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Original Research Paper

Attractivity and Global Attractivity for System of Fractional Functional and Nonlinear Fractional q-Differential Equations

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Abstract. In the current work, we present some innovative solutions for the attractivity of fractional functional q-differential equations involving Caputo fractional q-derivative in a k-dimensional system, by using some fixed point principle and the standard Schauder's fixed point theorem. Likewise, we look into the global attractivity of fractional q-differential equations involving classical Riemann-Liouville fractional q-derivative in a k-dimensional system, by employing the famous fixed point theorem of Krasnoselskii. Also, we must note that, this paper is mainly on the analysis of the model, with numerics used only to verify the analysis for checking the attractivity and global attractivity of solutions in the system. Lastly, we give two examples to illustrate our main results.

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1 Introduction

Fractional calculus and q-calculus are one of the significant branches in mathematical analysis and have countless applications [2, 3, 9, 26, 28, 33]. Similarly, the subject of fractional differential equations ranges from the theoretical views of existence and uniqueness of solutions to the analytical and mathematical methods for finding solutions [4, 6, 7, 10, 11, 13–16, 18, 24]. During the last two decades, the fractional differential equations and inclusions, in two type differential and q-differential, were developed intensively by many authors for a variety of subjects [1, 5, 8, 12, 19, 23, 32, 34, 38–42]. There are many published papers about the attractivity of solutions for fractional and fractional functional differential equations [17, 21, 22, 31, 43].

The subject of q-difference equations introduced in 1910 by Jackson [25]. In 2011, Chen *et al.* studied the attractivity of the fractional functional differential equation and the global attractivity of the nonlinear fractional differential equation with boundary value condition ${}^cD^\alpha x(t) = h_1(t, x_t)$, $x(t) = \varphi(t)$, and $D^\alpha u(t) = h_2(t, u(t))$, $D^{\alpha-1}u(t)|_{t=t_0} = u_0$, for each $t \geq t_0$ and all $t_0 - \delta \leq t \leq t_0$, respectively, where $t_0 \geq 0$, $\delta > 0$, $\alpha \in (0, 1)$, u_0 is a constant, ${}^cD$ is the standard Caputo fractional derivative, D is the standard Riemann-Liouvill fractional derivative, function φ belongs to $C([t_0 - \delta, t_0], \mathbb{R})$, h_1 and h_2 map $(t_0, \infty) \times C([-\delta, 0], \mathbb{R})$ and $(t_0, \infty) \times \mathbb{R}$ into $\mathbb{R}$, respectively, are function with some properties [21, 22].

In 2013, Baleanu *et al.* by using fixed-point methods, studied the existence and uniqueness of a solution for the nonlinear fractional differential equation boundary-value problem $D^\alpha u(t) = f(t, u(t))$ with a Riemann-Liouville fractional derivative via the different boundary-value problems $u(0) = u(T)$, and the three-point boundary condition $u(0) = \beta_1 u(b)$ and $u(T) = \beta_2 u(b)$, where $T > 0$, $t \in [0, T]$, $0 < \alpha < 1$, $b \in (0, T)$ and $0 < \beta_1 < \beta_2 < 1$ [18]. Also, Zhao *et al.* reviewed the nonlocal q-integral boundary value problem of nonlinear fractional q-derivatives equation

$$(D_q^\alpha f)(t) + T(t, f(t)) = 0,$$

with conditions $f(0) = 0$ and $f(1) = \mu I_q^\beta f(\eta)$, for $t \in (0, 1)$ and $q \in (0, 1)$, where $\alpha \in (1, 2]$, $\beta \in (0, 2]$, $\eta \in (0, 1)$, positive real number

μ, D_q^α is the q-derivative of Riemann-Liouville type of order α and T maps $[0,1]\times[0,\infty)$ to $[0,\infty)$ is continuous [41]. In 2015, Agarwal *et al.* analyzed the existence of solutions for the Caputo fractional differential inclusion with the boundary value conditions

$$\begin{cases} {}^cD^\sigma u(t) \in F(t, u(t), {}^cD^\alpha u(t)), \\ u(0) = 0, \qquad u(1) + u'(1) = \int_0^\eta u(s)\mathrm{d}s, \end{cases}$$

such that $0<\eta<1$, $1<\sigma\le 2$, $0<\alpha<1$, $\sigma-\alpha>1$ and ${}^cD^\sigma u(t) \in F(t,u(t))$ under conditions $u(0) = a\int_0^\nu u(s)\mathrm{d}s$ and $x(1) = b\int_0^\eta u(s)\mathrm{d}s$, where $\nu,\eta\in(0,1)$, $\sigma\in(1,2]$, $a,b\in\mathbb{R}$ [4]. In 2016, Ahmad *et al.* investigate the existence of solutions for a a q-antipriodic boundary value problem of fractional q-difference inclusions given by

$$\begin{cases} {}^cD_q^\alpha f(t) \in T\left(t, f(t), D_q f(t), D_q^2 f(t)\right), \\ f(0) + f(1) = 0, \\ D_q f(0) + D_q f(1) = 0, \\ D_q^2 f(0) + D_q^2 f(1) = 0, \end{cases}$$

for $t\in[0,1]$, where $\alpha\in(2,3]$, $\beta\in[0,3]$, ${}^cD_q^\alpha$ denote Caputo fractional q-derivative, $q\in(0,1)$ and T maps $[0,1]\times A$ to $\mathcal{P}(\mathbb{R})$ is a multivalued map with $\mathcal{P}(\mathbb{R})$ a class of all subsets of $\mathbb{R}$, where $A=\mathbb{R}^3$ [5].

In 2017, Losada *et al.* by applying the Schauder fixed point theorem in conjunction with the technique of measure of non-compactness, presented some alternative results concerning with the existence and attractivity dependence of solutions for the following of nonlinear fractional functional differential equations

$$\begin{cases} {}^CD^\alpha u(t) = \sum_{i=1}^m {}^CD^{\alpha_i} T_i(t, u_t) + f_0(t, u_t), & t\in(t_0,\infty), \\ u(t) = \varphi(t), & t\in[t_0-\delta, t_0], \end{cases}$$

where ${}^CD^\alpha$ and ${}^CD^{\alpha_i}$ denote Caputo's fractional derivative of order $\alpha>0$ and $\alpha_i\in(0,\alpha)$, respectively, δ is a positive constant, φ belongs to $C([t_0-\delta,t_0],\mathbb{R})$ and for all $i\in\{1,2,\ldots,m\}$ and T_i maps $I\times C([\delta,0],\mathbb{R})$ into $\mathbb{R}$, such that $I=[t_0,\infty)$, is a given function [31]. After that, in 2018, Zhou *et al.* studied the existence and attractivity of fractional evolution equations with Riemann-Liouville fractional derivative

$${}_LD_{0^+}^\alpha u(t) = Au(t) + f(t, u(t)),$$

$I_{0^+}^{1-\alpha}u(0) = u_0$, for all $t \in [0, \infty)$, where ${}_L D_{0^+}^{\alpha}$ and $I_{0^+}^{1-\alpha}$ are is Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$, and Riemann-Liouville fractional integral of order $1 - \alpha$, respectively, A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{\tau(t)\}_{t \geq 0} \subset X$, f maps $[0, \infty) \times X$ into X is a given function, and $u_0 \in X$ where X is a Banach space [43].

In 2019, Balkani *et al.* studied the existence of approximate solutions for the fractional q-difference equation

$$({}^cD_q^{\sigma}u)(t) = w(t, u(t), I_q^{\alpha}u(t)),$$

with the q-integral boundary value conditions $u(0)u(1) = 0$, where ${}^cD_q^{\sigma}$ denote the fractional q-derivative of the Caputo type of order σ, $t \in [0, 1]$, $\sigma \in (1, 2]$, $q \in (0, 1)$ $\alpha \in (0, 2]$ and $w : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous map [19]. In Addition to, Samei *et al.* discussed the fractional hybrid q-differential inclusions

$${}^cD_q^{\alpha}\left(\frac{x}{f\left(t, x, I_q^{\alpha_1}x, \cdots, I_q^{\alpha_n}x\right)}\right) \in F\left(t, x, I_q^{\beta_1}x, \cdots, I_q^{\beta_k}x\right),$$

with the boundary conditions $x(0) = x_0$ and $x(1) = x_1$, where $1 < \alpha \leq 2$, $q \in (0, 1)$, $x_0, x_1 \in \mathbb{R}$, $\alpha_i > 0$, for $i = 1, 2, \ldots, n$, $\beta_j > 0$, for $j = 1, 2, \ldots, k$, $n, k \in \mathbb{N}$, ${}^cD_q^{\alpha}$ denotes Caputo type q-derivative of order α, I_q^{β} denotes Riemann-Liouville type q-integral of order β, $f : J \times \mathbb{R}^n \to (0, \infty)$ is continuous and $F : J \times \mathbb{R}^k \to P(\mathbb{R})$ is multifunction [40]. Also, Ntouyas *et al.* [32] studied the existence and uniqueness of solutions for a multi-term nonlinear fractional q-integro-differential equations under some boundary conditions

$$\begin{aligned}{}^cD_q^{\alpha}x(t) = w\big(&t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t),\\ &{}^cD_q^{\beta_1}x(t), {}^cD_q^{\beta_2}x(t), \ldots, {}^cD_q^{\beta_n}x(t)\big).\end{aligned}$$

Similar results have been presented in other studies [27, 30, 36, 37, 39].

In this article, motivated by [17, 31], among these achievements, we are working to stretch out the analytical and computational methods of check of attractivity of fractional functional q-differential equations in a

k-dimensional system with boundary value conditions

$$\begin{cases} {}^cD_q^{\alpha_1}u_1(t) = F_1\left(t, u_{1_t}, u_{2_t}, \dots, u_{k_t}\right), & t \in J, \\ \quad u_1(t) = \varphi_1(t), & t \in [t_0-\delta, t_0], \\ {}^cD_q^{\alpha_2}u_2(t) = F_2\left(t, u_{1_t}, u_{2_t}, \dots, u_{k_t}\right), & t \in J, \\ \quad u_2(t) = \varphi_2(t), & t \in [t_0-\delta, t_0], \\ \qquad \vdots \\ {}^cD_q^{\alpha_k}u_k(t) = F_k\left(t, u_{1_t}, u_{2_t}, \dots, u_{k_t}\right), & t \in J, \\ \quad u_k(t) = \varphi_k(t), & t \in [t_0-\delta, t_0], \end{cases} \tag{1}$$

where $\alpha_i \in I = (0,1)$, $t_0 \in \overline{J} = [t_0, \infty)$, $\delta > 0$ is a real constant, ${}^cD_q$ is the standard Caputo fractional type of q-derivative, functions φ_i in $C\left([t_0-\delta, t_0], \mathbb{R}^n\right)$, and $F_i : J \times \mathcal{C}_k \to \mathbb{R}^n$ is a function, for any i belongs to $N_k = \{1, 2, \dots, k\}$, where $J = (t_0, \infty)$ and

$$\mathcal{C}^k = \prod_{i \in N_k} C(\overline{J}_{-\delta}, \mathbb{R}^n),$$

where $\overline{J}_{-\delta} = [-\delta, 0]$. We define function u_t by $u_t(\eta) = u(t+\eta)$ for u in $C\left(\overline{J}_{-\delta}^{\infty}, \mathbb{R}^n\right)$, where $\eta \in \overline{J}_{-\delta}$, $t \in \overline{J}$, and $\overline{J}_{-\delta}^{\infty} = [t_0-\delta, \infty)$. Also, we investigate the global attractivity of nonlinear fractional q-differential equations in a k-dimensional system with boundary value conditions

$$\begin{cases} D_q^{\alpha_1}u_1(t) = G_1\left(t, u_1(t), u_2(t), \dots, u_k(t)\right), & t \in J, \\ \quad D_q^{\alpha_1-1}u_1(t) = u_1^0, & t = t_0, \\ D_q^{\alpha_2}u_2(t) = G_2\left(t, u_1(t), u_2(t), \dots, u_k(t)\right), & t \in J, \\ \quad D_q^{\alpha_2-1}u_2(t) = u_2^0, & t = t_0, \\ \qquad \vdots \\ D_q^{\alpha_k}u_k(t) = \theta_k\left(t, u_1(t), u_2(t), \dots, u_k(t)\right), & t \in J, \\ \quad D_q^{\alpha_k-1}u_k(t) = u_k^0, & t = t_0, \end{cases} \tag{2}$$

where $\alpha_i \in I$, $t \in J$, D_q is the Riemann-Liouville fractional q-derivative, u_i^0 are constants for all $i \in N_k$, and $G_i : J \times \mathcal{R}_k \to \mathbb{R}^n$ is an integrable function where $\mathcal{R}^k = \prod_{i \in N_k} \mathbb{R}^n$. The functions F_i and G_i in Eq. (1)

and (2) have some properties for $i \in N_k$ which will be defined in Sec. 3.

The rest of the paper is arranged as follows: In Sec. 2, we recall some preliminary concepts, fundamental results of q-calculus and some theorems which were used in the our results. Sec. 3 is devoted to the main results, while example illustrating the obtained results and algorithm for the problems are presented in Sec. 4. Finally in Sec. 5, we state the conclusion.

2 Preliminaries

Below, we recall some known facts on the fractional q-calculus and fundamental results of it (for more information, consider [2, 9, 25, 33]). Then, some well-known theorems of fixed point theorem and definition are expressed.

Let $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [25]. The power function $(a-b)_q^n$ with $n \in \mathbb{N}_0$ is

$$(a-b)_q^{(n)} = \prod_{k=0}^{n-1} (a - bq^k)$$

and $(a-b)_q^{(0)} = 1$ where $a, b \in \mathbb{R}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ [33]. Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(a-b)_q^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}}.$$

If $b = 0$, then it is clear that $a^{(\alpha)} = a^\alpha$ (Algorithm 1). The q-Gamma function is given by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}},$$

where $x \in \mathbb{R} \backslash \{0, -1, -2, \cdots\}$ [25]. Note that, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$. The value of q-Gamma function, $\Gamma_q(x)$, for input values q and x with counting the number of sentences n in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating q-Gamma function of order n which show in Algorithm 2.

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^\alpha * s$
end if

Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

For any positive number α and β, the q-Beta function define by

$$B_q(\alpha, \beta) = \int_0^1 (1 - qs)_q^{(\alpha-1)} s^{\beta-1} \, \mathrm{d}_q s. \tag{3}$$

For function f, the q-derivative is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ which is shown in Algorithm 3 [2].

Input: n, $q \in (0,1)$, $x \in \mathbb{R}\backslash\{0, -1, 2, \cdots\}$

$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1 - q^{k+1})(1 - q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$

Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

Input: $q \in (0,1)$, $f(x)$, x

```
syms z
if x = 0 then
    g ← lim((f(z) − f(q ∗ z))/((1 − q)z), z, 0)
else
    g ← (f(x) − f(q ∗ x))/((1 − q)x)
end if
```

Output: $(D_q f)(x)$

Also, the higher order q-derivative of a function f is defined by

$$(D_q^n f)(x) = D_q(D_q^{n-1} f)(x),$$

for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [2]. The q-integral of a function f defined on $[0, b]$ is define by

$$I_q f(x) = \int_0^x f(s)\,\mathrm{d}_q s = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k),$$

for $x \in [0, b]$, provided that the sum converges absolutely [2]. If $a \in [0, b]$, then

$$\int_a^b f(u)\,\mathrm{d}_q u = I_q f(b) - I_q f(a) = (1-q)\sum_{k=0}^{\infty} q^k \left[bf(bq^k) - af(aq^k) \right],$$

whenever the series exists. The operator I_q^n is given by $(I_q^0 f)(x) = f(x)$ and $(I_q^n f)(x) = (I_q(I_q^{n-1} f))(x)$ for all $n \geq 1$ [2]. It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ [2]. The fractional Riemann-Liouville type q-integral of the function f on $[0, 1]$, of $\alpha \geq 0$ is given by $(I_q^0 f)(x) = f(x)$ and

$$\begin{aligned}(\mathcal{I}_q^{\alpha} f)(x) &= \frac{1}{\Gamma_q(\alpha)} \int_0^x (t - qs)^{(\alpha-1)} f(s)\,\mathrm{d}_q s \\ &= t^{\alpha}(1-q)^{\alpha} \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}\left(1 - q^{\alpha+i}\right)}{\prod_{i=1}^{k-1}\left(1 - q^{i+1}\right)} f(xq^k),\end{aligned}$$

for $x \in [0,1]$ and $\alpha > 0$ [5, 9, 23]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned}\left({}^{c}D_q^{\alpha} f\right)(x) &= \left(I_q^{[\alpha]-\alpha}(D_q^{[\alpha]} f)\right)(x)\\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^x (x-qs)^{([\alpha]-\alpha-1)} \left(D_q^{[\alpha]} f\right)(s)\, \mathrm{d}_q s,\end{aligned} \tag{4}$$

for $x \in [0,1]$ and $\alpha > 0$ [5, 23]. It has been proved that $(I_q^{\beta}(I_q^{\alpha} f))(x) = (I_q^{\alpha+\beta} f)(x)$, and $(D_q^{\alpha}(I_q^{\alpha} f))(x) = f(x)$, where $\alpha, \beta \geq 0$ [23]. By using Algorithm 2, we can calculate $(I_q^{\alpha} f)(x)$ which is shown in Algorithm 4.

Algorithm 4 The proposed method for calculated $(\mathcal{I}_q^{\sigma} f)(x)$

Input: $q \in (0,1)$, σ, x, n, $f(x)$,

```
p ← 0
for k = 0 to n do
    s1 ← 1
    s2 ← 1
    for i = 0 to k − 1 do
        s1 ← s1 × (1 − q^(i+σ))
        s2 ← s2 × (1 − q^(i+1))
    end for
    p ← P + q^k ∗ s1 ∗ f(x ∗ q^k)/s2
end for
g ← round((x^σ) ∗ ((1 − q)^σ) ∗ p, 6)
```

Output: $(I_q^{\alpha} f)(x)$

In the following, we point out and improvement of the well-known fixed point theorem of Schauder and Krasnoselskii, respectively, due to Burton which one can get those in [20, 29, 35].

Theorem 2.1. *Consider a nonempty subset A of the Banach space $\mathcal{X}$. The completely continuous self-map $\Theta : A \to \mathcal{X}$ has a fixed point, whenever A is closed, bounded and convex.*

Theorem 2.2. *The operator equation $H_1(u) + H_2(u) = u$ has a solution in a nonempty A subset of Banach space $\mathcal{X}$ whenever A is closed, convex and bounded, where self-map H_1 define on $\mathcal{X}$ is a contraction with constant $k < 1$, function H_2 maps A into $\mathcal{X}$ is a continuous which $H_2(A)$ resides in a compact subset of $\mathcal{X}$ such that $u = H_1(u) + H_2(v)$ and $v \in A$ implies $u \in A$.*

The solution $(u_1(t), u_2(t), \ldots, u_k(t))$ of the problem (1) and $u(t)$ of the problem (2) are said to be attractive and globally attractive, whenever if there exists a constant $c_i^0(t_0) > 0$ such that $|\varphi_i(s)| \leq c_i^0$ for all $i \in N_k$, $s \in \overline{J}_{-\delta}^{t_0} = [t_0 - \delta, t_0]$, then $\lim_{t\to\infty} u_i(t, t_0, \varphi_i)$ tends to zero and each solution tends to zero as $t \to \infty$, respectively. We consider the Banach space of all continuous functions define on J into $\mathbb{R}^n$ endowed with the norm $\|u\| = \sup_{t\in J} |u(t)|$, and denote by $\mathcal{A} = C(J, \mathbb{R}^n)$, where $|.|$ is a norm on $\mathbb{R}^n$ somehow that is suitable complete. It is readable that the product space $\left(\mathcal{A}^k, \|.\|_*\right)$ is also a Banach space, where $\mathcal{A}^k = \prod_{i\in N_k} \mathcal{A}$ and

$$\|(u_1, u_2, \ldots, u_k)\|_* = \sum_{i=1}^{k} \|u_i\|.$$

3 Main results

In this section by using the last two results and basic definition, we investigate attractive solutions and zero solution of the problem (1) and (2), respectively.

3.1 Attractivity of solution for the problem (1)

Let $\overline{J}_{-\delta}^{0} = [-\delta, 0]$. Consider the problem (1) and the supremum norm

$$\|u_t\| = \sup\left\{|u(t+\eta)| \,:\, \eta \in \overline{J}_{-\delta}^{0}\right\},$$

for almost all $t \in J$, Lebesgue measurable functions $F_i(t, u_{1_t}, x_{2_t}, \ldots, u_{k_t})$ with respect to t on $\overline{J}$ and continuous functions $F_i\left(t, \varphi_1, \varphi_2, \ldots, \varphi_k\right)$ with respect to φ_j on $\mathcal{C}$ for i and j belong to N_k. For finding the attractivity of solution of problem (1), we consider the equivalent system of equations

$$u_i(t) = \varphi_i(t_0) + \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \widetilde{F}_i\left(s, u_{i_s}\right) \mathrm{d}_q s,$$

and $u_i(t) = \varphi_i(t)$ for almost all $t \in J$ and each $t \in \overline{J}^{t_0}_{-\delta}$, respectively, or

$$u_i(t) = \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \times \left[\frac{\varphi_i(t_0)}{\Gamma_q(1 - \alpha_i)} (s - t_0)^{-\alpha_i} + \widetilde{F}_i\left(s, u_{i_s}\right) \right] \mathrm{d}_q s,$$

and $u_i(t) = \varphi_i(t)$ for almost all $t \in J$ and all $t \in \overline{J}^{t_0}_{-\delta}$, respectively, for $i \in N_k$, where

$$\widetilde{F}_i\left(s, u_{i_s}\right) = F_i\left(s, u_{1_s}, u_{2_s}, \ldots, u_{k_s}\right).$$

We take the operator $\Theta : \mathcal{A}^k \to \mathcal{A}^k$ by

$$\Theta\left(u_1, u_2, \ldots, u_k\right)(t) = \left(\widetilde{\theta}_1(t), \widetilde{\theta}_2(t), \cdots, \widetilde{\theta}_k(t)\right),$$

where

$$\widetilde{\theta}_i(t) = \varphi_i(t_0) + \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \widetilde{F}_i\left(s, u_{i_s}\right) \mathrm{d}_q s,$$

and $\widetilde{\theta}_i(t) = \varphi_i(t)$ whenever $t \in J$ and $t \in \overline{J}^{t_0}_{-\delta}$, respectively, for $i \in N_k$, where $\widetilde{\theta}_i(t) = \theta_i\left(u_1, u_2, \ldots, u_k\right)(t)$. By simple to go over that, we accept $(u_1(t), u_2(t), \ldots, u_k(t))$ is a solution of the problem (1) if and only if $(u_1(t), u_2(t), \ldots, u_k(t))$ is a fixed point of the operator Θ.

Theorem 3.1. *Let $\overline{J}^{\infty}_{-\delta} = [t_0 - \delta, \infty)$. The problem (1) has at least one attractive solution $(u_1, u_2, \ldots, u_k)$ with $u_i \in C(\overline{J}^{\infty}_{-\delta}, \mathbb{R}^n)$ for all $i \in N_k$, whenever, for each $i \in N_k$, there exist $\beta_{1i} > 0$ and κ_{1i} belong to $(0, \alpha_i)$ such that*

$$\left| \varphi_i(t_0) + \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \widetilde{F}_i\left(s, u_{i_s}\right) \mathrm{d}_q s \right| \leq (t - t_0)^{-\beta_{1i}},$$

for all $t \in J$ and $F_i \in L^{\frac{1}{\kappa_{1i}}}\left(J \times \mathcal{C}^k\right)$ where

$$\widetilde{F}_i\left(s, u_{i_s}\right) = F_i\left(s, u_{1_s}, u_{2_s}, \ldots, u_{k_s}\right).$$

Proof. Let Ω_1 is the set of all $(u_1, u_2, \ldots, u_k)$ with $u_i \in C(\overline{J}_{-\delta}^{\infty}, \mathbb{R}^n)$, such that $|u_i(t)| \leq (t - t_0)^{-\beta_{1i}}$, for all $i \in N_k$ and $t \in [\tau_1, \infty)$, where $\tau_1 > t_0$ is a constant. It is clear that $\Omega_1 \subseteq \mathcal{R}^k$ is a closed, bounded and convex. We show that the operator Θ has a fixed point in S_1. This implies that the problem (1) has a solution. Note that,

$$|\theta_i\left(u_1, u_2, \ldots, u_k\right)(t)| \leq (t - t_0)^{-\beta_{1i}},$$

for all i in N_k and so $\Theta(\Omega_1) \subset \Omega_1$. At present, we prove that Θ is continuous. Assume that $(u_1^m, u_2^m, \ldots, u_k^m)$ for all $m \geq 1$, and $(u_1, u_2, \ldots, u_k)$ belong to Ω_1 such that $\lim_{m\to\infty} |u_i^m(t) - u_i(t)| = 0$ for all $i \in N_k$. Then, we have

$$\lim_{m\to\infty} F_i\left(t, u_{1_t}^m, u_{2_t}^m, \ldots, u_{k_t}^m\right) = F_i\left(t, u_{1_t}, u_{2_t}, \ldots, u_{k_t}\right),$$

for all i and $t \in J$. Choose $\tilde{\tau}_1 \in J$ such that $(t - t_0)^{-\beta_{1i}} < \frac{\varepsilon}{2}$ whenever $\tilde{\tau}_1 \leq t$, where $\varepsilon > 0$ be given. Let $\lambda_{1i} = \frac{\alpha_i - 1}{1 - \kappa_{1i}}$ and note that $1 + \lambda_{1i} > 0$ for $i \in N_k$. Also, we obtain

$$\begin{aligned}
\left|\tilde{\theta}_i\left(u_i^m\right)(t)\right. & \left. -\tilde{\theta}_i\left(u_i\right)(t)\right| \\
& \leq \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \left|\tilde{F}_i\left(u_{i_s}^m\right) - \tilde{F}_i\left(u_{i_s}\right)\right| \mathrm{d}_q s \\
& \leq \frac{1}{\Gamma_q(\alpha_i)} \left[\int_{t_0}^{t} \left[(t - qs)^{(\alpha_i - 1)}\right]^{\frac{1}{1-\kappa_{1i}}} \mathrm{d}_q s\right]^{1-\kappa_{1i}} \\
& \quad \times \left[\int_{t_0}^{t} \left|\tilde{F}_i\left(u_{i_s}^m\right) - \tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}} \mathrm{d}_q s\right]^{\kappa_{1i}} \\
& \leq \frac{1}{\Gamma_q(\alpha_i)} \left[\frac{1}{1 + \lambda_{1i}} (t - t_0)^{1+\lambda_{1i}}\right]^{1-\kappa_{1i}} \\
& \quad \times \left[\int_{t_0}^{\tilde{\tau}_1} \left|\tilde{F}_i\left(u_{i_s}^m\right) - \tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}} \mathrm{d}_q s\right]^{\kappa_{1i}} \\
& \leq \frac{1}{\Gamma_q(\alpha_i)} \left[\frac{1}{1 + \lambda_{1i}} (\tilde{\tau}_1 - t_0)^{1+\lambda_{1i}}\right]^{1-\kappa_{1i}} (\tilde{\tau}_1 - t_0)^{\kappa_{1i}} \\
& \quad \times \sup_{s\in[t_0, \tilde{\tau}_1]} \left|\tilde{F}_i\left(u_{i_s}^m\right) - \tilde{F}_i\left(u_{i_s}\right)\right|,
\end{aligned}$$

for $t \in (t_0, \tilde{\tau}_1]$, where

$$\tilde{\theta}_i\left(u_i^m\right)(t) = \left(u_1^m, u_2^m, \ldots, u_k^m\right)(t),$$

$\tilde{\theta}_i\left(u_i\right)(t) = \theta_i\left(u_1, u_2, \ldots, u_k\right)(t)$, and

$$\begin{aligned} \tilde{F}_i\left(u_{i_s}^m\right) &= F_i\left(s, u_{1_s}^m, u_{2_s}^m, \ldots, u_{k_s}^m\right), \\ \tilde{F}_i\left(u_{i_s}\right) &= F_i\left(s, u_{1_s}, u_{2_s}, \ldots, u_{k_s}\right). \end{aligned}$$

Thus, for all $t_0 < t \leq \tilde{\tau}_1$, we have

$$\lim_{m \to \infty} \left| \tilde{\theta}_i\left(u_i^m\right)(t) - \tilde{\theta}_i\left(u_i\right)(t) \right| = 0.$$

Also, we obtain

$$\begin{aligned} \left| \tilde{\theta}_i\left(u_i^m\right)(t) - \tilde{\theta}_i\left(u_i\right)(t) \right| &= \left| \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \tilde{F}_i\left(u_{i_s}^m\right) \mathrm{d}_q s \right. \\ &\quad \left. - \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \tilde{F}_i\left(u_{i_s}\right) \mathrm{d}_q s \right| \\ &\leq 2(t - t_0)^{-\beta_{1i}} \leq \varepsilon, \end{aligned}$$

for $\tilde{\tau}_1 < t$. Hence, for almost all $t \in J$, we have

$$\lim_{m \to \infty} \left| \tilde{\theta}_i\left(u_i^m\right)(t) - \tilde{\theta}_i\left(u_i\right)(t) \right| = 0.$$

Therefore, we conclude that θ_i is continuous for $i \in N_k$ and so Θ is continuous. Assume that $\varepsilon > 0$ be given. Since for i in N_k, we have $\lim_{t \to \infty}(t - t_0)^{-\beta_{1i}} = 0$, there is a $\tilde{\tau}_2 \in J$ such that $(t - t_0)^{-\beta_{1i}} < \frac{\varepsilon}{2}$ for all $\tilde{\tau}_2 < t$ and $i \in N_k$. Let ν_1 and ν_2 belong to J somehow that $\nu_1 < \nu_2$. At present, we consider three cases.

1) If $\nu_1, \nu_2 \in (t_0, \tilde{\tau}_2]$, then

$$
\begin{aligned}
&\left|\tilde{\theta}_i\ (u_i)(\nu_2) - \tilde{\theta}_i(u_i)(\nu_1)\right| \\
&\quad\leq \left|\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{\nu_2}(\nu_2 - qs)^{(\alpha_i-1)}\tilde{F}_i\left(u_{i_s}\right)\,\mathrm{d}_q s \right.\\
&\qquad \left. - \frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{\nu_1}(\nu_1 - qs)^{(\alpha_i-1)}\tilde{F}_i\left(u_{i_s}\right)\,\mathrm{d}_q s\right| \\
&\quad\leq \frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{\nu_1}\left[(\nu_1 - qs)^{(\alpha_i-1)} - (\nu_2 - qs)^{(\alpha_i-1)}\right]|\tilde{F}_i\left(u_{i_s}\right)|\,\mathrm{d}_q s \\
&\qquad + \frac{1}{\Gamma_q(\alpha_i)}\int_{\nu_1}^{\nu_2}(\nu_2 - qs)^{(\alpha_i-1)}|\tilde{F}_i\left(u_{i_s}\right)|\,\mathrm{d}_q s \\
&\quad\leq \frac{1}{\Gamma_q(\alpha_i)}\left[\int_{t_0}^{\nu_1}\left[(\nu_1 - qs)^{(\alpha_i-1)} - (\nu_2 - qs)^{(\alpha_i-1)}\right]^{\frac{1}{1-\kappa_{1i}}}\,\mathrm{d}_q s\right]^{1-\kappa_{1i}} \\
&\qquad \times\left[\int_{t_0}^{\nu_1}\left|\tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}}\,\mathrm{d}_q s\right]^{\kappa_{1i}} \\
&\qquad + \frac{1}{\Gamma_q(\alpha_i)}\left[\int_{\nu_1}^{\nu_2}(\nu_2 - qs)^{(\frac{\alpha_i-1}{1-\kappa_{1i}})}\,\mathrm{d}_q s\right]^{1-\kappa_{1i}} \\
&\qquad \times\left[\int_{\nu_1}^{\nu_2}\left|\tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}}\,\mathrm{d}s\right]^{\kappa_{1i}} \\
&\quad\leq \frac{1}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda_{1i}}\right]^{1-\kappa_{1i}} \\
&\qquad \times\left[(\nu_1 - t_0)^{\frac{\alpha_i-1}{1-\kappa_{1i}}+1} + (\nu_2 - \nu_1)^{\frac{\alpha_i-1}{1-\kappa_{1i}}+1} - (\nu_2 - t_0)^{\frac{\alpha_i-1}{1-\kappa_{1i}}+1}\right]^{1-\kappa_{1i}} \\
&\qquad \times\left[\int_{t_0}^{\tilde{\tau}_2}\left|\tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}}\,\mathrm{d}s\right]^{\kappa_{1i}} \\
&\qquad + \frac{1}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda_{1i}}\right]^{1-\kappa_{1i}}\left[(\nu_2 - \nu_1)^{\frac{\alpha_i-1}{1-\kappa_{1i}}+1}\right]^{1-\kappa_{1i}} \\
&\qquad \times\left[\int_{t_0}^{\tilde{\tau}_2}\left|\tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}}\,\mathrm{d}s\right]^{\kappa_{1i}} \\
&\quad\leq \frac{1}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda_{1i}}\right]^{1-\kappa_{1i}}\left[\int_{t_0}^{\tilde{\tau}_2}\left|\tilde{F}_i\left(u_{i_s}\right)\right|^{\frac{1}{\kappa_{1i}}}\,\mathrm{d}s\right]^{\kappa_{1i}}(\nu_2 - \nu_1)^{\alpha_i - \kappa_{1i}},
\end{aligned}
$$

and so $\lim_{\nu_2\to\nu_1}\left|\tilde{\theta}_i\left(u_i\right)\left(\nu_2\right)-\tilde{\theta}_i\left(u_i\right)\left(\nu_1\right)\right|=0$.

2) If $\nu_1,\nu_2\in(\tilde{\tau}_2,\infty)$, then

$$\begin{aligned}\left|\tilde{\theta}_i\left(u_i\right)\left(\nu_2\right)-\tilde{\theta}_i\left(u_i\right)\left(\nu_1\right)\right| &= \left|\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{\nu_2}(\nu_2-qs)^{(\alpha_i-1)}\tilde{F}_i\left(u_{i_s}\right)\mathrm{d}_qs\right.\\ &\quad\left.-\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{\nu_1}(\nu_1-qs)^{(\alpha_i-1)}\tilde{F}_i\left(u_{i_s}\right)\mathrm{d}_qs\right|\\ &\le(\nu_2-t_0)^{-\beta_{1i}}+(\nu_1-t_0)^{-\beta_{1i}}\\ &\le\varepsilon.\end{aligned}$$

3) If $\nu_1\in(t_0,\tilde{\tau}_2)$ and $\nu_2\in(\tilde{\tau}_2,\infty)$, then by triangle inequality

$$\begin{aligned}\left|\tilde{\theta}_i\left(u_i\right)\left(\nu_2\right)-\tilde{\theta}_i\left(u_i\right)\left(\nu_1\right)\right| &\le \left|\tilde{\theta}_i\left(u_i\right)\left(\nu_2\right)-\tilde{\theta}_i\left(u_i\right)\left(\tilde{\tau}_2\right)\right|\\ &\quad+\left|\tilde{\theta}_i\left(u_i\right)\left(\tilde{\tau}_2\right)-\tilde{\theta}_i\left(u_i\right)\left(\nu_1\right)\right|,\end{aligned}$$

we get $\lim_{\nu_2\to\nu_1}\left|\tilde{\theta}_i\left(u_1\right)\left(\nu_2\right)-\tilde{\theta}_i\left(u_1\right)\left(\nu_1\right)\right|=0$.

Regarding all cases, we conclude that the set $\Theta(\Omega_1)$ is equi-continuous. So, $\Theta(\Omega_1)$ is relatively compact, because $\Theta(\Omega_1)\subset\Omega_1$ is uniformly bounded. At present, by employing Theorem 2.1, we have the problem (1) has a solution $u(t)=(u_1(t),u_2(t),\ldots,u_k(t))\in\Omega_1$, which is fixed point of Θ. Hence, $\lim_{t\to\infty}u(t)=0$. Indeed, $u(t)$ is an attractive solution for the problem (1). □

Theorem 3.2. *The k-dimensional system (1) has at least one attractive solution $(u_1,u_2,\ldots,u_k)$ with $u_i\in C(\overline{J}_{-\delta}^{\infty},\mathbb{R}^n)$ for all $i\in N_k$, whenever, for each i there exist $\beta_{2i}>0$, $\kappa_{2i}\in(0,\alpha_i)$ and $\mu_i\in L^{\frac{1}{\kappa_{2i}}}(J,(0,\infty))$ such that*

$$\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t}(t-qs)^{(\alpha_i-1)}\mu_i(s)(s-t_0)^{-\beta_{2i}}\mathrm{d}_qs\le(t-t_0)^{-\beta_{2i}},$$

and

$$\left|\frac{\varphi_i(t_0)}{\Gamma_q(1-\alpha_i)}(t-t_0)^{-\alpha_i}+\tilde{F}_i\left(u_{i_t}\right)\right|\le\mu_i(t)\|u_{i_t}\|,$$

for each $i\in N_k$, $t\in J$ and $u_i\in C(\overline{J}_{-\delta}^{\infty},\mathbb{R}^n)$, where

$$\tilde{F}_i\left(u_{i_t}\right)=F_i\left(t,u_{1_t},u_{2_t},\ldots,u_{k_t}\right).$$

Proof. By similarly techniques of proof in Theorem 3.1, sufficient we consider the set Ω_2 of all $(u_1, u_2, \ldots, u_k)$ with $u_i \in C(\overline{J}_{-\delta}^{\infty}, \mathbb{R}^n)$ such that $\|u_{i_t}\| \le (t - t_0)^{-\beta_{2i}}$ for $i \in N_k$ and t belongs to $[\tau, \infty)$, where $\tau > t_0$ is a constant, one can show that $\Theta(\Omega_2) \subset \Omega_2$, Θ is continuous and $\Theta(\Omega_2)$ is relatively compact. At present, by applying Theorem 2.1, we conclude that the problem (1) has a solution $u(t) = (u_1(t), u_2(t), \ldots, u_k(t)) \in \Omega_2$ which is a fixed point of Θ. Hence, $u(t)$ is an attractive solution, because $\lim_{t\to\infty} u(t) = 0$. □

Theorem 3.3. *The k-dimensional system (1) has at least one attractive solution $(u_1, u_2, \ldots, u_k)$ with $u_i \in C(\overline{J}_{-\delta}^{\infty}, \mathbb{R}^n)$, whenever for each $i \in N_k$ there exists $\kappa'_{1i} \in (\alpha_i, 1)$ such that*

$$\left| \frac{\varphi_i(t_0)}{\Gamma_q(1-\alpha_i)} (t - t_0)^{-\alpha_i} + \tilde{F}_i\left(u_{i_t}\right) \right| \le \frac{\Gamma_q(1 + \alpha_i - \kappa'_{1i})}{\Gamma_q(1 - \beta'_{1i})} (t - t_0)^{-\beta'_{1i}},$$

for all $t \in J$, where $\tilde{F}_i\left(u_{i_t}\right) = f_i(t, x_{1_t}, x_{2_t}, \ldots, x_{k_t})$.

Proof. We consider the set Ω_3 of all $(u_1, u_2, \ldots, u_k)$ and $u_i \in C(\overline{J}_{-\delta}^{\infty}, \mathbb{R}^n)$, such that $|u_i(t)| \le (t - t_0)^{\beta'_{1i} - \alpha_i}$, for all $i \in N_k$ and $t \in [\tau, \infty)$, where constant τ is more than t_0. Now, by employing a similar techniques in proof of Theorem 3.1, we conclude that $\Theta(\Omega_3)$ subset of Ω_3, Function Θ is continuous and $\Theta(\Omega_3)$ is relatively compact. Hence, by employing the Theorem 2.1, we get the problem (1) has a solution

$$u(t) = (u_1(t), u_2(t), \ldots, u_k(t)) \in \Omega_3,$$

which is a fixed point of Θ. Thus, $u(t)$ is an attractive solution, because, $\lim_{t\to\infty} u(t) = 0$. □

3.2 Global attractivity of solution for the system (2)

In the second part, we discuss to global attractivity of the k-dimensional system (2). Let us, we consider the integrable function

$$G_i(t, u_1(t), u_2(t), \ldots, u_k(t))$$

is Lebesgue measurable with respect to t on $\overline{J}$ and there exists a constant κ_{1i} in $(0, \alpha_i)$ such that $G_i \in L^{\frac{1}{\kappa_{1i}}}(J \times \mathcal{R}^k)$ and $G_i(t, u_1(t), u_2(t), \ldots, u_k(t))$

is continuous with respect to u_j on $\overline{J}$, for any i and j belong to N_k. For finding the global attractivity of solution of problem (2), we consider the equivalent system of equations

$$\begin{aligned} u_i(t) &= \frac{u_i^0}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i-1} \\ &\quad + \frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t}(t-qs)^{(\alpha_i-1)}\widetilde{G}_i\left(s,u_i(s)\right)\,\mathrm{d}_q s, \end{aligned}$$

for all $t \in J$ and $i \in N_k$ where

$$\widetilde{G}_i(s,u_i(s)) = G_i\left(s,u_1(s),u_2(s),\ldots,u_k(s)\right).$$

We define the operator Θ on $\mathcal{A}^k$ to $\mathcal{A}^k$ by

$$\Theta(u_1,u_2,\ldots,u_k)(t) = \big(\widetilde{\theta}_1(t),\widetilde{\theta}_2(t),\cdots,\widetilde{\theta}_k(t)\big),$$

where

$$\begin{aligned} \widetilde{\theta}_i(t) &= \frac{u_i^0}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i-1} \\ &\quad + \frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t}(t-qs)^{(\alpha_i-1)}\widetilde{G}_i\left(s,u_i(s)\right)\,\mathrm{d}_q s, \end{aligned}$$

for each $i \in N_k$, where $\widetilde{\theta}_i(t) = \theta_i(u_1,u_2,\ldots,u_k)(t)$. At present, we define two operators

$$\begin{aligned} H_1(u_1,u_2,\ldots,u_k)(t) &= \left(\tilde{h}_{11}(u_i),\tilde{h}_{12}(u_i),\ldots,\tilde{h}_{1k}(u_i)\right), \\ H_2(u_1,u_2,\ldots,u_k)(t) &= \left(\tilde{h}_{21}(u_i),\tilde{h}_{22}(u_i),\ldots,\tilde{h}_{2k}(u_i)\right), \end{aligned}$$

where

$$\begin{aligned} \tilde{h}_{1i}(u_i) &= h_{1i}(u_1,u_2,\ldots,u_k)(t) = \frac{u_i^0}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i-1}, \\ \tilde{h}_{2i}(u_i) &= h_{2i}\left(u_1,u_2,\ldots,u_k\right)(t) \\ &= \frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t}(t-qs)^{(\alpha_i-1)}\widetilde{G}_i\left(s,u_i(s)\right)\,\mathrm{d}_q s, \end{aligned}$$

for all $i \in N_k$. Therefor, it can be concluded that

$$(u_1(t), u_2(t), \dots, u_k(t)),$$

is a solution of the k-dimensional system (2) if and only if it is a fixed point of the operator Θ. As you can see, the contraction constant of the operator H_1 is zero.

Theorem 3.4. *The zero solution of the k-dimensional system (2) is globally attractive, whenever there exist $\kappa'_{1i} \in (\alpha_i, 1)$ and positive real number $p_i \geq 0$ such that*

$$\left| \widetilde{G}_i \left(s, u_i(s)\right) \right| \leq p_i (t - t_0)^{-\kappa'_{1i}},$$

for all $t \in J$ and $u_i \in C(J, \mathbb{R}^n)$, for each $i \in N_k$.

Proof. We define the set Ω'_1 of all $(u_1, u_2, \dots, u_k)$ with u_i belong to $C(J, \mathbb{R}^n)$ such that

$$|u_i(t)| \leq (t - t_0)^{-\kappa'_{1i}},$$

for all $i \in N_k$ and $t \in [t_0 + \tau, \infty)$, where $\beta'_{1i} = \frac{1}{2}(\kappa'_{1i} - \alpha_i)$ and τ is chosen such that

$$\begin{aligned} |u_i^0| \Gamma_q(1 + \alpha_i - \kappa'_{1i}) \tau^{\frac{1}{2}(\alpha_i - 1)} &+ p_i \Gamma_q(1 - \kappa'_{1i}) \Gamma_q(\alpha_i) \\ &\leq \Gamma_q(\alpha_i) \Gamma_q(1 + \alpha_i - \kappa'_{1i}), \end{aligned}$$

for each i belongs to N_k. Foremost, we prove that H_2 is self-maps on Ω'_1. It is leisurely to get over that the subset Ω'_1 of $\mathcal{R}^k$ is a bounded, closed and convex. On the other hand,

$$\begin{aligned} &\left| h_{2i} \left(v_1, v_2, \dots, v_k\right)(t) \right| \\ &\quad \leq \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \left| \widetilde{G}_i \left(s, u_i(s)\right) \right| \mathrm{d}_q s \\ &\quad \leq \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} p_i (s - t_0)^{-\kappa'_{1i}} \mathrm{d}s \\ &\quad \leq \frac{p_i \Gamma_q(1 - \kappa'_{1i})}{\Gamma_q(1 + \alpha_i - \kappa'_{1i})} (t - t_0)^{-(\kappa'_{1i} - \alpha_i)}, \end{aligned}$$

and

$$\begin{aligned}\frac{p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}(t-t_0)^{-\frac{1}{2}(\kappa'_{1i}-\alpha_i)} &\le \frac{p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}\tau^{-\frac{1}{2}(\kappa'_{1i}-\alpha_i)}\\ &\le 1,\end{aligned}$$

for all $i \in N_k$ and $t \in [t_0+\tau, \infty)$. Thus,

$$\begin{aligned}|h_{2i}(v_1, v_2, \ldots, v_k)(t)| &\le \left[\frac{p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}(t-t_0)^{-\frac{1}{2}(\kappa'_{1i}-\alpha_i)}\right]\\ &\quad\times(t-t_0)^{-\frac{1}{2}(\kappa'_{1i}-\alpha_i)}\\ &\le (t-t_0)^{-\beta'_{1i}},\end{aligned}$$

for almost all $t \in [t_0+\tau, \infty)$ and for all $i \in N_k$. Hence, $H_2(\Omega'_1) \subset \Omega'_1$. Let $(v_1^m, v_2^m, \ldots, v_k^m)$ for all natural numbers m, and $(v_1, v_2, \ldots, v_k)$ belong to Ω'_1 somehow that $\lim_{m\to\infty}|v_i^m(t) - v_i(t)| = 0$. Then, one can get

$$\lim_{m\to\infty} G_i\left(t, v_1^m(t), v_2^m(t), \ldots, v_k^m(t)\right) = G_i\left(t, v_1(t), v_2(t), \ldots, v_k(t)\right),$$

for all t belongs to $[t_0+\tau, \infty)$. Choose $\tau_1 \in [t_0+\tau, \infty)$ such that

$$\frac{p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}(\tau_1-t_0)^{-(\kappa'_{1i}-\alpha_i)} < \frac{\varepsilon}{2},$$

for all $t \in (\tau_1, \infty)$, where $\varepsilon > 0$ be given. Take $\lambda'_{1i} = \frac{\alpha_i-1}{1-\kappa'_{1i}}$ for $i \in N_k$. Therefore, we get

$$\begin{aligned}\left|\tilde{h}_{2i}(v_i^m)(t) - \tilde{h}_{2i}(v_i)(t)\right| &\le \frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t}(t-qs)^{(\alpha_i-1)}\\ &\quad\times\left|\widetilde{G}_i(s, v_i^m(s)) - \widetilde{G}_i(s, v_i(s))\right| \mathrm{d}_q s\\ &\le \frac{1}{\Gamma_q(\alpha_i)}\left[\int_{t_0}^{t}\left[(t-qs)^{(\alpha_i-1)}\right]^{\frac{1}{1-\kappa'_{1i}}} \mathrm{d}_q s\right]^{1-\kappa'_{1i}}\\ &\quad\times\left[\int_{t_0}^{t}\left|\widetilde{G}_i(s, v_i^m(s)) - \widetilde{G}_i(s, v_i(s))\right|^{\frac{1}{\kappa'_{1i}}} \mathrm{d}_q s\right]^{\kappa'_{1i}}\\ &\le \frac{1}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda'_{1i}}(t-t_0)^{1+\lambda'_{1i}}\right]^{1-\kappa'_{1i}}\end{aligned}$$

$$
\begin{aligned}
&\times \left[\int_{t_0}^{\tau_2} \left| \widetilde{G}_i\left(s, v_i^m(s)\right) - \widetilde{G}_i\left(s, v_i(s)\right) \right|^{\frac{1}{\kappa'_{1i}}} \, \mathrm{d}s \right]^{\kappa'_{1i}} \\
&\leq \frac{1}{\Gamma_q(\alpha_i)} \left[\frac{1}{1+\lambda'_{1i}} (\tilde{\tau}_2 - t_0)^{1+\lambda'_{1i}} \right]^{1-\kappa'_{1i}} (\tau_1 - t_0)^{\kappa'_{1i}} \\
&\times \sup_{s \in [t_0, \tau_1]} \left| \widetilde{G}_i\left(s, v_i^m(s)\right) - \widetilde{G}_i\left(s, v_i(s)\right) \right|,
\end{aligned}
$$

for all $t \in [t_0 + \tau, \tau_1]$, where

$$
\tilde{h}_{2i}\left(v_i^m\right)(t) = h_{2i}\left(v_1^m, v_2^m, \ldots, v_k^m\right)(t),
$$

$\tilde{h}_{2i}\left(v_i\right)(t) = h_{2i}\left(v_1, v_2, \ldots, v_k\right)(t)$, and

$$
\begin{aligned}
\widetilde{G}_i\left(s, v_i^m(s)\right) &= G_i\left(s, v_1^m(s), v_2^m(s), \ldots, v_k^m(s)\right), \\
\widetilde{G}_i\left(s, v_i(s)\right) &= G_i\left(s, v_1(s), v_2(s), \ldots, v_k(s)\right).
\end{aligned}
$$

Hence,

$$
\lim_{m \to \infty} \left| \tilde{h}_{2i}\left(v_i^m\right)(t) - \tilde{h}_{2i}\left(v_i\right)(t) \right| = 0,
$$

for each $t \in [t_0 + \tau, \tau_1]$. Also,

$$
\begin{aligned}
\left| \tilde{h}_{2i}\left(v_i^m\right)(t) - \tilde{h}_{2i}\left(v_i\right)(t) \right| &\leq \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \\
&\times \left| \widetilde{G}_i\left(s, v_i^m(s)\right) - \widetilde{G}_i\left(s, v_i(s)\right) \right| \mathrm{d}_q s \\
&\leq \frac{1}{\Gamma_q(\alpha_i)} \left[\int_{t_0}^{t} \left[(t - qs)^{(\alpha_i - 1)} \right]^{\frac{1}{1-\kappa'_{1i}}} \, \mathrm{d}_q s \right]^{1-\kappa'_{1i}} \\
&\times \left[\int_{t_0}^{t} \left| \widetilde{G}_i\left(s, v_i^m(s)\right) - \widetilde{G}_i\left(s, v_i(s)\right) \right|^{\frac{1}{\kappa'_{1i}}} \, \mathrm{d}_q s \right]^{\kappa'_{1i}} \\
&\leq \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} \left[\left| \widetilde{G}_i\left(s, v_i^m(s)\right) \right| \right. \\
&\left. + \left| \widetilde{G}_i\left(s, v_i(s)\right) \right| \right] \mathrm{d}_q s \\
&\leq \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t - qs)^{(\alpha_i - 1)} [2 p_i (s - t_0)^{-\kappa'_{1i}}] \, \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{2p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}(t-t_0)^{-(\kappa'_{1i}-\alpha_i)}\\
&\le \frac{2p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}(\tau_1-t_0)^{-(\kappa'_{1i}-\alpha_i)}\\
&\le \varepsilon,
\end{aligned}
$$

for all $t \in [\tau_1, \infty)$, and so

$$\lim_{m\to\infty}\left|\tilde{h}_{2i}\left(v_i^m\right)(t)-\tilde{h}_{2i}\left(v_i\right)(t)\right|=0,$$

for all $t \in [t_0+\tau, \infty)$. Hence, for i in N_k, function h_{2i} is continuous on $[t_0+\tau, \infty)$. Therefore, H_2 is also continuous on $[t_0+\tau, \infty)$. Since

$$\lim_{t\to\infty}(t-t_0)^{-\beta'_{1i}}=0,$$

there exists $\tau'_1 \in (t_0+\tau, \infty)$ somehow that $(t-t_0)^{-\beta'_{1i}} < \frac{\varepsilon}{2}$ for $t \in (\tau'_1, \infty)$, where $\varepsilon > 0$ be given. Assume that $t_1, t_2 \in [t_0+\tau, \infty)$ such that $t_2 > t_1$. We consider three cases.

1) If $t_1, t_2 \in [t_0+\tau, \tau'_1]$, then we have

$$
\begin{aligned}
&\left|\tilde{h}_{2i}\ (v_i)(t_2)-\tilde{h}_{2i}(v_i)(t_1)\right|\\
&\quad=\left|\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t_2}(t_2-qs)^{(\alpha_i-1)}\widetilde{G}_i\left(s,v_i(s)\right)\,\mathrm{d}_qs\right.\\
&\quad\quad\left.-\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha_i-1)}\widetilde{G}_i\left(s,v_i(s)\right)\,\mathrm{d}_qs\right|\\
&\quad\le\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t_1}\left[(t_1-qs)^{(\alpha_i-1)}-(t_2-qs)^{(\alpha_i-1)}\right]\\
&\quad\quad\times\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|\,\mathrm{d}_qs\\
&\quad\quad+\frac{1}{\Gamma_q(\alpha_i)}\int_{t_1}^{t_2}(t_2-qs)^{(\alpha_i-1)}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|\,\mathrm{d}_qs\\
&\quad\le\frac{1}{\Gamma_q(\alpha_i)}\left[\int_{t_0}^{t_1}\left[(t_1-qs)^{(\alpha_i-1)}-(t_2-qs)^{(\alpha_i-1)}\right]^{\frac{1}{1-\kappa'_{1i}}}\,\mathrm{d}_qs\right]^{1-\kappa'_{1i}}\\
&\quad\quad\times\left[\int_{t_0}^{t_1}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|^{\frac{1}{\kappa'_{1i}}}\,\mathrm{d}s\right]^{\kappa'_{1i}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma_q(\alpha_i)}\left[\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha_i-1}{1-\kappa'_{1i}}}\,\mathrm{d}s\right]^{1-\kappa'_{1i}}\\
&\times\left[\int_{t_0}^{t_1}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|^{\frac{1}{\kappa'_{1i}}}\,\mathrm{d}s\right]^{\kappa'_{1i}}\\
\leq{}&\frac{1}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda'_{1i}}\right]^{1-\kappa'_{1i}}\\
&\times\left[(t_1-t_0)^{1+\lambda'_{1i}}-(t_2-t_0)^{1+\lambda'_{1i}}+(t_2-t_1)^{1+\lambda'_{1i}}\right]^{1-\kappa'_{1i}}\\
&\times\left[\int_{t_0}^{\tilde{\tau}'_2}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|^{\frac{1}{\kappa'_{1i}}}\,\mathrm{d}s\right]^{\kappa'_{1i}}\\
&+\frac{1}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda'_{1i}}\right]^{1-\kappa'_{1i}}\left[(t_2-t_1)^{1+\lambda'_{1i}}\right]^{1-\kappa'_{1i}}\\
&\times\left[\int_{t_0}^{\tilde{\tau}'_2}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|^{\frac{1}{\kappa'_{1i}}}\,\mathrm{d}s\right]^{\kappa'_{1i}}\\
\leq{}&\frac{2}{\Gamma_q(\alpha_i)}\left[\frac{1}{1+\lambda'_{1i}}\right]^{1-\kappa'_{1i}}\left[\int_{t_0}^{\tilde{\tau}'_2}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|^{\frac{1}{\kappa'_{1i}}}\,\mathrm{d}s\right]^{\kappa'_{1i}}\\
&\times(t_2-t_1)^{\alpha_i-\kappa'_{1i}},
\end{aligned}
$$

and so $\lim_{t_2\to t_1}\left|\tilde{h}_{2i}\left(v_i\right)(t_2)-\tilde{h}_{2i}\left(v_i\right)(t_1)\right|=0$.

2) If $t_1,t_2\in[\tau'_1,\infty)$, then

$$
\begin{aligned}
\left|\tilde{h}_{2i}\left(v_i\right)(t_2)\right.&\left.-\tilde{h}_{2i}\left(v_i\right)(t_1)\right|\\
&\leq\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t_2}(t_2-qs)^{(\alpha_i-1)}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|\,\mathrm{d}_qs\\
&\quad+\frac{1}{\Gamma_q(\alpha_i)}\int_{t_0}^{t_1}(t_1-qs)^{(\alpha_i-1)}\left|\widetilde{G}_i\left(s,v_i(s)\right)\right|\,\mathrm{d}_qs\\
&\leq(t_2-t_0)^{-\beta'_{1i}}+(t_1-t_0)^{-\beta'_{1i}}\\
&\leq\varepsilon.
\end{aligned}
$$

3) If $t_1\in[t_0+\tau,\tau'_1)$ and $t_2\in(\tau'_1,\infty)$, then by triangular inequality,

we get

$$\left|\tilde{h}_{2i}(v_i)(t_2) - \tilde{h}_{2i}(v_i)(t_1)\right| \leq \left|\tilde{h}_{2i}(v_i)(t_2) - \tilde{h}_{2i}(v_i)(\tilde{\tau}_2')\right| + \left|\tilde{h}_{2i}(v_i)(\tilde{\tau}_2') - \tilde{h}_{2i}(v_i)(t_1)\right|,$$

and so $\lim_{t_2\to t_1}\left|\tilde{h}_{2i}(v_i)(t_2) - \tilde{h}_{2i}(v_i)(t_1)\right| = 0$.

Thus, by regarding all cases, we conclude that $H_2(\Omega_1')$ is equi-continuous. Thus, $H_2(\Omega_1')$ is relatively compact, because the subset $H_2(\Omega_1')$ of Ω_1' is uniformly bounded. At present, we consider $u = (u_1, u_2, \ldots, u_k)$ and $v = (v_1, v_2, \ldots, v_k)$ belong to $\prod_{i\in N_k} C(J, \mathbb{R}^n)$ and Ω_1', respectively, such that $u = H_1u + H_2v$. Then,

$$\begin{aligned}
|u_i(t)| &\leq |H_{1i}(u_1, u_2, \ldots, u_k)(t)| + |H_{2i}(v_1, v_2, \ldots, v_k)(t)| \\
&\leq \frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i-1} + \frac{1}{\Gamma_q(\alpha_i)} \\
&\quad \times \int_{t_0}^{t}(t-qs)^{(\alpha_i-1)}\left|G_i(s, v_1(s), v_2(s), \ldots, v_k(s))\right| \mathrm{d}_qs \\
&\leq \frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i-1} \\
&\quad + \frac{p_i\Gamma_q(1-\kappa_{1i}')}{\Gamma_q(1+\alpha_i-\kappa_{1i}')}(t-t_0)^{-(\kappa_{1i}'-\alpha_i)}.
\end{aligned}$$

Since a non-zero element κ_{1i}' belongs to $(\alpha_i, 1)$ for $i \in N_k$, we get

$$\begin{aligned}
|u_i^0|(t-t_0)^{\frac{1}{2}(\alpha_i-1)}\Gamma_q(1+\alpha_i-\kappa_{1i}') &\\
+ p_i\Gamma_q(1-\kappa_{1i}')\Gamma_q(\alpha_i)(t-t_0)^{-\frac{1}{2}(\kappa_{1i}'-\alpha_i)} &\\
\leq |u_i^0|\tau^{\frac{1}{2}(\alpha_i-1)}\Gamma_q(1+\alpha_i-\kappa_{1i}') &\\
+ p_i\Gamma_q(1-\kappa_{1i}')\Gamma_q(\alpha_i)\tau^{-\frac{1}{2}(\kappa_{1i}'-\alpha_i)} &\\
\leq 1. &
\end{aligned}$$

Therefore,

$$\begin{aligned}|u_i(t)| &\leq \left[\frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\frac{1}{2}(\alpha_i-1)}\right.\\&\quad\left.+\frac{p_i\Gamma_q(1-\kappa'_{1i})}{\Gamma_q(1+\alpha_i-\kappa'_{1i})}(t-t_0)^{-\frac{1}{2}(\kappa'_{1i}-\alpha_i)}\right](t-t_0)^{-\beta'_{1i}}\\&\leq (t-t_0)^{-\beta'_{1i}},\end{aligned}$$

for all $t \in [t_0+\tau,\infty)$ and i in N_k. We conclude that $u(t) \in \Omega'_1$, for all $t \in [t_0+\tau,\infty)$. Therefore, by employing Theorem 2.2, the system (2) has a solution, which is a fixed point Θ in Ω'_1. Hence, the zero solution of the k-dimension system (2) is globally attractive, because all elements of the set Ω'_1 tend to 0 as $t \to \infty$. □

Theorem 3.5. *The zero solution of the problem* (2) *is globally attractive, whenever for all $i \in N_k$ there exist $\kappa'_{2i} \in (\alpha_i \frac{1}{2}(1+\alpha_i))$ and $p_i \geq 0$ such that*

$$|G_i(t,u_1(t),u_2(t),\dots,u_k(t))| \leq p_i(t-t_0)^{-\kappa'_{2i}}|u_i(t)|,$$

for any $t \in J$ and $u_i \in C(J,\mathbb{R}^n)$.

Proof. We just take the set Ω'_2 of all $(u_1,u_2,\dots,u_k)$ with $u_i \in C(J,\mathbb{R}^n)$ such that $|u_i(t)| \leq (t-t_0)^{-\beta'_{2i}}$ for each $i \in N_k$ and almost all $t \in [t_0+\tau,\infty)$, where $\beta'_{2i} = \frac{1}{2}(1-\alpha_i)$ and τ is chosen such that

$$\begin{aligned}&|u_i^0|\tau^{\frac{1}{2}(\alpha_i-1)}\Gamma_q(1+\alpha_i-\kappa'_{2i}-\beta'_{2i})\\&\qquad+p_i\Gamma_q(1-\kappa'_{2i}-\beta'_{2i})\Gamma_q(\alpha_i)\tau_3^{-(\kappa'_{2i}-\alpha_i)}\\&\qquad\leq \Gamma_q(\alpha_i)\Gamma_q(1+\alpha_i-\kappa'_{2i}-\beta'_{2i}).\end{aligned}$$

With the same use proof of Theorem 3.4, we conclude that Ω'_2 is a bounded, closed and convex set, H_2 is a self-maps on Ω'_2, the set $H_2(\Omega'_2)$ is relatively compact and H_2 is continuous on $[t_0+\tau,\infty)$. Let $u = (u_1,u_2,\dots,u_k)$ and $v = (v_1,v_2,\dots,v_k)$ belong to $\prod_{i\in N_k} C(J,\mathbb{R}^n)$ and

Ω_2, respectively, somehow that $u = H_1 u + H_2 v$. Then,

$$\begin{aligned}
|u_i(t)| &\le |h_{1i}(u_1, u_2, \ldots, u_k)(t)| + |h_{2i}(v_1, v_2, \ldots, v_k)(t)| \\
&\le \frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i - 1} \\
&\quad + \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t-qs)^{(\alpha_i - 1)} |\widetilde{G}\,(s, v_i(s))\,|\,\mathrm{d}_q s \\
&\le \frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i - 1} \\
&\quad + \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t-qs)^{(\alpha_i - 1)} p_i (s-t_0)^{-\kappa'_{2i}} |v_i(s)|\,\mathrm{d}_q s \\
&\le \frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i - 1} \\
&\quad + \frac{1}{\Gamma_q(\alpha_i)} \int_{t_0}^{t} (t-qs)^{(\alpha_i - 1)} p_i (s-t_0)^{-\kappa'_{2i} - \beta'_{2i}}\,\mathrm{d}s \\
&\le \frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\alpha_i - 1} \\
&\quad + \frac{p_i \Gamma_q(1 - \kappa'_{2i} - \beta'_{2i})}{\Gamma_q(1 + \alpha_i - \kappa'_{2i} - \beta'_{2i})}(t-t_0)^{-(\kappa'_{2i} - \beta'_{2i} - \alpha_i)}
\end{aligned}$$

for all i belongs to N_k. Since a non-zero element

$$\kappa'_{2i} \in (\alpha_i, \frac{1}{2}(1+\alpha_i)),$$

we get

$$\begin{aligned}
&\frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\frac{1}{2}(\alpha_i - 1)} + \frac{p_i \Gamma_q(1 - \kappa'_{2i} - \beta'_{2i})}{\Gamma_q(1 + \alpha_i - \kappa'_{2i} - \beta'_{2i})}(t-t_0)^{-(\kappa'_{2i} - \alpha_i)} \\
&\quad \le \frac{|u_i^0|}{\Gamma_q(\alpha_i)} \tau^{\frac{1}{2}(\alpha_i - 1)} + \frac{p_i \Gamma_q(1 - \kappa'_{2i} - \beta'_{2i})}{\Gamma_q(1 + \alpha_i - \kappa'_{2i} - \beta'_{2i})} \tau^{-(\kappa'_{2i} - \alpha_i)} \\
&\quad \le 1.
\end{aligned}$$

Thus,

$$\begin{aligned}|u_i(t)| &\le \left[\frac{|u_i^0|}{\Gamma_q(\alpha_i)}(t-t_0)^{\frac{1}{2}(\alpha_i-1)}\right.\\ &\qquad \left.+\frac{p_i\Gamma_q(1-\kappa'_{2i}-\beta'_{2i})}{\Gamma_q(1+\alpha_i-\kappa'_{2i}-\beta'_{2i})}(t-t_0)^{-(\kappa'_{2i}-\alpha_i)}\right](t-t_0)^{-\beta'_{2i}}\\ &\le (t-t_0)^{-\beta'_{2i}},\end{aligned}$$

for each $t \in [t_0+\tau, \infty)$ and $i \in N_k$. Hence, we conclude that $u(t) \in \Omega_2$, for almost all $t \ge t_0 + \tau_3$. Thus, the zero solution of the k-dimension system (2) is globally attractive, because all elements of the set Ω'_2 tend to zero as $t \to \infty$. □

4 Examples and algorithms for the problem

In this part, we give a complete computational techniques for checking working to exists the attractivity of solutions for fractional functional q-differential equations, and the global attractivity for nonlinear fractional q-differential equations in k-dimensional system with the boundary value conditions (1) and (2), respectively, and present numerical examples. Foremost, we present a simplified analysis can be executed to calculate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in summation. To this aim, we consider a pseudo-code description of the method for calculated q-Gamma function of order n in Algorithm 2 (for more details, see the following link `https://en.wikipedia.org/wiki/Q-gamma_function`).

Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with less values of n. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third columns of Table 2. Also, Table 3 is the same as Table 2, but x values increase in 3. Similarly, the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. Furthermore, we provided algorithms 3

Table 1: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$ of Algorithm 2.

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	2.340263	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	11.257095	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	64.350881
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2.

n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$	n	$q = \frac{1}{3}$	$q = \frac{1}{2}$	$q = \frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

and 4 which calculated $(\mathcal{D}_q^\alpha f)(x)$ and $(\mathcal{I}_q^\sigma f)(x)$, respectively.

Here, we provide two example to illustrate our results.

Example 4.1. For $k = 3$, $t_0 = 0$, $\delta = 1$, $J = (0, \infty)$, $\overline{J} = [0, \infty)$, $\overline{J}_{-\delta} = [-1, 0]$, $\overline{J}_{-\delta}^{t_0} = [-1, 0]$ and $\overline{J}_{-\delta}^{\infty} = [-1, \infty)$ in k-dimensional system (1),

Table 3: Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2.

n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	<u>11.257095</u>	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	<u>49.065751</u>	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	<u>259.967394</u>
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

we consider

$$\begin{cases} {}^cD_q^{\frac{2}{3}} u_1(t) = A_1(t+3)^{\frac{-6}{7}} \dfrac{u_3(t-1)\cos^2(u_1(t-1))}{(1+(u_2(t-1))^2)(1+|u_3(t-1)|)}, \\ {}^cD_q^{\frac{3}{5}} u_2(t) = A_2\,(t+2)^{\frac{-9}{10}} \dfrac{\sin^4(u_1(t-1))}{1+\cos^2(u_3(t-1))+|u_2(t-1)|}, \\ {}^cD_q^{\frac{1}{4}} u_3(t) = A_3(t+1)^{\frac{-5}{8}} \dfrac{(u_1(t-1))^4}{1+((u_1(t-1))^4+6|u_2(t-1)|^3}, \end{cases} \tag{5}$$

for any $t \in (0, \infty)$ and $u_1(t) = u_2(t) = u_3(t) = t$ for almost all $t \in [-1, 0]$, where $q \in (0, 1)$, and

$$A_i = \begin{bmatrix} \frac{\Gamma_q(\frac{17}{21})}{\Gamma_q(\frac{1}{7})} \\ \frac{\Gamma_q(\frac{7}{10})}{\Gamma_q(\frac{1}{10})} \\ \frac{\Gamma_q(\frac{5}{8})}{\Gamma_q(\frac{1}{4})} \end{bmatrix}, \qquad \varphi_i = \begin{bmatrix} t \\ t \\ t \end{bmatrix}.$$

Define the maps

$$\begin{aligned}
F_1\left(t, u_{1_t}, u_{2_t}, u_{3_t}\right) &= A_1(t+3)^{\frac{-6}{7}} \frac{u_3(t-1)\cos^2\left(u_1(t-1)\right)}{(1+(u_2(t-1))^2)(1+|u_3(t-1)|)} \\
&\in L^{\frac{1}{\kappa_{11}}}(J \times \mathcal{C}^3), \\
F_2\left(t, u_{1_t}, u_{2_t}, u_{3_t}\right) &= A_2\,(t+2)^{\frac{-9}{10}} \frac{\sin^4\left(u_1(t-1)\right)}{1+\cos^2\left((u_3(t-1)) + |u_2(t-1)|\right.} \\
&\in L^{\frac{1}{\kappa_{12}}}(J \times \mathcal{C}^3), \\
F_3\left(t, u_{1_t}, u_{2_t}, u_{3_t}\right) &= A_3(t+1)^{\frac{-5}{8}} \frac{\left(u_1(t-1)\right)^4}{1+\left((u_1(t-1)\right)^4 + 6|u_2(t-1)|^3} \\
&\in L^{\frac{1}{\kappa_{13}}}(J \times \mathcal{C}^3).
\end{aligned}$$

On the other hand, by using (3), the facts

$$\frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)^{(\alpha-1)}(s-a)^{\beta}\, \mathrm{d}_q s = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)}(t-a)^{\alpha+\beta},$$

with $a=0$, and $B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$, we obtain

$$\begin{aligned}
&\left| \int_{t_0}^t \frac{(t-qs)^{(\frac{-1}{3})}}{\Gamma_q(\frac{2}{3})} F_1\left(s, u_{1_s}, u_{2_s}, u_{s_t}\right)\, \mathrm{d}_q s \right| \\
&\qquad \le \int_0^t \frac{(t-qs)^{(\frac{-1}{3})}}{\Gamma_q(\frac{2}{3})} \left(A_1(s+3)^{\frac{-6}{7}}\right) \mathrm{d}_q s \\
&\qquad \le \frac{A_1}{\Gamma_q(\frac{2}{3})} \int_0^t (t-qs)^{(\frac{-1}{3})} s^{\frac{-6}{7}}\, \mathrm{d}_q s \\
&\qquad = \frac{A_1}{\Gamma_q(\frac{2}{3})} t^{\frac{-4}{21}} \int_0^1 (1-qs)^{(\frac{-1}{3})} s^{\frac{-6}{7}}\, \mathrm{d}_q s \\
&\qquad = \frac{A_1}{\Gamma_q(\frac{2}{3})} t^{\frac{-4}{21}} B_q\left(\frac{1}{7}, \frac{2}{3}\right) = t^{\frac{-4}{21}}, \\
&\left| \int_{t_0}^t \frac{(t-qs)^{(\frac{-2}{5})}}{\Gamma_q(\frac{3}{5})} F_2\left(s, u_{1_s}, u_{2_s}, u_{3_s}\right)\, \mathrm{d}_q s \right| \\
&\qquad \le \int_0^t \frac{(t-qs)^{(\frac{-2}{5})}}{\Gamma_q(\frac{3}{5})} \left(A_2(s+2)^{\frac{-9}{10}}\right) \mathrm{d}_q s
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{A_2}{\Gamma_q(\frac{3}{5})}\int_0^t (t-qs)^{(\frac{-2}{5})} s^{\frac{-9}{10}}\,\mathrm{d}_q s\\
&= \frac{A_2}{\Gamma_q(\frac{3}{5})} t^{\frac{-3}{10}} \int_0^1 (t-qs)^{(\frac{-2}{5})} s^{\frac{-9}{10}}\,\mathrm{d}_q s\\
&= \frac{A_2}{\Gamma_q(\frac{3}{5})} t^{\frac{-3}{10}} B_q\left(\frac{1}{10},\frac{3}{5}\right) = t^{\frac{-3}{10}},
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{t_0}^t \frac{(t-qs)^{(\frac{-3}{4})}}{\Gamma_q(\frac{1}{4})} F_3\left(s,u_{1_s},u_{2_s},u_{3_s}\right)\,\mathrm{d}_q s\right|\\
&\qquad \leq \int_0^t \frac{(t-qs)^{(\frac{-3}{4})}}{\Gamma_q(\frac{1}{4})}\left(A_3(s+1)^{\frac{-5}{8}}\right)\mathrm{d}_q s\\
&\qquad \leq \frac{A_3}{\Gamma_q(\frac{1}{4})}\int_0^t (t-qs)^{(\frac{-3}{4})} s^{\frac{-5}{8}}\,\mathrm{d}_q s\\
&\qquad = \frac{A_3}{\Gamma_q(\frac{1}{4})} t^{\frac{-3}{8}} \int_0^1 (1-qs)^{(\frac{-3}{4})} s^{\frac{-5}{8}}\,\mathrm{d}_q s\\
&\qquad = \frac{A_3}{\Gamma_q(\frac{1}{4})} t^{\frac{-3}{8}} B_q\left(\frac{1}{4},\frac{3}{8}\right) = t^{\frac{-3}{8}}.
\end{aligned}$$

Note that, $\beta_{11}=\frac{4}{21}, \beta_{12}=\frac{3}{10}$, and $\beta_{13}=\frac{3}{8}$. Now, we take

$$\kappa_{1i} = \begin{bmatrix} \frac{1}{7} \\ \frac{1}{10} \\ \frac{1}{8} \end{bmatrix}.$$

Then,

$$\begin{aligned}
\int_{t_0}^{\infty} \left|F_1\left(t,u_{1_t},u_{2_t},u_{3_t}\right)\right|^7 \mathrm{d}t &\leq \int_0^{\infty}\left[A_1(t+3)^{\frac{-6}{7}}\right]^7 \mathrm{d}t\\
&= \frac{1}{1215}A_1^7 = \eta_1,
\end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{\infty} \left|F_2\left(t,u_{1_t},u_{2_t},u_{3_t}\right)\right|^{10} \mathrm{d}t &\leq \int_0^{\infty}\left[A_2(t+2)^{\frac{-9}{10}}\right]^{10} \mathrm{d}t\\
&= \frac{1}{2048}A_2^{10} = \eta_2,
\end{aligned}$$

Table 4: Some numerical results of η_i in Example 4.1 where $q = \frac{1}{8}$ by Algorithmic 2.

n	A_1	A_2	A_3	η_1	η_2	η_3
1	0.32041	0.24551	0.56155	0	0	0.00247
2	0.31762	0.24328	0.55871	0	0	0.00237
3	0.31727	0.24301	0.55836	0	0	0.00236
4	0.31723	0.24297	0.55831	0	0	0.00236
5	0.31722	0.24297	0.55831	0	0	0.00236
6	0.31722	0.24297	0.55831	0	0	0.00236
7	0.31722	0.24297	0.55831	0	0	0.00236

Table 5: Some numerical results of η_i in Example 4.1 where $q = \frac{1}{2}$ by Algorithmic 2.

n	A_1	A_2	A_3	η_1	η_2	η_3
1	0.26678	0.20349	0.3422	0	0	0.00005
2	0.24071	0.1844	0.34571	0	0	0.00005
3	0.22985	0.17648	0.35676	0	0	0.00007
4	0.22486	0.17283	0.36973	0	0	0.00009
5	0.22246	0.17109	0.38255	0	0	0.00011
6	0.22129	0.17023	0.3944	0	0	0.00015
7	0.22071	0.1698	0.40503	0	0	0.00018
8	0.22042	0.16959	0.41441	0	0	0.00022
9	0.22027	0.16949	0.42263	0	0	0.00025
10	0.2202	0.16944	0.4298	0	0	0.00029
11	0.22016	0.16941	0.43606	0	0	0.00033
12	0.22015	0.1694	0.44153	0	0	0.00036
13	0.22014	0.16939	0.44631	0	0	0.00039

$$
\begin{aligned}
\int_{t_0}^{\infty} \left|F_3\left(t, u_{1_t}, u_{2_t}, u_{3_t}\right)\right|^8 \mathrm{d}t &\leq \int_0^{\infty} \left[A_3 (t+1)^{\frac{-5}{8}}\right]^8 \mathrm{d}t \\
&= \frac{1}{4} A_3^8 = \eta_3.
\end{aligned}
$$

Tables 4, 5 and 6 show the some numerical value of η_1, η_2 and η_3 for $q = \frac{1}{8}$, $\frac{1}{2}$ and $\frac{8}{9}$, respectively. Thus, all conditions of Theorem 3.1 hold and so this system of fractional functional differential equations have an attractive solution.

Table 6: Some numerical results of η_i in Example 4.1 where $q = \frac{8}{9}$ by Algorithmic 2.

n	A_1	A_2	A_3	η_1	η_2	η_3
1	0.52085	0.37029	0.73247	0.00001	0	0.02071
2	0.41238	0.29787	0.64117	0	0	0.00714
3	0.35289	0.25793	0.58681	0	0	0.00351
4	0.31506	0.23239	0.55021	0	0	0.0021
5	0.28886	0.21461	0.52377	0	0	0.00142
6	0.26968	0.20153	0.50376	0	0	0.00104
7	0.25508	0.19153	0.48813	0	0	0.00081
8	0.24365	0.18368	0.47562	0	0	0.00065
9	0.2345	0.17738	0.46542	0	0	0.00055
10	0.22705	0.17224	0.457	0	0	0.00048
11	0.22091	0.16799	0.44996	0	0	0.00042
12	0.21578	0.16444	0.44403	0	0	0.00038
13	0.21147	0.16144	0.43898	0	0	0.00034
14	0.20781	0.1589	0.43467	0	0	0.00032
15	0.20469	0.15673	0.43097	0	0	0.0003
16	0.20201	0.15486	0.42777	0	0	0.00028
17	0.1997	0.15325	0.425	0	0	0.00027
18	0.1977	0.15185	0.42259	0	0	0.00025
19	0.19597	0.15064	0.42049	0	0	0.00024
20	0.19446	0.14959	0.41866	0	0	0.00024
21	0.19314	0.14867	0.41705	0	0	0.00023
22	0.19199	0.14786	0.41564	0	0	0.00022
23	0.19097	0.14715	0.4144	0	0	0.00022
24	0.19009	0.14653	0.41332	0	0	0.00021
25	0.18931	0.14598	0.41236	0	0	0.00021
26	0.18862	0.1455	0.41151	0	0	0.00021
27	0.18801	0.14508	0.41076	0	0	0.0002
28	0.18748	0.1447	0.4101	0	0	0.0002
29	0.18701	0.14437	0.40952	0	0	0.0002
30	0.18659	0.14408	0.409	0	0	0.0002

Example 4.2. Consider the k-dimensional system of (2) for $k = 3$,

$$\begin{cases} D_q^{\alpha_1} u_1(t) = \dfrac{p_1 u_3(t)\cos^2(u_2(t))}{1.5 + |u_3(t)| + |u_2(t)|}(t-a)^{-\kappa'_{11}}, \\ D_q^{\alpha_2} u_2(t) = \dfrac{p_2 t^2 (u_1(t))^2}{(7+5t^2)(1+2(u_1(t))^2 + (u_3(t))^2)}(t-a)^{-\kappa'_{12}}, \\ D_q^{\alpha_3} u_3(t) = \dfrac{p_3 \cos^3(u_2(t))}{8 + 3(u_2(t))^2 + |u_3(t)|^3}(t-a)^{-\kappa'_{13}}, \end{cases} \tag{6}$$

for almost all $t \in J$ and

$$D_q^{\alpha_i - 1} u_i(t) = u_i^0,$$

for $t = t_0$, where $\alpha_i \in (0,1)$, $p_i \in [0,\infty)$, $\kappa'_{1i} \in (\alpha_i, 1)$ and u_i^0 is a constant for $i = 1,2,3$. If we define maps

$$\begin{aligned} G_1(t, u_1(t), u_2(t), \dots, u_k(t)) &= \frac{p_1 u_2(t)\cos^2(u_3(t))}{1.5 + |u_2(t)| + |u_3(t)|}(t-a)^{-\kappa'_{11}}, \\ G_2(t, u_1(t), u_2(t), \dots, u_k(t)) &= \frac{p_2 t^2 (u_1(t))^2}{(7+5t^2)(1+2(u_1(t))^2 + (u_3(t))^2)} \\ &\quad \times (t-a)^{-\kappa'_{12}}, \\ G_3(t, u_1(t), u_2(t), \dots, u_k(t)) &= \frac{p_3 \cos^3(u_2(t))}{8 + 3(u_2(t))^2 + |u_3(t)|^3}(t-a)^{-\kappa'_{31}}, \end{aligned}$$

then, with a simple check, we will conclude that all conditions of Theorem 3.4 hold and so this system of fractional q-differential equations has a globally attractive solution.

5 Conclusions

The attractive and global attractivity solutions of the system of fractional q-differential equations and their applications represent a matter of high interest in the area of fractional q-calculus and its applications in diverse fields of science and engineering. In this manuscript, we focused on the attractivity and global attractivity of solutions for two k-dimensional systems of fractional q-differential equations. Two illustrative examples demonstrate the pertinence of the suggested methods.

The techniques of the reported results can be applied to investigating the attractivity and global attractivity of solutions of q-differential systems of (singular) fractional q-differential equations.

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