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Note on the Covariance Coset of the Moore-Penrose Inverses in C*-Algebras

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Abstract. We will introduce and study several algebraic properties of the covariance cosets in C^*-algebras. Indeed, we will characterize the covariance coset in terms of commutators. Also, we will show that for an invertible element b, the covariance coset of b^{-1} coincides with the covariance coset of b^*. Moreover, if b is normal, then the covariance coset of b coincides with the covariance coset of b^*. In addition, we will prove that the covariance coset is a cone.

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1. Introduction

The concept of a generalized inverse seems to have been first mentioned in 1903 by Fredholm and the class of all pseudoinverses was characterized in 1912 by Hurwitz [2]. Generalized inverses of differential and integral operators thus antedated the generalized inverses of matrices, whose existence was first introduced and studied by Moore [13, 14] during the years 1910-1920. This notion was rediscovered by Penrose [9] in 1955, and is nowadays called the Moore-Penrose inverse. In recent years, generalized inverses and their properties have received a lot of attention (see for example [2, 3, 4, 10], and the references therein). The notion of generalized inverses in C^*-algebra, was introduced in the seminal paper by Harte and Mbekhta [4]. Harte and Mbekhta have shown that many important properties of Moore-Penrose inverses in C^*-algebra. A part of

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literature has been dedicated to weighted Moore-Penrose inverse, spectral theory, closed range operators, linear preserver problems and numerical computations in the area of optimization, statistics and ill-posed problem, etc. (see [2, 5, 10, 14]). All these reasons have convinced many mathematicians, to start research in this rich and important branch of mathematics.

Throughout this paper $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is called regular if it has a generalized inverse (in the sense of von Neumann) in $\mathcal{A}$, i.e., there exists $b \in \mathcal{A}$ such that

$$aba = a.$$

An element $a \in \mathcal{A}$ is Moore-Penrose invertible if there exists $b \in \mathcal{A}$ such that

$$aba = a, \quad bab = b, \quad (ab)^* = ab \quad \text{and} \quad (ba)^* = ba.$$

It is well known that each regular element in a C^*-algebra, has the Moore-Penrose inverse (denoted by MP− inverse from now on.). Generally MP− inverse is uniquely determined in $\mathcal{A}$ if it exists. We will denote the MP–inverse of a by $a^\dagger$.

In the following, we will denote by $\mathcal{A}^{-1}$ and $\mathcal{A}^\dagger$ the set of all invertible and MP− invertible elements of $\mathcal{A}$, respectively. An element a in $\mathcal{A}$ is called idempotent if $a^2 = a$. A projection $p \in \mathcal{A}$ satisfies $p = p^* = p^2$. It should be noticed that if $x \in \mathcal{A}^\dagger$, then $xx^\dagger$ and $x^\dagger x$ are projections. Moreover,

$$\left(xx^\dagger\right)^\dagger = xx^\dagger \quad \text{and} \quad \left(x^\dagger x\right)^\dagger = x^\dagger x.$$

The commutator of a pair of elements x and y in $\mathcal{A}$ is defined by

$$[x, y] = xy - yx.$$

Obviously $[x, y] = 0$ if and only if x and y commute.

Assume that a is an element in $\mathcal{A}^{-1}$. Its inverse a^{-1} is *covariant* with respect to $\mathcal{A}^{-1}$, i.e., for all $b \in \mathcal{A}^{-1}$ we have

$$(bab^{-1})^{-1} = ba^{-1}b^{-1}.$$

In general, the elements of $\mathcal{A}^{\dagger}$ are not covariant under $\mathcal{A}^{-1}$ (see [1]). For a given element $a \in \mathcal{A}^{\dagger}$ with MP–inverse $a^{\dagger}$, we will denote the *covariance set* by $\mathcal{C}(a)$ and define

$$\mathcal{C}(a) = \{b \in \mathcal{A}^{-1} : (bab^{-1})^{\dagger} = ba^{\dagger}b^{-1}\}. \tag{1}$$

Covariance set was studied by [1], [6], [11] and [13].
In this note we introduce the notion of *covariance coset* of the Moore-Penrose inverses in C^*-algebras. In fact we define this set by reversing the roles of a and b in $\mathcal{C}(a)$ and denote it by $\mathcal{B}(b)$. i.e.,

$$\mathcal{B}(b) = \left\{a \in \mathcal{A}^{\dagger} : (bab^{-1})^{\dagger} = ba^{\dagger}b^{-1}\right\}. \tag{2}$$

The notion of covariance coset was introduced by Robinson in [12] for matrices. In this paper we will characterize the covariance coset in terms of commutators. Also, we will show that for an invertible element $b \in \mathcal{A}$ we have $\mathcal{B}\left(b^{-1}\right) = \mathcal{B}\left(b^{*}\right).$ Moreover, if b is normal, then $\mathcal{B}\left(b\right) = \mathcal{B}\left(b^{-1}\right) = \mathcal{B}\left(b^{*}\right).$ In Proposition 4 and related corollaries we will describe the main properties of the covariance coset in C^*-algebras. We conclude the results by showing that for any non-zero scalar λ, $\mathcal{B}\left(b\right) = \mathcal{B}\left(\lambda b\right).$

2. Main Results

In the following proposition we characterize $\mathcal{B}(b)$ in terms of commutators.

Proposition 2.1. *Assume $b \in \mathcal{A}^{-1}$. Then the following statements are equivalent:*

(i) $a \in \mathcal{B}\left(b\right);$

(ii) $\left[a^{\dagger}a, b^{*}b\right] = 0$ *and* $\left[aa^{\dagger}, b^{*}b\right] = 0.$

Proof. (i) $\Longrightarrow$ (ii): Suppose $a \in \mathcal{B}\left(b\right).$ Therefore $ba^{\dagger}b^{-1}$ is the MP–inverse of bab^{-1}. Thus, $\left(ba^{\dagger}ab^{-1}\right)^{*} = ba^{\dagger}ab^{-1}$. Therefore, $\left(b^{*}\right)^{-1}a^{\dagger}ab^{*}b = ba^{\dagger}a$. From here one can conclude that $\left[a^{\dagger}a, b^{*}b\right] = 0$. In a similar manner from $\left(baa^{\dagger}b^{-1}\right)^{*} = ba^{\dagger}ab^{-1}$ we get that $\left[aa^{\dagger}, b^{*}b\right] = 0$.

(ii) $\Longrightarrow$ (i): Since $a^\dagger$ is MP–inverse of a, it suffices to show that $\left(ba^\dagger ab^{-1}\right)^* = ba^\dagger ab^{-1}$ and $\left(baa^\dagger b^{-1}\right)^* = baa^\dagger b^{-1}$. By the assumptions $\left[a^\dagger a, b^*b\right] = 0$. From this we obtain $(b^*)^{-1}a^\dagger ab^*b = ba^\dagger a$. Thus, $\left(ba^\dagger ab^{-1}\right)^* = ba^\dagger ab^{-1}$. In a similar manner from $\left[aa^\dagger, b^*b\right] = 0$ we get $\left(baa^\dagger b^{-1}\right)^* = baa^\dagger b^{-1}$. □

Note that if $a \in \mathcal{A}^\dagger$ with MP–inverse $a^\dagger$ and $b \in \mathcal{A}^{-1}$, then from the above proposition and Lemma 2.1 in [1], we conclude that

$$b \in \mathcal{C}(a) \quad \text{if and only if} \quad a \in \mathcal{B}\left(b\right).$$

Also, we remark that $\mathcal{C}(a) \subset \mathcal{A}^{-1} \subset \mathcal{B}\left(b\right) \subset \mathcal{A}^\dagger$ for all $a \in \mathcal{A}^\dagger$ and for each $b \in \mathcal{A}^{-1}$.

Proposition 2.2. *Assume that $b \in \mathcal{A}^{-1}$. Then $\mathcal{B}\left(b^*\right) = \mathcal{B}\left(b^{-1}\right)$.*

Proof. By Proposition 2.1.

$$a \in \mathcal{B}\left(b^*\right) \text{ if and only if } \left[a^\dagger a, bb^*\right] = 0 \textit{ and } \left[aa^\dagger, bb^*\right] = 0.$$

This is equivalent to

$$a^\dagger abb^* = bb^*a^\dagger a \quad \text{and} \quad aa^\dagger bb^* = bb^*aa^\dagger. \tag{3}$$

Multiply (3) from left and right by $\left(bb^*\right)^{-1}$, we get

$$[a^\dagger a, \left(b^{-1}\right)^* b^{-1}] = 0 \quad \text{and} \quad [aa^\dagger, \left(b^{-1}\right)^* b^{-1}] = 0. \tag{4}$$

Again Proposition 1.1, shows that (4) holds if and only if $a \in \mathcal{B}\left(b^{-1}\right)$. □

Proposition 2.3. *Assume that $b \in \mathcal{A}^{-1}$ and b is normal. Then $\mathcal{B}\left(b\right) = \mathcal{B}\left(b^{-1}\right)$.*

Proof. Since b is normal, the equality is an immediate consequence of Proposition 2.2. □

We recall that a set $K \subset \mathcal{A}$ is a cone if $x \in K$ implies $\lambda x \in K$ for each $\lambda \geqslant 0$. Also, an element $a \in \mathcal{A}$ is called simply polar [4] if it has a commuting generalized inverse, that is, there exists a generalized inverse c of a, such that $[a, c] = 0$.

In the following proposition we collect some main properties of the covariance coset.

Proposition 2.4. *Assume that* $b \in \mathcal{A}^{-1}$*. Then, the following statements are equivalent:*

(i) $a \in \mathcal{B}(b)$;

(ii) $a^{\dagger} \in \mathcal{B}(b)$;

(iii) $a^{*} \in \mathcal{B}(b)$;

(iv) $aa^{\dagger} \in \mathcal{B}(b)$ *and* $a^{\dagger}a \in \mathcal{B}(b)$;

(v) $\lambda a \in \mathcal{B}(b)$ *for any non-zero scalar* λ.

Proof. First we show that (i) and (ii) are equivalent: By Proposition 1.1, $a \in \mathcal{B}(b)$ if and only if

$$\left[a^{\dagger}a, b^{*}b\right] = 0 \quad and \quad \left[aa^{\dagger}, b^{*}b\right] = 0. \tag{5}$$

Since $\left(a^{\dagger}\right)^{\dagger} = a$. Thus (5) is equivalent to

$$\left[a^{\dagger}\left(a^{\dagger}\right)^{\dagger}, b^{*}b\right] = 0 \quad and \quad \left[\left(a^{\dagger}\right)^{\dagger}a^{\dagger}, b^{*}b\right] = 0. \tag{6}$$

Again, Proposition 1.1, shows that (6) holds if and only if $a^{\dagger} \in \mathcal{B}(b)$.

(i) $\Longleftrightarrow$ (iii): In a similar manner, since $a^{\dagger}a$ and $aa^{\dagger}$ are normal elements we infer that $a^{\dagger}a = a^{*}\left(a^{*}\right)^{\dagger}$ and $aa^{\dagger} = \left(a^{*}\right)^{\dagger}a^{*}$ by applying Proposition 1.1, we get $a \in \mathcal{B}(b) \iff a^{*} \in \mathcal{B}(b)$.

(i) $\Longrightarrow$ (iv): $a \in \mathcal{B}(b)$ if and only if (5) holds. Since $aa^{\dagger} = aa^{\dagger}aa^{\dagger}$ and $\left(aa^{\dagger}\right)^{\dagger} = aa^{\dagger}$. Thus (5) is equivalent to

$$[aa^{\dagger}(aa^{\dagger})^{\dagger}, b^{*}b] = 0 \quad \text{and} \quad [(aa^{\dagger})^{\dagger}aa^{\dagger}, b^{*}b] = 0. \tag{7}$$

This implies that $aa^{\dagger} \in \mathcal{B}(b)$. Similarly we get

$$[a^{\dagger}a(a^{\dagger}a)^{\dagger}, b^{*}b] = 0 \;\; \text{and} \;\; [(a^{\dagger}a)^{\dagger}a^{\dagger}a, b^{*}b] = 0. \tag{8}$$

Thus, $a^{\dagger}a \in \mathcal{B}(b)$.

For the proof of (iv) $\Longrightarrow$ (i); It is easy to verify that (iv) satisfies if and only if (5) and (8) hold. These together imply (5), that is, (i) holds.

(i) $\iff$ (v): Since $\lambda \neq 0$, $(\lambda a)^{\dagger} = \frac{1}{\lambda} a^{\dagger} = (a\lambda)^{\dagger}$. Now applying the Proposition 1.1, we obtain the result. $\square$

Corollary 2.5. *If $b \in \mathcal{A}^{-1}$, then $\mathcal{B}(b)$ is a cone.*
It is well known that every normal element is simply polar. Hence

Corollary 2.6. *If a is normal, then*

$$a \in \mathcal{B}(b) \iff aa^{\dagger} \in \mathcal{B}(b) \iff a^{\dagger}a \in \mathcal{B}(b).$$

Proposition 2.7. *Assume that $b \in \mathcal{A}^{-1}$ and $\lambda \neq 0$ is any scalar. Then $\mathcal{B}(b) = \mathcal{B}(\lambda b)$.*

Proof. By Proposition 1.1, $a \in \mathcal{B}(b)$ if and only if (5) satisfies which is equivalent to

$$\left[a^{\dagger}a, (\lambda b)^{*}(\lambda b)\right] = 0 \quad \textit{and} \quad \left[aa^{\dagger}, (\lambda b)^{*}(\lambda b)\right] = 0.$$

This holds if and only if $a \in \mathcal{B}(\lambda b)$. $\square$

References

[1] M. H. Alizadeh, *On the covariance of generalized inverse in C^*-algebra*, J. Numer. Anal. Indust. Appl. Math., 5 (2011) 135-139.

[2] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications,* 2nd Edn. Springer-Verlag, New York, 2003.

[3] J. Benítez and V. Rakočević, *Invertibility of the commutator of an element in a C^*-algebra and its Moore-Penrose inverse*, Studia Math., 200 (2010), 163-174.

[4] R. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math., 103 (1992), 71-77.

[5] Z. Khodadadi and B. Tarami, *Robust empirical bayes estimation of the elliptically countoured covariance matrix*, Journal of Mathematical Extension, 5 (2011), 31-46.

[6] A. R. Meenakshi and V. Chinadurai, *Some remarks on the covariance of the Moore-Penrose inverse*, Houtson J. Math., 18 (1992), 167-174.

[7] E. H. Moore, On the reciprocal of the general algebraic matrix, *Bull. Amer. Math. Soc.*, 26 (1920), 394-395.

[8] E. H. Moore and R. W. Barnard, General Analysis, Memories of the American Philosophical Society, I. American Philosophical Society, Philadelphia, Pennsylvania, Part 1, (1935), 197-209.

[9] R. Penrose, A generalized inverse for matrices. Proc. *Cambridge Philos. Soc.,* 51 (1955), 406-413.

[10] C. R. Rao and S. K. Mitra, *Generalized inverse of matrices and its applications*, Wiley and Sons, New York, 1971.

[11] D. W. Robinson, *On the covariance of the Moore-Penrose inverse*, Linear Algebra Appl., 61 (1984), 91-99.

[12] D. W. Robinson, *Covariance of the Moore-Penrose inverses with respect to an invertible matrix*, Linear Algebra Appl., 71 (1985), 275-281.

[13] H. Schwerdtfeger, *On the covariance of Moore-Penrose inverse*, Linear Algebra Appl., 52/53 (1983), 629-643.

[14] Q. Xu, Y. Wei, and Y. Gu, *Sharp norm-estimations for Moore-Penrose inverses of stable perturbations of Hilbert C*-module operators*, SIAM J. Numer. Anal., 47 (2010), 4735-4758.

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