(x)$ and $f'_{gH}(x)$ are the different:*

$$f''_{ii.gH}(x_0;\alpha)=[(f^+)''(x_0;\alpha)\ ,\ (f^-)''(x_0;\alpha)],\quad 0\leq\alpha\leq 1$$

Theorem 2.9. *[3] Let $f(x,t):D\to R_F$ be [gH-p]-differentiable with respect to t and $c\in R^+$. Then*

$$\partial_{t_{gH}}(c\odot f)(x,t)\ \textit{exist and}\ \partial_{t_{gH}}(c\odot f)(x,t)=c\odot\partial_{t_{gH}}f(x,t).$$

Definition 2.10. *[3] Let $f(x,t):D\to\mathbb{R}_f,(x_0,t_0)\in D$ and $f^-(x,t;\alpha)$, $f^+(x,t;\alpha)$ are real value function and partial differentiable w.r.t. x we say that*

- $f(x,t)\ \ is\ \ [(i)-p]-differentiable\ \ w.r.t\ \ x\ \ at\ \ (x_0,t_0)\ \ if$

$$\partial_{x_i.gH}f(x_0,t_0;\alpha)=[\partial_x f^-(x_0,t_0;\alpha),\partial_x f^+(x_0,t_0;\alpha)].\qquad(4)$$

- $f(x,t)$ *is* $[(ii)-p]-differentiable$ *w.r.t* x *at* (x_0,t_0) *if*

$$\partial_{x_{ii.gH}} f(x_0,t_0;\alpha) = [\partial_x f^+(x_0,t_0;\alpha), \partial_x f^-(x_0,t_0;\alpha)]. \quad (5)$$

Definition 2.11. *[3] For any fixed ξ_0, we say that $(\xi_0,t) \in D$ is a switching points for the differentiability of $f(x,t)$ with respect to x, if in any neighborhood V of (ξ_0,t) there exist points $(x_1,t) < (\xi_0,t) < (x_2,t)$ such that*

type(I) *at (x_1,t)(4) holds while (5) does not hold and at (x_2,t)(5) holds and (4) does not hold for all t, or*

type(II) *at (x_1,t)(5) holds while (4) does not hold and at (x_2,t)(4) holds and (5) does not hold for all t.*

Definition 2.12. *[3] Let $f(x,t) : D \to R_F$ and $\partial_x f(x,t)$ are [gH-p]-differentiable at $(x_0,t_0) \in D$ with respect to x, moreover there is not any switching point on D. We say that*

- *$\partial_{xx_{gH}} f(x,t)$is $[(i)-p]$-differentiable w.r.t x if the type of $[gH-p]$-differentiability of both $f(x,t)$ and $\partial_{x_{gH}} f(x,t)$ are the same:*

$$\partial_{xx_{i.gH}} f(x_0,t_0;\alpha) = [\partial_{xx} f^-(x_0,t_0;\alpha) \ \ \partial_{xx} f^+(x_0,t_0;\alpha)].$$

- *$\partial_{xxgH} f(x,t)$ is $[(ii)-p]$-differentiable w.r.t x if the type of $[gH-p]$-differentiability $f(x,t)$ and $\partial_{xgH} f(x,t)$are different:*

$$\partial_{xx_{ii.gH}} f(x_0,t_0;\alpha) = [\partial_{xx} f^+(x_0,t_0;\alpha) \ \ \partial_{xx} f^-(x_0,t_0;\alpha)].$$

Lemma 2.13. *[3] Consider $f : D \to R_f$ as a fuzzy continuous function. Assume that f is [gH-p]-differentiable with respect to x , with no switching point in the interval $[a,s]$ and fuzzy continuous, then we have*

$$\int_a^s \partial_{t_{gH}} f(x,t)dt = f(x,s) \ominus_{gH} f(x,a).$$

In the following section, we propose fundamental properties of fuzzy laplace transform for fuzzy partial derivatives.

3 Fuzzy laplace transform for fuzzy functions and their partial derivatives by using generalized Hukuhara differentiality

Given a fuzzy function u(x,t) defined for all $t > 0$ and assumed to be bounded we can apply the fuzzy laplace transform in t considering x as a parameter.

$$L[u(x,t)] = \int_0^{\infty} e^{-st} \odot u(x,t)dt = U(x,s)$$

or

$$L[u(x,t)] = (\int_0^{\infty} e^{-st}\bar{u}(x,t,\alpha)dt, \int_0^{\infty} e^{-st}\overset{+}{u}(x,t,\alpha)dt)$$

Also by using the definition of classical laplace transform:
$l[\bar{u}(x,t,\alpha)] = \int_0^{\infty} e^{-st}\bar{u}(x,t,\alpha)dt$ and $l[\overset{+}{u}(x,t,\alpha)] = \int_0^{\infty} e^{-st}\overset{+}{u}(x,t,\alpha)dt$
Then we follow:

$$L[u(x,t)] = (l[\bar{u}(x,t,\alpha)], l[\overset{+}{u}(x,t,\alpha)])$$

In application to FPDES we need the following:

$$L[\partial_{t_{gH}} u(x,t)] = \int_0^{\infty} e^{-st}\partial_{t_{gH}} u(x,t)dt = \partial_{t_{gH}} U(x,s)$$

Also, by using theorm 2.1, we have:
$L[\partial_{t_{gH}} u(x,t)] = s \odot L[u(x,t)] \ominus_{gH} u(x,0)$ where u is (i)-generalized Hukuhara differentiable
or
$L[\partial_{t_{gH}} u(x,t)] = \ominus u(x,0) \ominus_{gH} (\ominus s \odot L[u(x,t)])$
where u is (ii)generalized Hukuhara differentiable.
And, by using the theorem (2.1) we obtain the following alternatives for solving:
Case I: If we consider u and $\partial_{t_{gH}} u$ by using (i)-generalized Hukuhara differentiable, then

$$L[\partial_{tt_{gH}} u(x,t)] = s^2 \odot L[u(x,t)] \ominus_{gH} s \odot u(x,0) \ominus_{t_{gH}} \partial_{t_{gH}} u(x,0).$$

Case II: were u is (ii)generalized Hukuhara differentiable and $\partial_{t_{gH}} u$ is (ii)- generalized Hukuhara differentiable then

$$L[\partial_{tt_{gH}} u(x,t)] = s^2 \odot L[u(x,t)] \ominus_{gH} s \odot u(x,0) \ominus \partial_{gH} u(x,0).$$

Case III: were u is (i)– generalized Hukuhara differentiable and $\partial_{t_{gH}} u$ is (ii)-generalized Hukuhara differentiable then

$$L[\partial_{tt_{gH}} u(x,t)] = \ominus_{gH}(\ominus s^2) \odot L[u(x,t)] \ominus s \odot u(x,0) \ominus \partial_{gH} u(x,0).$$

Case IV:

$$L[\partial_{tt_{gH}} u(x,t)] = \ominus_{gH}(\ominus s^2) \odot L[u(x,t)] \ominus s \odot u(x,0) \ominus_{gH} \partial_{t_{gH}} u(x,0)$$

. Where u is (ii)-generalized Hukuhara differentiable and $\partial_{t_{gH}} u$ is (i)-generalized Hukuhara differentiable.
We also need the corresponding transforms of the x derivatives

$$L[\partial_{x_{gH}} u(x,t)] = \int_0^\infty e^{-st} \partial_{x_{gH}} u(x,t) dt = \partial_{x_{gH}} U(x,s),$$

or

$$L[\partial_{x_{gH}} u(x,t)] = (\int_0^\infty e^{-st} \partial_{x_{gH}} \bar{u}(x,t,\alpha) dt \quad , \int_0^\infty e^{-st} \partial_{x_{gH}} \overset{+}{u}(x,t,\alpha) dt)$$

Also by using the definition of classical Laplace transform:

$$L[\partial_{x_{gH}} \bar{u}(x,t,\alpha)] = \int_0^\infty e^{-st} \partial_{x_{gH}} \bar{u}(x,t,\alpha) dt = \partial_{x_{gH}} \bar{U}(x,s,\alpha)$$

and

$$L[\partial_{x_{gH}} \overset{+}{u}(x,t,\alpha)] = \int_0^\infty e^{-st} \partial_{x_{gH}} \overset{+}{u}(x,t,\alpha) dt = \partial_{x_{gH}} \overset{+}{U}(x,s,\alpha)$$

If number of (ii)-generalized Hukuhara derivative is even, then we have:

$$L[\partial_{xx_{gH}} u(x,t)] = \int_0^\infty e^{-st} \partial_{xx_{gH}} u(x,t) dt$$

$$= (\int_0^\infty e^{-st}\partial_{xx_{gH}}\overset{-}{u}(x,t,\alpha)dt, \int_0^\infty e^{-st}\partial_{xx_{gH}}\overset{+}{u}(x,t,\alpha)dt)$$

$$= (l[\partial_{xx_{gH}}\overset{-}{u}(x,t,\alpha)], l[\partial_{xx_{gH}}\overset{+}{u}(x,t,\alpha)])$$

$$= (\partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha), \partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha))$$

If number of(ii)-generalized Hukuhara derivative is odd, then we have:

$$\begin{aligned} L[\partial_{xx_{gH}}u(x,t)] &= \int_0^\infty e^{-st}\partial_{xx_{gH}}u(x,t)dt \\ &= (\int_0^\infty e^{-st}\partial_{xx_{gH}}\overset{+}{u}(x,t,\alpha)dt, \int_0^\infty e^{-st}\partial_{xx_{gH}}\overset{-}{u}(x,t,\alpha)dt) \\ &= (l[\partial_{xx_{gH}}\overset{+}{u}(x,t,\alpha)], l[\partial_{xx_{gH}}\overset{-}{u}(x,t,\alpha)]) \\ &= (\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha), \partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha)) \end{aligned}$$

Using this representation, we have the following examples.

4 Solving fuzzy partial generalized Hukuhara differential equations by using fuzzy laplace transform

There is a variety of heat conduction problems, which although linear, are not readily solved using the separation of variables method. Fortunately, fuzzy laplace method not based on separation of variables are available for the solution of these heat conduction problems. In this case, examples are used to illustrate the application of the method. The laplace transform method makes use of the laplace transform to convert the original heat equation into ordinary differential equations which is more readily solved. In this section, we present examples to illustrate the laplace trans- form method for solving FPDE.

Example 1. Consider the problem

$$\partial_{x_{gH}}u(x,t) + \partial_{t_{gH}}u(x,t) = \tilde{1}x, \quad x > 0, \quad t > 0$$

With boundary and initial condition

$$u(0,t) = \frac{\tilde{1}}{2} \quad t > 0 \quad and \quad u(x,0) = 0, \quad x > 0$$

$$Where: \tilde{1} = (.8 + .2\alpha, 1.5 - .5\alpha) \quad and \quad \frac{\tilde{1}}{2} = (\frac{1}{2}\alpha, 1 - \frac{1}{2}\alpha)$$

By using fuzzy laplace transform method , we have:

$$L[\partial_{x_{gH}} u(x,t)] \oplus L[\partial_{t_{gH}} u(x,t)] = L[\tilde{1}x] \tag{6}$$

$$L[u(0,t)] = \frac{1}{s}(\frac{\tilde{1}}{2})$$
$$L[u(x,0)] = 0$$

The solutions of Eq.(6) can be divided into four cases.
Case I : If we consider $\partial_{x_{gH}} u_(x,t)$ is (i)- generalized Hukuhara differentiable and $\partial_{t_{gH}} u_(x,t)$ is (i)- generalized Hukuhara differentiable then we have:

$$\partial_{x_{gH}} U(x,s) \oplus (s \odot U(x,s)) \ominus_{gH} u(x,0) = \tilde{1}\frac{x}{s}$$

or

$$\begin{cases} i)\ (\partial_{x_{gH}} \overset{-}{U}(x,s,\alpha), \partial_{x_{gH}} \overset{+}{U}(x,s,\alpha)) \oplus (s\overset{-}{U}(x,s,\alpha) - \overset{-}{u}(x,0,\alpha), s\overset{+}{U}(x,s,\alpha) \\ \quad - \overset{+}{u}(x,0,\alpha)) = (\overset{-}{1}\frac{x}{s}, \overset{+}{1}\frac{x}{s}) \\ ii)\ (\partial_{x_{gH}} \overset{-}{U}(x,s,\alpha), \partial_{x_{gH}} \overset{+}{U}(x,s,\alpha)) \oplus (s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha), s\overset{-}{U}(x,s,\alpha) \\ \qquad - \overset{-}{u}(x,0,\alpha)) = (\overset{-}{1}\frac{x}{s}, \overset{+}{1}\frac{x}{s}) \end{cases}$$

For (i), We have two crisp linear systems which can be displayed as follows:

$$\begin{cases} \partial_{x_{gH}} \overset{-}{U}(x,s,\alpha) + s\overset{-}{U}(x,s,\alpha) - \overset{-}{u}(x,0,\alpha) = \overset{-}{1}\frac{x}{s} \\ \partial_{x_{gH}} \overset{+}{U}(x,s,\alpha) + s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha) = \overset{+}{1}\frac{x}{s} \end{cases}$$

This is a constant coefficient first order ODE. We solve it by finding the integrating factor

$$\mu = e^{\int s dx} = e^{sx}$$

Thus we have:

$$\partial_{x_{gH}}[e^{sx}\overset{-}{U}(x,s,\alpha)] = e^{sx}(\overset{-}{1}\frac{x}{s})$$

and

$$\partial_{x_{gH}}[e^{sx}\overset{+}{U}(x,s,\alpha)] = e^{sx}(\overset{+}{1}\frac{x}{s})$$

We integrate both sides to get

$$\overset{-}{U}(x,s,\alpha) = \overset{-}{1}\frac{e^{-sx}}{s}(\int e^{sx}xdx) + c_1(\alpha)e^{-sx}$$

$$\overset{+}{U}(x,s,\alpha) = \overset{+}{1}\frac{e^{-sx}}{s}(\int e^{sx}xdx) + c_2(\alpha)e^{-sx}$$

We can use integration by parts to evaluate the integral

$$\overset{-}{U}(x,s,\alpha) = \overset{-}{1}\frac{e^{-sx}}{s}(\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}) + c_1(\alpha)e^{-sx} = \frac{\overset{-}{1}x}{s^2} - \frac{\overset{-}{1}}{s^3} + c_1(\alpha)e^{-sx}$$

$$\overset{+}{U}(x,s,\alpha) = \overset{+}{1}\frac{e^{-sx}}{s}(\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}) + c_2(\alpha)e^{-sx} = \frac{\overset{+}{1}x}{s^2} - \frac{\overset{+}{1}}{s^3} + c_2(\alpha)e^{-sx}.$$

We can evaluate the constants $c_1(\alpha), c_2(\alpha)$ using the boundary condition

$$\frac{\overset{-}{1}}{2} = \overset{-}{U}(0,s,\alpha) = -\frac{\overset{-}{1}}{s^3} + c_1(\alpha) \Rightarrow c_1(\alpha) = \frac{1}{2s}\alpha + \frac{\overset{-}{1}}{s^3}$$

$$\frac{\overset{+}{1}}{2} = \overset{+}{U}(0,s,\alpha) = -\frac{\overset{+}{1}}{s^3} + c_2(\alpha) \Rightarrow c_2(\alpha) = (1-\frac{1}{2}\alpha)\frac{1}{s} + \frac{\overset{+}{1}}{s^3}$$

So we have :

$$\overset{-}{U}(x,s,\alpha) = \frac{\overset{-}{1}}{s^2}x - \frac{\overset{-}{1}}{s^3} + (\frac{1}{2}\alpha + \frac{\overset{-}{1}}{s^3})e^{-sx}$$

$$\overset{+}{U}(x,s,\alpha) = \frac{\overset{+}{1}}{s^2}x - \frac{\overset{+}{1}}{s^3} + \Big((1-\frac{1}{2}\alpha)\frac{1}{s} + \frac{\overset{+}{1}}{s^3}\Big)e^{-sx}$$

Taking the inverse laplace transform we have:

$$L^{-1}(\bar{U}(x,s,\alpha)) = L^{-1}[\frac{\bar{1}}{s^2}x - \frac{\bar{1}}{s^3} + (\frac{1}{2}\alpha + \frac{\bar{1}}{s^3})e^{-sx}]$$

$$\bar{u}(x,t,\alpha) = \bar{1}xt - \frac{\bar{1}}{2}t^2 + \frac{1}{2}\alpha x H(t-x) + \frac{\bar{1}}{2}(t-x)^2 H(t-x) \quad (7)$$

And

$$L^{-1}[(\overset{+}{U}(x,s,\alpha)] = L^{-1}[\frac{\overset{+}{1}}{s^2}x - \frac{\overset{+}{1}}{s^3} + \Big((1-\frac{1}{2}\alpha)\frac{1}{s} + \frac{\overset{+}{1}}{s^3}\Big)e^{-sx}]$$

$$\overset{+}{u}(x,t,\alpha) = \overset{+}{1}xt - \frac{\overset{+}{1}}{2}t^2 + (1-\frac{1}{2}\alpha)xH(t-x) + \frac{\overset{+}{1}}{2}(t-x)^2 H(t-x) \quad (8)$$

With

$$H(t-x) = \begin{cases} 0 & t < x \\ 1 & t \geq x \end{cases}$$

for the solution in Eq.(7) and Eq.(8) to be valid, the conditions
$\overset{+}{u}(x,t,\alpha) > \bar{u}(x,t,\alpha)$
$\partial_{x_{gH}}\overset{+}{u}(x,t,\alpha) > \partial_{x_{gH}}\bar{u}(x,t,\alpha)$
$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha) > \partial_{t_{gH}}\bar{u}(x,t,\alpha)$
So, we check the above conations on our results :
$\overset{+}{u}(x,t,\alpha) - \bar{u}(x,t,\alpha) = 0.7xt(1-\alpha) + \frac{0.7}{2}t^2(\alpha-1) + (1-\alpha)(\frac{1}{t-x})H(t-x) + \frac{0.7}{2}(1-\alpha)(t-x)^2H(t-x)$

$\partial_{x_{gH}}\overset{+}{u}(x,t,\alpha) - \partial_{x_{gH}}\bar{u}(x,t,\alpha) = 0.7t(1-\alpha) + \frac{1}{(t-x)^2}(1-\alpha)H(t-x) + 0.7(\alpha-1)(t-x)H(t-x)$

$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha) - \partial_{t_{gH}}\bar{u}(x,t,\alpha) = 0.7x(1-\alpha) + 0.7t(\alpha-1) + 0.7(1-\alpha)(t-x)H(t-x) + (\alpha-1)\frac{1}{(t-x)^2}H(t-x)$.
From the above calculation, we can see that the solutions are valid for all $x, t > 0$ such that $\frac{t}{2} - x < 0$.

Results obtained for this case are illustrated in figs 1 and 2.
The values of t and x are fixed at (fig. 1) $x = 1$,$t = \frac{2}{3}$ and (fig.2) x=2, $t = \frac{2}{3}$

Table 1: The answer of FPDE by the fuzzy laplace transform $x = 1, t = 2/3$

r	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\overset{-}{u}$	0.356	0.365	0.374	0.383	0.392	0.4	0.41	0.418	0.427	0.436	0.445
$\overset{+}{u}$	0.667	0.645	0.623	0.6	0.578	0.556	0.534	0.512	0.489	0.467	0.445

Table 2: The answere of FPDE by the fuzzy laplace transform $x = 2, t = 2/3$

r	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$\overset{-}{u}$	0.883	0.910	0.932	0.955	0.977	0.999	1.021	1.043	1.065	1.088	1.11
$\overset{+}{u}$	1.665	1.61	1.554	1.498	1.443	1.387	1.332	1.276	1.221	1.165	1.11

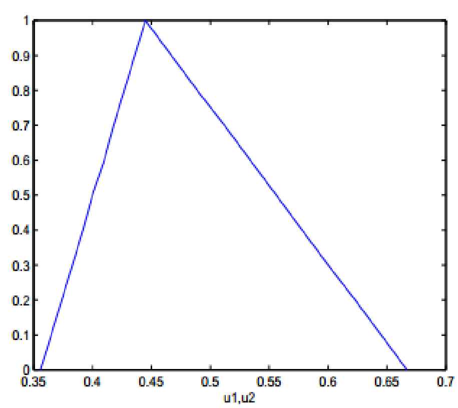

Figure 1: The answer of FPDE at the point $x = 1, t = 2/3$

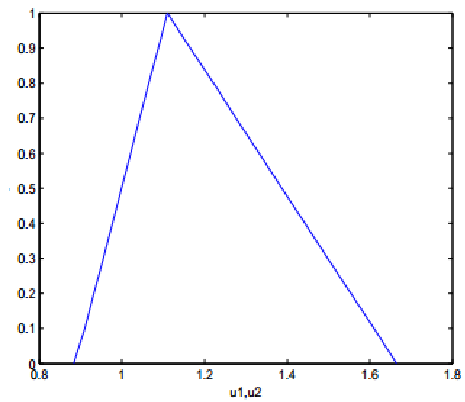

Figure 2: The answer of FPDE at the point $x = 2, t = 2/3$

For (ii), In the following it is shown that how a classical differential equation is created

$$\begin{cases} \partial_{x_{gH}} \overset{-}{U}(x,s,\alpha) + s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha) = \overset{-}{1}\frac{x}{s} \\ \partial_{x_{gH}} \overset{+}{U}(x,s,\alpha) + s\overset{-}{U}(x,s,\alpha) - \overset{-}{u}(x,0,\alpha) = \overset{+}{1}\frac{x}{s} \end{cases}$$

$$\overset{-}{U}(x,s,\alpha) = c_1(\alpha)e^{sx} + c_2(\alpha)^{-sx} + \frac{\overset{+}{1}}{s^2}x - \frac{\overset{-}{1}}{s^3}$$

$$\overset{+}{U}(x,s,\alpha) = -c_1(\alpha)e^{sx} + c_2(\alpha)^{-sx} + \frac{\overset{-}{1}}{s^2}x - \frac{\overset{+}{1}}{s^3}$$

If $c_1(\alpha) = \frac{1}{2}[\frac{\alpha-1}{s} + \frac{-0.7+0.7\alpha}{s^3}]$, $c_2(\alpha) = \frac{1}{2}[\frac{1}{s} + \frac{2.3-0.3\alpha}{s^3}]$ then:

$$\overset{-}{u}(x,t,\alpha) = \frac{-1}{4}[\alpha - 1 + \frac{-0.7+0.7\alpha}{s^3}]\delta(x) + \frac{1}{2(t-x)}H(t-x)$$

$$+\frac{1}{4}(2.3-0.3\alpha)(t-x)^2 + H(t-x) + \overset{+}{1}xt - \frac{\overset{-}{1}}{2}t^2$$

$$\overset{+}{u}(x,t,\alpha) = +\frac{1}{4}[\alpha - 1 + \frac{-0.7+0.7\alpha}{s^3}]\delta(x) + \frac{1}{2(t-x)}xH(t-x)$$
$$+\frac{1}{4}(2.3-0.3\alpha)(t-x)^2H(t-x) + \bar{1}xt - \frac{\overset{+}{1}}{2}t^2 \qquad (9)$$

For acqurinig fuzzy number by $\overset{+}{u}(x,t,\alpha)\overset{-}{u}(x,t,\alpha)$ the below conditions must exist

$\overset{+}{u}(x,t,\alpha) > \overset{-}{u}(x,t,\alpha)$

$\partial_{x_{gH}}\overset{+}{u}(x,t,\alpha) > \partial_{x_{gH}}\overset{-}{u}(x,t,\alpha)$

$\partial_{x_{gH}}\overset{+}{u}(x,t,\alpha) > \partial_{t_{gH}}\overset{-}{u}(x,t,\alpha)$

Then the above conditions must be checked on the result

$\overset{+}{u}(x,t,\alpha) - \overset{-}{u}(x,t,\alpha) = 0.7xt(\alpha-1) + 0.7t^2(\alpha-1) + \frac{1}{2}(1-\alpha)[1+\frac{0.7}{s^3}]\delta(x)$

$\partial_{x_{gH}}\overset{+}{u}(x,t,\alpha) - \partial_{x_{gH}}\overset{-}{u}(x,t,\alpha) = 0.7t(\alpha-1)$

$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha) - \partial_{t_{gH}}\overset{-}{u}(x,t,\alpha) = 0.7x(1-\alpha) + 0.7t(1-\alpha)$.

We conculde that a fuzzy number is not defined.

Case II: If $\partial_{x_{gH}}u(x,t)$ is (ii)-generated Hukuhara differentiable and $\partial_{t_{gH}}u(x,t)$ is (i)-generalized Hukuhara differentiable then we have:

$\partial_{x_{gH}}U(x,s) \oplus (s \odot U(x,s)) \ominus_{gH} u(x,0) = \tilde{1}\frac{x}{s}$

or

$$\begin{cases} i)\ (\partial_{x_{gH}}\overset{+}{U}(x,s,\alpha), \partial_{x_{gH}}\overset{-}{U}(x,s,\alpha)) \oplus (s\overset{-}{U}(x,s,\alpha) - \overset{-}{u}(x,0,\alpha), s\overset{+}{U}(x,s,\alpha) \\ \qquad - \overset{+}{u}(x,0,\alpha)) = (\overset{-}{1}\frac{x}{s}, \overset{+}{1}\frac{x}{s}) \\ ii)\ (\partial_{x_{gH}}\overset{+}{U}(x,s,\alpha), \partial_{x_{gH}}\overset{-}{U}(x,s,\alpha)) \oplus (s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha), s\overset{-}{U}(x,s,\alpha) \\ \qquad - \overset{-}{u}(x,0,\alpha)) = (\overset{-}{1}\frac{x}{s}, \overset{+}{1}\frac{x}{s}) \end{cases}$$

For (i), Then a differential equation system like below is created:

$$\begin{cases} \partial_{x_{gH}}\overset{+}{U}(x,s,\alpha) + s\overset{-}{U}(x,s,\alpha) - \overset{-}{u}(x,0,\alpha) = \overset{-}{1}\frac{x}{s} \\ \partial_{x_{gH}}\overset{-}{U}(x,s,\alpha) + s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha) = \overset{+}{1}\frac{x}{s} \end{cases}$$

We can use integration by parts to evaluate the integral

$$\bar{U}(x,s,\alpha) = -c_1(\alpha)e^{sx} + c_2(\alpha)e^{-sx} - \frac{\overset{+}{1}}{s^3} + \frac{\bar{1}}{s^2}x$$

$$\overset{+}{U}(x,s,\alpha) = c_1(\alpha)e^{sx} + c_2(\alpha)e^{-sx} - \frac{\bar{1}}{s^3} + \frac{\overset{+}{1}}{s^2}x$$

$$c_1(\alpha) = \frac{1}{2}[\frac{1-\alpha}{s} + \frac{-0.7+0.7\alpha}{s^3}], c_2(\alpha) = \frac{1}{2}[\frac{1}{s} + \frac{2.3-0.3\alpha}{s^3}]$$

$$\overline{u}(x,t,\alpha) = +\frac{1}{4}[1-\alpha+\frac{-0.7+0.7\alpha}{s^3}]\delta(x) + \frac{1}{2(t-x)}H(t-x)$$
$$+\frac{1}{4}(2.3-0.3\alpha)(t-x)^2H(t-x) + \bar{1}xt - \frac{\overset{+}{1}}{2}t^2$$

$$\overset{+}{u}(x,t,\alpha) = -\frac{1}{4}[1-\alpha+\frac{-0.7+0.7\alpha}{s^3}]\delta x + \frac{1}{2(t-x)}H(t-x)$$
$$+\frac{1}{4}(2.3-0.3\alpha)(t-x)^2H(t-x) + \overset{+}{1}xt - \frac{\bar{1}}{2}t^2$$

For acqurinig fuzzy number by $\overset{+}{u}(x,t,\alpha)\bar{u}(x,t,\alpha)$ the below conditions must exist
$\overset{+}{u}(x,t,\alpha) > \bar{u}(x,t,\alpha)$
$\partial_{x_{gH}}\bar{u}(x,t,\alpha) > \partial_{x_{gH}}\overset{+}{u}(x,t,\alpha)$
$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha) > \partial_{t_{gH}}\bar{u}(x,t,\alpha)$
So we check this conditions on our results:
$\overset{+}{u}(x,t,\alpha) - \bar{u}(x,t,\alpha) = 0.7xt(1-\alpha) + \frac{0.7}{2}t^2(1-\alpha) + \frac{1}{2}[(\alpha-1) + \frac{0.7}{s^3}(1-\alpha)]\delta(x)$
$\partial_{x_{gH}}\bar{u}(x,t,\alpha) - \partial_{x_{gH}}\overset{+}{u}(x,t,\alpha) = 0.7t(1-\alpha)$
$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha) - \partial_{t_{gH}}\bar{u}(x,t,\alpha) = 0.7x(1-\alpha) + 0.7t(1-\alpha)$
According to a bove calculated relationships, we conculde that a fuzzy number is not defined.
For (ii), Then a general differential equation system like below is created

$$\begin{cases} \partial_{x_{gH}}\overset{+}{U}(x,s,\alpha) + s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha) = \bar{1}\frac{x}{s} \\ \partial_{x_{gH}}\bar{U}(x,s,\alpha) + s\bar{U}(x,s,\alpha) - \bar{u}(x,0,\alpha) = \overset{+}{1}\frac{x}{s} \end{cases}$$

So we replace this integrals with equation (10) and (11)

$$\overset{-}{U}(x,s,\alpha) = \overset{+}{1}\frac{e^{-sx}}{s}(\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}) + c_1(\alpha)e^{-sx} = \frac{\overset{+}{1}}{s^2}x - \frac{\overset{+}{1}}{s^3} + c_1(\alpha)e^{-sx} \quad (10)$$

$$\overset{+}{U}(x,s,\alpha) = \overset{-}{1}\frac{e^{-sx}}{s}(\frac{xe^{sx}}{s} - \frac{e^{sx}}{s^2}) + c_2(\alpha)e^{-sx} = \frac{\overset{-}{1}}{s^2}x - \frac{\overset{-}{1}}{s^3} + c_2(\alpha)e^{-sx} \quad (11)$$

If

$$c_1(\alpha) = \frac{1}{2s}\alpha + \frac{\overset{+}{1}}{s^3}, \quad c_2(\alpha) = (1 - \frac{1}{2}\alpha)\frac{1}{s} + \frac{\overset{-}{1}}{s^3}.$$

Then

$$\overset{-}{U}(x,s,\alpha) = \frac{\overset{+}{1}}{s^2}x - \frac{\overset{+}{1}}{s^3} + (\frac{1}{2s}\alpha + \frac{\overset{+}{1}}{s^3})e^{-sx}$$

$$\overset{+}{U}(x,s,\alpha) = \frac{\overset{-}{1}}{s^2}x - \frac{\overset{-}{1}}{s^3} + ((1 - \frac{1}{2}\alpha)\frac{1}{s} + \frac{\overset{+}{1}}{s^3})e^{-sx}.$$

Taking the inverse laplace transform we have:

$$\overset{-}{u}(x,t,\alpha) = \overset{+}{1}xt - \frac{\overset{+}{1}}{2}t^2 + \frac{1}{2(t-x)}\alpha H(t-x) + \frac{\overset{+}{1}}{2}(t-x)^2 H(t-x),$$

$$\overset{+}{u}(x,t,\alpha) = \overset{-}{1}xt - \frac{\overset{-}{1}}{2}t^2 + (1 - \frac{1}{2}\alpha)(\frac{1}{t-x})H(t-x) + \frac{\overset{-}{1}}{2}(t-x)^2 H(t-x).$$

For acqurinig fuzzy number by $\overset{+}{u}(x,t,\alpha)\overset{-}{u}(x,t,\alpha)$ the below conditions must exist

$\overset{+}{u}(x,t,\alpha) > \overset{-}{u}(x,t,\alpha)$

$\partial_{x_{gH}}\overset{-}{u}(x,t,\alpha) > \partial_{x_{gH}}\overset{+}{u}(x,t,\alpha)$

$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha) > \partial_{t_{gH}}\overset{-}{u}(x,t,\alpha)$

So we check this conditions on our results:

$$\overset{+}{u}(x,t,\alpha) - \overset{-}{u}(x,t,\alpha) = 0.7(xt - \frac{1}{2}t^2)(\alpha - 1) + H(t-x)(\frac{0.7}{2})(t-x)^2(\alpha - 1) + (\frac{1}{t-x})(1-\alpha),$$

$$\partial_{x_{gH}}\overset{-}{u}(x,t,\alpha)-\partial_{x_{gH}}\overset{+}{u}(x,t,\alpha)=$$
$$0.7t(\alpha-1)+(\alpha-1)\frac{1}{(t-x)^2}H(t-x)+0.7(t-x)(\alpha-1)H(t-x),$$

$$\partial_{t_{gH}}\overset{+}{u}(x,t,\alpha)-\partial_{t_{gH}}\overset{-}{u}(x,t,\alpha)$$
$$=0.7(x-t)(1-\alpha)+0.7(t-x)(1-\alpha)H(t-x)+(\alpha-1)\frac{1}{(t-x)^2}H(t-x).$$

According to calculated relationships,we conculde that a fuzzy number is not defined.

Case III: If $\partial_{x_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable and $\partial_{t_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable then we have:

$$\partial_{x_{gH}}U(x,s)\oplus[\ominus u(x,0)\ominus_{gH}(\ominus s\odot L[u(x,t)])]=\tilde{1}\frac{x}{s}.$$

Or

$$\begin{cases} i)\ (\partial_{x_{gH}}\overset{+}{U}(x,s,\alpha),\partial_{x_{gH}}\overset{-}{U}(x,s,\alpha))\oplus(s\overset{+}{U}(x,s,\alpha) \\ \qquad -\overset{+}{u}(x,0,\alpha),s\overset{-}{U}(x,s,\alpha)-\overset{-}{u}(x,0,\alpha))=(\overset{-}{1}\frac{x}{s},\overset{+}{1}\frac{x}{s}) \\ ii)\ (\partial_{x_{gH}}\overset{+}{U}(x,s,\alpha),\partial_{x_{gH}}\overset{-}{U}(x,s,\alpha))\oplus(s\overset{-}{U}(x,s,\alpha) \\ \qquad -\overset{-}{u}(x,0,\alpha),s\overset{+}{U}(x,s,\alpha)-\overset{+}{u}(x,0,\alpha))=(\overset{-}{1}\frac{x}{s},\overset{+}{1}\frac{x}{s}) \end{cases}$$

It is solved by the second method.

Case IV: If $\partial_{x_{gH}}u(x,t)$ is (i)- generalized Hukuhara differentiable and $\partial_{t_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable then we have:

$$\begin{cases} i)\ (\partial_{x_{gH}}\overset{-}{U}(x,s,\alpha),\partial_{x_{gH}}\overset{+}{U}(x,s,\alpha))\oplus(s\overset{+}{U}(x,s,\alpha) \\ \qquad -\overset{+}{u}(x,0,\alpha),s\overset{-}{U}(x,s,\alpha)-\overset{-}{u}(x,0,\alpha))=(\overset{-}{1}\frac{x}{s},\overset{+}{1}\frac{x}{s}) \\ ii)\ (\partial_{x_{gH}}\overset{-}{U}(x,s,\alpha),\partial_{x_{gH}}\overset{+}{U}(x,s,\alpha))\oplus(s\overset{-}{U}(x,s,\alpha) \\ \qquad -\overset{-}{u}(x,0,\alpha),s\overset{+}{U}(x,s,\alpha)-\overset{+}{u}(x,0,\alpha))=(\overset{-}{1}\frac{x}{s},\overset{+}{1}\frac{x}{s}) \end{cases}$$

It is solved by the first method.

Example 2. Consider the problem of finding $u(x,t)$ in $0 < x < \infty$ saisfying

$$\partial_{t_{gH}} u(x,t) \ominus_{gH} K\partial_{xx_{gH}} u(x,t) = F(x,t)$$

with boundary and initial condition

$$F(x,t) = 0$$

$$u(x,0) = (\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha),\ \ x > 0\ \ and\ \ u(x,0) = (98+2\alpha, 102-2\alpha),\ \ t > 0$$

$$u(x \to \infty, t) = (\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha).$$

By using fuzzy Laplace transform method, we have:

$$\begin{aligned} L[\partial_{t_{gH}} u(x,t)] &= KL[\partial_{xx_{gH}} U(x,t)],\ U(0,s) \\ &= \frac{1}{s}(98+2\alpha, 102-2\alpha),\ U(x \to \infty, s) \\ &= \frac{1}{s}(\frac{1}{2}\alpha, 1-\frac{1}{2}\alpha) \end{aligned}$$

Case I : If $\partial_{t_{gH}} u(x,t)$ is (i)- generalized Hukuhara differentiable and $\partial_{xx_{gH}} u(x,t)$ is (i)- generalized Hukuhara differentiable or $\partial_{t_{gH}} u(x,t)$ is (ii)- generalized Hukuhara differentiable and $\partial_{xx_{gH}} u(x,t)$ is (ii)- generalized Hukuhara differentiable we have:

$$s \odot U(x,s) \ominus_{gH} u(x,0) = K(\partial_{xx_{gH}} \overset{+}{U}(x,s,\alpha), \partial_{xx_{gH}} \overset{-}{U}(x,s,\alpha))$$

Or

$$\begin{aligned} &(s\overset{-}{U}(x,s,\alpha)) - \overset{-}{u}(x,0,\alpha), s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha)) \\ &\qquad = K(\partial_{xx_{gH}} \overset{-}{U}(x,s,\alpha), \partial_{xx_{gH}} \overset{+}{U}(x,s,\alpha)) \end{aligned}$$

Then, we have two crisp linear systems which can be displayed as follows:

$$\partial_{xx_{gH}} \overset{-}{U}(x,s,\alpha) - \frac{s}{K}\overset{-}{U}(x,s,\alpha) = -\frac{1}{K}\overset{-}{u}(x,0,\alpha)$$

$$\partial_{xx_{gH}} \overset{+}{U}(x,s,\alpha) - \frac{s}{K}\overset{+}{U}(x,s,\alpha) = -\frac{1}{K}\overset{+}{u}(x,0,\alpha)$$

the solution of this problem is

$$\overset{-}{U}(x,s,\alpha)=\frac{1}{s}(98+\frac{3}{2}\alpha)e^{-\sqrt{\frac{5}{K}}x}+\frac{\alpha}{2s}$$

$$\overset{+}{U}(x,s,\alpha)=\frac{1}{s}(103-\frac{5}{2}\alpha)e^{-\sqrt{\frac{5}{k}}x}+\frac{1}{s}(1-\frac{1}{2}\alpha)$$

Taking the inverse Laplace transform we have:

$$L^{-1}\left[\overset{-}{U}(x,s,\alpha)\right]=L^{-1}\left[f^{-}(s,\alpha)e^{-\sqrt{\frac{5}{K}}x}\right]+L^{-1}\left[\frac{\alpha}{2s}\right]$$

$$L^{-1}\left[\overset{+}{U}(x,s,\alpha)\right]=L^{-1}\left[f^{+}(s,\alpha)e^{-\sqrt{\frac{5}{K}}x}\right]+L^{-1}\left[\frac{1}{s}(1-\frac{1}{2}\alpha)\right]$$

where $f^{-}(s,\alpha)=\frac{1}{s}(98+\frac{3}{2}\alpha)$ and$f^{+}(s,\alpha)=\frac{1}{s}(103-\frac{5}{2}\alpha)$

Inversion then produces

$$\overset{-}{u}(x,t,\alpha)=\frac{x}{\sqrt{4\pi K}}\int_{\tau=0}^{t}\frac{f^{-}(\tau,\alpha)}{(t-\tau)^{(\frac{3}{2})}}exp(-\frac{x^2}{4K(t-\tau)})d\tau+\frac{\alpha}{2}$$

$$\overset{+}{u}(x,t,\alpha)=\frac{x}{\sqrt{4\pi K}}\int_{\tau=0}^{t}\frac{f^{+}(\tau,\alpha)}{(t-\tau)^{(\frac{3}{2})}}exp(-\frac{x^2}{4K(t-\tau)})d\tau+(1-\frac{\alpha}{2})$$

if $f^{-}(s,\alpha)=\frac{1}{s}(98+\frac{3}{2}\alpha)$ and$f^{+}(s,\alpha)=\frac{1}{s}(103-\frac{5}{2}\alpha)$ the solution is :

$$\overset{-}{u}(x,t,\alpha)=(98+\frac{3}{2}\alpha)erfc(\frac{x}{2\sqrt{kt}})+\frac{\alpha}{2}$$

$$\overset{+}{u}(x,t,\alpha)=(103-\frac{5}{2}\alpha)erfc(\frac{x}{2\sqrt{kt}})+(1-\frac{1}{2})\alpha$$

Case II : If $\partial_{t_{gH}}u(x,t)$ is (i)- generalized Hukuhara differentiable and $\partial_{xx_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable or $\partial_{t_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable and $\partial_{xx_{gH}}u(x,t)$ is (i)- generalized Hukuhara differentiable we have:

$$s\odot U(x,s)\ominus_{gH}u(x,0)=K(\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha),\partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha))$$

Or

$$(s\overset{-}{U}(x,s,\alpha)) - \overset{-}{u}(x,0,\alpha), s\overset{+}{U}(x,s,\alpha)) - \overset{+}{u}(x,0,\alpha))$$
$$= K(\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha), \partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha))$$

Then, we have two crisp linear systems which can be display as follows:

$$\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha) - \frac{s}{K}\overset{-}{U}(x,s,\alpha) = -\frac{1}{K}\overset{-}{u}(x,0,\alpha)$$
$$\partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha) - \frac{s}{K}\overset{+}{U}(x,s,\alpha) = -\frac{1}{K}\overset{+}{u}(x,0,\alpha)$$

the solution of this problem is:

$$\overset{-}{U}(x,s,\alpha) =(\frac{3}{2}\alpha - \frac{3}{2})\frac{1}{s}e^{i\sqrt{\frac{s}{K}}x} - (\frac{3}{2}\alpha - \frac{3}{2})\frac{1}{s}e^{i\sqrt{\frac{s}{K}}(2l-x)}$$
$$+ (\frac{199}{2s})(\frac{e^{\sqrt{\frac{s}{K}}(l-x)} - e^{\sqrt{\frac{s}{K}}(x-l)}}{e^{\sqrt{\frac{s}{K}}l} - e^{-\sqrt{\frac{s}{K}}l}})$$

$$\overset{+}{U}(x,s,\alpha) = - (\frac{3}{2}\alpha - \frac{3}{2})\frac{1}{s}e^{i\sqrt{\frac{s}{K}}x} + (\frac{3}{2}\alpha - \frac{3}{2})\frac{1}{s}e^{i\sqrt{\frac{s}{K}}(2l-x)}$$
$$+ (\frac{199}{2s})(\frac{e^{\sqrt{\frac{s}{K}}(l-x)} - e^{\sqrt{\frac{s}{K}}(x-l)}}{e^{\sqrt{\frac{s}{K}}l} - e^{-\sqrt{\frac{s}{K}}l}})$$

Taking the inverse Laplace transform we have:

$$\overset{-}{u}(x,t,\alpha) =(\frac{3}{2}\alpha - \frac{3}{2})erfc(\frac{-xi}{2\sqrt{Kt}}) - (\frac{3}{2}\alpha - \frac{3}{2})erfc(\frac{-i(2l-x)}{2\sqrt{Kt}})$$
$$+ \frac{199}{\pi}(\sum_{n=1}^{\infty}\frac{-1}{n}e^{\frac{-Kn^2\pi^2t}{l^2}}\sin(\frac{n\pi x}{l}))$$

$$\overset{+}{u}(x,t,\alpha) = - (\frac{3}{2}\alpha - \frac{3}{2})erfc(\frac{-xi}{2\sqrt{Kt}}) + (\frac{3}{2}\alpha - \frac{3}{2})erfc(\frac{-i(2l-x)}{2\sqrt{Kt}})$$
$$+ \frac{199}{\pi}(\sum_{n=1}^{\infty}\frac{-1}{n}e^{\frac{-Kn^2\pi^2t}{l^2}}\sin(\frac{n\pi x}{l}))$$

Example 3. Consider the problem of finding $u(x,t)$ in $0 < x < \infty$ saisfying

$$\partial_{t_{gH}} u(x,t) = K\partial_{xx_{gH}} u(x,t)$$

with boundary and initial condition

$$u(x,0) = 0 \quad x > 0 \quad and \quad u(0,t) = \tilde{f}(t) \quad t > 0$$

$$u(x \to \infty, t) = 0$$

By using fuzzy Laplace transform, we have:

$$\partial_{t_{gH}} U(x,s) = K\partial_{xx_{gH}} U(x,s), \quad U(0,s) = \tilde{f}(s), \quad U(x \to \infty, s) = 0$$

Case I: If $\partial_{t_{gH}} u(x,t)$ is (i)- generalized Hukuhara differentiable and $\partial_{xx_{gH}} u(x,t)$ is (i)- generalized Hukuhara differentiable or $\partial_{t_{gH}} u(x,t)$ is (ii)- generalized Hukuhara differentiable and $\partial_{xx_{gH}} u(x,t)$ is (ii)- generalized Hukuhara differentiable we have:

$$s \odot U(x,s) \ominus_{gH} u(x,0) = K(\partial_{xx_{gH}} \overset{-}{U}(x,s,\alpha), \partial_{xx_{gH}} \overset{+}{U}(x,s,\alpha))$$

or

$$(s\overset{-}{U}(x,s,\alpha)) - \overset{-}{u}(x,0,\alpha), s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha)) \\ = K(\partial_{xx_{gH}} \overset{-}{U}(x,s,\alpha), \partial_{xx_{gH}} \overset{+}{U}(x,s,\alpha))$$

Then, we have two crisp linear systems which can be displayed as follows:

$$\partial_{xx_{gH}} \overset{-}{U}(x,s,\alpha) - \frac{s}{K}\overset{-}{U}(x,s,\alpha) = -\frac{1}{K}\overset{-}{u}(x,0,\alpha),$$

$$\partial_{xx_{gH}} \overset{+}{U}(x,s,\alpha) - \frac{s}{K}\overset{+}{U}(x,s,\alpha) = -\frac{1}{K}\overset{+}{u}(x,0,\alpha).$$

The solution of this problem is

$$\overset{-}{U}(x,s,\alpha) = f^{-}(s,\alpha)e^{-\sqrt{\frac{s}{k}}x},$$

$$\overset{+}{U}(x,s,\alpha) = f^{+}(s,\alpha)e^{-\sqrt{\frac{s}{k}}x}.$$

Inversion then produces

$$\overset{-}{u}(x,t,\alpha) = \frac{x}{\sqrt{4\pi k}}\int_{\tau=0}^{t}\frac{f^{-}(\tau,\alpha)}{(t-\tau)^{\frac{3}{2}}}exp(-\frac{x^2}{4k(t-\tau)})d\tau,$$

$$\overset{+}{u}(x,t,\alpha) = \frac{x}{\sqrt{4\pi k}}\int_{\tau=0}^{t}\frac{f^{+}(\tau,\alpha)}{(t-\tau)^{\frac{3}{2}}}exp(-\frac{x^2}{4k(t-\tau)})d\tau.$$

Where $f^{-}(t,\alpha) = \frac{\alpha}{2}$ *and* $f^{+}(t,\alpha) = (1-\frac{1}{2}\alpha)$ the solotion is :

$$\overset{-}{u}(x,t,\alpha) = \frac{\alpha}{2}erfc(\frac{x}{2\sqrt{kt}})$$

$$\overset{+}{u}(x,t,\alpha) = (1-\frac{1}{2}\alpha)erfc(\frac{x}{2\sqrt{kt}})$$

Case II : If $\partial_{t_{gH}}u(x,t)$ is (i)- generalized Hukuhara differentiable and $\partial_{xx_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable or $\partial_{t_{gH}}u(x,t)$ is (ii)- generalized Hukuhara differentiable and $\partial_{xx_{gH}}u(x,t)$ is (i)- generalized Hukuhara differentiable we have:

$$s \odot U(x,s) \ominus_{gH} u(x,0) = K(\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha), \partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha)),$$

or

$$(s\overset{-}{U}(x,s,\alpha)) - \overset{-}{u}(x,0,\alpha), s\overset{+}{U}(x,s,\alpha) - \overset{+}{u}(x,0,\alpha))$$
$$= K(\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha), \partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha)).$$

Then, we have two crisp linear systems which can be display as follows:

$$\partial_{xx_{gH}}\overset{+}{U}(x,s,\alpha) - \frac{s}{K}\overset{-}{U}(x,s,\alpha) = \frac{-1}{K}\overset{-}{u}(x,0,\alpha),$$

$$\partial_{xx_{gH}}\overset{-}{U}(x,s,\alpha) - \frac{s}{K}\overset{+}{U}(x,s,\alpha) = \frac{-1}{K}\overset{+}{u}(x,0,\alpha).$$

The solution of this problem is:

$$\bar{U}(x,s,\alpha)=$$
$$\frac{\alpha-1}{2s}e^{i\sqrt{\frac{s}{K}}x}-\frac{\alpha-1}{2s}e^{i\sqrt{\frac{s}{K}}(2l-x)}+(\frac{1}{2s})(\frac{e^{\sqrt{\frac{s}{K}}(l-x)}-e^{\sqrt{\frac{s}{K}}(x-l)}}{e^{\sqrt{\frac{s}{K}}l}-e^{-\sqrt{\frac{s}{K}}l}}),$$

$$\overset{+}{U}(x,s,\alpha)=$$
$$-\frac{\alpha-1}{2s}e^{i\sqrt{\frac{s}{K}}x}+\frac{\alpha-1}{2s}e^{i\sqrt{\frac{s}{K}}(2l-x)}+(\frac{1}{2s})(\frac{e^{\sqrt{\frac{s}{K}}(l-x)}-e^{\sqrt{\frac{s}{K}}(x-l)}}{e^{\sqrt{\frac{s}{K}}l}-e^{-\sqrt{\frac{s}{K}}l}}).$$

Taking the inverse Laplace transform we have:

$$\bar{u}(x,t,\alpha)=\frac{\alpha-1}{2}erfc(\frac{-xi}{2\sqrt{Kt}})-\frac{\alpha-1}{2}erfc(\frac{-i(2l-x)}{2\sqrt{Kt}})+\frac{1}{\pi}(\sum_{n=1}^{\infty}\frac{-1}{n}e^{\frac{-Kn^2\pi^2t}{l^2}}\sin(\frac{n\pi x}{l})),$$

$$\overset{+}{u}(x,t,\alpha)=-\frac{\alpha-1}{2}erfc(\frac{-xi}{2\sqrt{Kt}})+\frac{\alpha-1}{2}erfc(\frac{-i(2l-x)}{2\sqrt{Kt}})+\frac{1}{\pi}(\sum_{n=1}^{\infty}\frac{-1}{n}e^{\frac{-Kn^2\pi^2t}{l^2}}\sin(\frac{n\pi x}{l})).$$

5 Conclusion

Developing fuzzy Laplace transform, we provided solutions to fuzzy partial differential equation which is interpreted by using the generalized Hukuhara derivative concept. in this paper, we have studied Fuzzy Laplace Transform(FLT) for solving FPDE. Then, a procedure for obtaining solutions of FPDEs using FLT are constructed. Two examples has been demonstrated to describe the usage of FLT on FPDEs.

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