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Efficiency of Two-Stage Systems in Stochastic DEA

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Abstract. In the present world, there are many two-stage systems which provide information of inputs, outputs and intermediate measures which are imprecise, such as, (stochastic, fuzzy, interval etc). In these conditions, a two-stage data envelopment analysis or a (two-stage DEA method) cannot evaluate the efficiencies of these systems. In several two-stage systems, the simultaneous presence of stages is necessary for the final product. Hence, in this paper, we shall propose the stochastic multiplicative model and the deterministic equivalent, to measure the efficiencies of these systems, primarily, in the presence of stochastic data, under the constant returns to scale (CRS) assumption, by using the non-compensatory property of the multiplication operator.Then, we will use the reparative property of the additive operation to propose the additive models as well as the deterministic equivalents, to calculate the

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efficiencies of two-stage systems, in presence of stochastic data, under the constant returns to scale (CRS) and variable returns to scale (VRS) assumptions. This is to illustrate that a simultaneous presence of the stages is not necessary for the final product and one stage compensates the shortcomings of another stage. Likewise, we shall convert each of these deterministic equivalents to quadratic programming problems. Based on the proposed stochastic models, the whole system is efficient if and only if, the first and the second stages are efficient. Ultimately, in the proposed multiplicative model, we will illustrate the proposed multiplicative model, by employing the data of the Taiwanese non-life insurance companies, which has been extracted from the extant literature.

Keywords and Phrases: Data Envelopment Analysis, Efficiency, Two stage system, Stochastic Data, Multiplier Form, Additive Form.

1 Introduction

DEA is a non-parametric mathematical approach that evaluates the efficiency and the performance of decision making units (DMUs). Initially, DEA was presented by Charnes, Cooper, and Rhodes their first proposed model was called CCR [2]. Then onwards, many models have been proposed, that measure the efficiency of DMUs, by considering DMUs, as 'black box systems. In variety of applications, data may not be precise such as, stochastic data. Stochastic DEA (SDEA) was presented, so as to measure the efficiency of black box systems in the presence of stochastic data, by extending the classical DEA. In this field, some researchers rendered the Stochastic models (see examples [5], [6], [11] and [12]). These authors considered the envelopment form of DEA models and proposed the stochastic DEA models by utilizing the chance constrained programming method. In addition, Mirbolouki et al. [15] also utilized the chance constrained programming method and offered a stochastic DEA model, based on the multiplier form of DEA, which measures the stochastic efficiency of black box systems. To do this, they solved two problems, (the existing equivalent constraints and the random variable in the objective function). In real applications, there are systems with an internal structure such as, network systems. Hence, a group of DEA models was presented in order to assess the efficiency of these systems. These models were called Network DEA (NDEA) models (see examples [1], [3], [4], [7], [9], [8], [10], [14], [13] and [16]). The special factor of these network systems is their two-stage system. Therefore, in this paper, we will combine SDEA and NDEA to propose the stochastic multiplicative and additive models that measure the stochastic efficiency of two-stage systems in presence of stochastic data. Note, that in the proposed multiplicative model, a simultaneous presence of stages, is necessary in the final product and the shortcoming (default) of one stage is not compensated by another stage. Moreover, it models the overall efficiency of the system in the mathematical average of the efficiencies of the stages. In this case, the first and the second stages present and evaluateing the overall efficiency. This paper is organized as follows: In section 2, firstly, we review production possibility sets T_c, T_v. Then, we briefly review the Kao and Hwang [9] and Chen et al. [3] models that measure the efficiency of two-stage systems. In section 3, firstly we propose the structure of stochastic efficiency of the two-stage systems in presence of stochastic data. Then, we apply the chance-constrained programming method on the Kao and Hwang [9] model and determine corresponding deterministic equivalent form; and also, the stochastic versions of Chen et al. [3] models including the deterministic equivalents, which are given. In section 4, the introduced stochastic models are illustrated in the form of a case-study in relative to 10 Taiwanese non-life insurance companies. Finally, section 5 presents our conclusions and future research directions.

2 Preliminaries

In this section, two production possibility sets are presented. Then, in order to measure the CRS and VRS efficiency of two-stage systems, the multiplicative and additive models are introduced.

2.1 Production possibility sets

Consider n DMUs where each $DMU_j(j = 1, \ldots, n)$ consume m inputs $x_{ij}(i = 1, \ldots, m)$ to produce s outputs $y_{rj}(r = 1, \ldots, s)$. Production Possibility Sets T_c, T_v are defined as follows regarding the prevalence of

CRS and VRS assumptions of the production technology, respectively:

$$T_c = \left\{ (x,y) \middle| \sum_{j=1}^{n} \lambda_j x_j \leq x,\ \sum_{j=1}^{n} \lambda_j y_j \geq y,\ \lambda_j \geq 0,\ j=1,\ldots,n \right\}$$

$$T_v = \left\{ (x,y) \middle| \sum_{j=1}^{n} \lambda_j x_j \leq x,\ \sum_{j=1}^{n} \lambda_j y_j \geq y,\ \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \right. \\ \left. j=1,\ldots,n \right\}$$

2.2 Two-stage DEA models

In this subsection, firstly, we briefly present the models to evaluate the CRS and VRS efficiency of two-stage systems with deterministic data that were presented by Kao and Hwang [9] and Chen et al. [3]. Let us assume that there are n DMUs with a two-stage structure. Each DMU_j $(j = 1,\ldots,n)$ in the stage 1 consumes m input $x_{ij}(i = 1,\ldots,m)$ to produce D intermediate measure $z_{dj}(d = 1,\ldots,n)$. Then, stage 2, uses D intermediate measure $z_{dj}(d = 1,\ldots,n)$ to generate s output $y_{rj}(r = 1,\ldots,s)$. The structure of a two-stage system is shown in Figure 1.

Kao and Hwang [9] presented the following model that measures the overall efficiency of the system and the efficiency of stages under the CRS assumption simultaneously:

$$\begin{aligned} E_o^s = \max & \sum_{r=1}^{s} u_r y_{ro} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j=1,\ldots,n \\ & \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j=1,\ldots,n \\ & u_r, w_d, v_i \geq 0 \qquad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m \end{aligned} \tag{1}$$

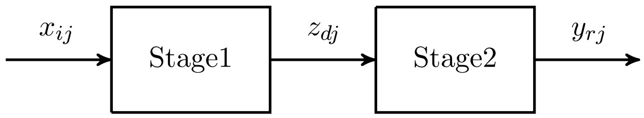

Figure 1. Two-stage system

If (u^*, v^*, w^*) is an optimal solution of this model,, we have:

$$E_o^s = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad E_o^I = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad E_o^{II} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}.$$

That E_o^s, E_o^I, E_o^{II} indicates the overall efficiency of the system and efficiency of the first and second stages respectively.

Theorem 2.1. *DMU_o is overall efficient if and only if $E_o^I = E_o^{II} = 1$.*

Proof. Refer to [9] □

Their proposed model cannot measure the VRS efficiency of two-stage systems. Chen et al. [3] proposed the models that calculate the overall efficiency of the system and efficiency of the stages under CRS and VRS. The following model is presented to measure the CRS efficiency of two-stage systems by Chen et al. [3]:

$$\begin{aligned} E_o^{(chen-CRS)s} &= \max \ w_1 \frac{\sum_{d=1}^{D} w_d z_{do}}{\sum_{i=1}^{m} v_i x_{io}} + w_2 \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{d=1}^{D} w_d z_{do}} \\ \text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \ldots, n \\ & \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\ & u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \end{aligned} \tag{2}$$

Note that

$$w_1 = (\sum_{i=1}^{m} v_i x_{io})/(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do}),$$
$$w_2 = (\sum_{d=1}^{D} w_d z_{do})/(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do})$$

are defined as Proportion of the aggregate input of stage 1 and stage 2 to the aggregate input of the whole system and demonstrate contribution of the performance of stages 1, 2. Actually, our argument is that the importance of a stage as measured by its weight.
Therefore, model (2) can be converted into the following form:

$$\begin{aligned} E_o^{(chen-CRS)s} = \max \ & \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} - \sum_{d=1}^{D} w_d z_{do} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \dots, n \\ & \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \dots, n \\ & u_r, w_d, v_i \geq 0 \quad r = 1, \dots, s \quad d = 1, \dots, D \quad i = 1, \dots, m \end{aligned} \tag{3}$$

If (u^*, v^*, w^*) is an optimal solution of this model, we have:

$$E_o^{(chen-CRS)s} = \frac{\sum_{r=1}^{s} u_r^* y_{ro} + \sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{d=1}^{D} w_d^* z_{do}}, \quad E_o^I = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}},$$
$$E_o^{II} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}.$$

$E_o^{(chen-CRS)s}$, E_o^I, E_o^{II} indicates the overall efficiency of the system and efficiency of the first and second stages respectively. We also have $E_o^{(chen-CRS)s} = w_1 E_o^I + w_2 E_o^{II}$. And also, Chen et al. [3], also suggested a model to compute the efficiency of a two-stage system under the VRS assumption. Their proposed model is as follows:

$$\begin{aligned}
E_o^{(chen-VRS)s} &= \max \ w_1 \frac{\sum_{d=1}^{D} w_d z_{do} + u_{01}}{\sum_{i=1}^{m} v_i x_{io}} + w_2 \frac{\sum_{r=1}^{s} u_r y_{ro} + u_{02}}{\sum_{d=1}^{D} w_d z_{do}} \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} \le 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} \le 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \ge 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& u_{01}, u_{02} \ \ free
\end{aligned} \tag{4}$$

By applying w_1, w_2 in this model, the following model is obtained:

$$\begin{aligned}
E_o^{(chen-VRS)s} &= \max \ \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} \le 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} \le 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \ge 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& u_{01}, u_{02} \ \ free
\end{aligned} \tag{5}$$

After solving this model, the overall efficiency of the system and efficiency of the stage 1, 2 ($E_o^{(chen-VRS)s}, E_o^I, E_o^{II}$) can be determined as follows:

$$E_o^{(chen-VRS)s} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro} + \sum\limits_{d=1}^{D} w_d^* z_{do} + u_{01} + u_{02}}{\sum\limits_{i=1}^{m} v_i^* x_{io} + \sum\limits_{d=1}^{D} w_d^* z_{do}}, \quad E_o^I = \frac{\sum\limits_{d=1}^{D} w_d^* z_{do} + u_{01}}{\sum\limits_{i=1}^{m} v_i^* x_{io}},$$

$$E_o^{II} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro} + u_{02}}{\sum\limits_{d=1}^{D} w_d^* z_{do}}.$$

Therefore, the relationship between $E_o^{(chen-VRS)s}$, E_o^I, E_o^{II} can be defined as follows: $E_o^{(chen-VRS)s} = w_1 E_o^I + w_2 E_o^{II}$.

3 Stochastic Efficiency of Two-Stage Systems

In many situations, the input, intermediate product and output vectors might be stochastic variables. Therefore, in this case, providing a stochastic model is necessary, in order to measure the efficiency of two-stage systems under CRS and VRS assumptions. Suppose we have n DMUs with a two-stage structure; corresponding to the first stage $DMU_j (j = 1, \dots, n)$, $\tilde{x}_j$, $\tilde{z}_j$ are the random inputs and intermediate measures vectors. Then, the second stage, consumes these intermediate measures to produce the random output vector $\tilde{y}_j$. without losing generality, we presume that all components of inputs, intermediate measures and output s have a normal distribution:

$$\tilde{x}_{ij} \sim N(x_{ij}, \sigma_{ij}^2), \quad \tilde{y}_{rj} \sim N(y_{rj}, \sigma_{rj}^2), \quad \tilde{z}_{dj} \sim N(z_{dj}, \sigma_{dj}^2)$$

Wherein, x_{ij}, y_{rj}, z_{dj} $(i = 1, \dots, m \quad r = 1, \dots, s \quad d = 1, \dots, D)$ are vectors of the expected values of inputs, intermediate measures and outputs of $DMU_j (j = 1, \dots, n)$.

3.1 Stochastic efficiency of the multiplicative model

In this subsection, we will initially propose a stochastic model of Kao and Hwang [9], using the multiplicative model. Then, the deterministic equivalent of the proposed model will be provided. The stochastic model that measures the efficiencies of the two-stage systems under CRS can be described as follows:

$$\begin{aligned}
\tilde{E}_o^s = \max \ & \sum_{r=1}^{s} u_r \tilde{y}_{ro} \\
\text{s.t.} \quad & p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1\right\} \geq (1-\alpha) \\
& p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0\right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& p\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0\right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{6}$$

In this model, p means probability. indicates the level of error that is predetermined. In the objective function of model [4], there is random variable and we also have the following expression in the first constraint of model [4] which is wrong:

$$p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} = 1\right\} = 0 \geq (1-\alpha) \Rightarrow \alpha \geq 1$$

For solving these problems, firstly we introduce an alternative form of the model (1). Note that we can replace the objective function of the model (1) by:

$$\begin{aligned}
\max \quad & k \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r \tilde{y}_{io} \geq k
\end{aligned}$$

Therefore, the following model is obtained:

$$
\begin{aligned}
\tilde{E}_o^s = \max \quad & k \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r \tilde{y}_{ro} \geq k \\
& \sum_{i=1}^{m} v_i \tilde{x}_{io} = 1 \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \qquad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{7}
$$

However, the mentioned approximation error has not been remedied. Hence, we replace the first constraint by $\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1$. Therefore, the following model can be constructed as an alternative form of the proposed model (7):

$$
\begin{aligned}
\tilde{E}_o^{s'} = \max \quad & k \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r \tilde{y}_{ro} \geq k \\
& \sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1 \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \qquad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{8}
$$

Note that the models (7) and (8) have equal objective values. Hence, we have the following theorem:

Theorem 3.1. *In the models* (7) *and* (8) *we have:* $\tilde{E}_o^s = \tilde{E}_o^{s'}$.

Proof. Suppose S, S' indicate the feasible regions related to the models (7) and (8) respectively. Note that $S \subseteq S'$ and $k \leq \sum_{i=1}^m v_i \tilde{x}_{io} \leq 1$. Since this is a maximization problem, in optimal solution we have $k = \sum_{i=1}^m v_i \tilde{x}_{io} = 1$. Hence, in optimality solutions of model (8) satisfy in the constraints of the model (7) and the proof is complete. $\square$

Now, we apply chance constrained problem and proposed the following stochastic model of model (8):

$$
\begin{aligned}
\tilde{E}_o^{s'} &= \max \ k \\
\text{s.t.} \quad & p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} \geq k\right\} \geq (1-\alpha) \\
& p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1\right\} \geq (1-\alpha) \\
& p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0\right\} \geq (1-\alpha), \quad j = 1, \dots, n \\
& p\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0\right\} \geq (1-\alpha), \quad j = 1, \dots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \dots, s \quad d = 1, \dots, D \quad i = 1, \dots, m
\end{aligned}
\tag{9}
$$

3.1.1 Deterministic equivalent of model (9)

In this subsection, we will exhibit a deterministic equivalent of model (9) using Cooper et al. [5]. Firstly, consider the following constraint:

$$p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1\right\} \geq (1-\alpha).$$

In order to achieve the equality constraint, we define $\zeta_1 \geq 0$ as an external slack:

$$p\left\{\sum_{r=1}^{s} u_i \tilde{y}_{ro} \geq k\right\} = (1-\alpha) + \zeta_1.$$

Thus, there is $S_1 \geq 0$ such that,

$$p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k \geq s_1\right\} = (1-\alpha).$$

Note that $\zeta_1 = 0$ if and only if $s_1 = 0$. By also, by defining $\zeta_2 \geq 0$ as an external slack, we have

$$p\left\{\sum_{r=1}^{m} v_r \tilde{x}_{io} \leq 1\right\} = (1-\alpha) + \zeta_2.$$

Hence there is $s_2 \geq 0$ such that

$$p\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 1 + s_2\right\} = (1-\alpha).$$

Corresponding to other constraints, we suppose there are $s_{3j}, s_{4j} \geq 0$ such that

$$p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq s_{3j}\right\} = (1-\alpha), \quad j = 1, \ldots, n$$
$$p\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq s_{4j}\right\} = (1-\alpha), \quad j = 1, \ldots, n$$

Now, we set:

$$E(\tilde{x}_{ij}) = x_{ij}, \quad E(\tilde{y}_{rj}) = y_{rj}, \quad E(\tilde{z}_{dj}) = z_{dj},$$
$$E(\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k) = \sum_{r=1}^{s} u_r y_{ro} - k$$

Hence: $p\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} - k \geq s_1\right\} = (1-\alpha)$ conclude that

$$p\left\{\sum_{r=1}^{s} u_r^* \tilde{y}_{ro} - k\right\} \leq s_1 = \alpha.$$

Thus,

$$p\left\{\frac{(\sum\limits_{r=1}^{s} u_r\tilde{y}_{ro}-k)-(\sum\limits_{r=1}^{s} u_r y_{ro}-k)}{\sqrt{\mathrm{var}(\sum\limits_{r=1}^{s} u_r\tilde{y}_{ro}-k)}} \leq \frac{s_1-\left(\sum\limits_{r=1}^{s} u_r y_{ro}-k\right)}{\sqrt{\mathrm{var}\left(\sum\limits_{r=1}^{s} u_r\tilde{y}_{ro}-k\right)}}\right\}=\alpha \tag{10}$$

By considering Φ as standard normal distribution function, we recall that $p(\tilde{Z}\leq z)=\alpha \Rightarrow \Phi(z)=\alpha \Rightarrow \Phi^{-1}(\alpha)=z$ Hence, (10) can be converted to

$$\frac{s_1-(\sum\limits_{r=1}^{s} u_r y_{ro}-k)}{\sqrt{\mathrm{var}(\sum\limits_{r=1}^{s} u_r\tilde{y}_{ro}-k)}}=\Phi^{-1}(\alpha).$$

In order to simplify, we denote

$$\begin{aligned}
(\sigma^o(k,u))^2 &= \mathrm{var}(\sum_{r=1}^{s} u_r\tilde{y}_{ro}-k)=\sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'}\mathrm{cov}(\tilde{y}_{ro},\tilde{y}_{r'o})\\
(\sigma^I(k,v))^2 &= \mathrm{var}(1-\sum_{i=1}^{m} v_i\tilde{x}_{io})=\mathrm{var}(\sum_{i=1}^{m} v_i\tilde{x}_{io})\\
&= \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'}\mathrm{cov}(\tilde{x}_{io},\tilde{x}_{i'o})\\
(\sigma_j(w,u))^2 &= \mathrm{var}(\sum_{d=1}^{D} w_d\tilde{z}_{dj}-\sum_{r=1}^{s} u_r\tilde{y}_{rj})=\sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'}\mathrm{cov}(\tilde{y}_{rj},\tilde{y}_{r'j})\\
&\quad+\sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\mathrm{cov}(\tilde{z}_{dj},\tilde{z}_{d'j})-2\mathrm{cov}(\sum_{d=1}^{D} w_d\tilde{z}_{dj},\sum_{r=1}^{s} u_r\tilde{y}_{rj})
\end{aligned}$$

$$(\sigma'_j(v,w))^2 = \mathrm{var}(\sum_{i=1}^{m} v_i\tilde{x}_{ij} - \sum_{d=1}^{D} w_d\tilde{z}_{dj}) = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\mathrm{cov}(\tilde{z}_{dj},\tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'}\mathrm{cov}(\tilde{x}_{ij},\tilde{x}_{i'j}) - 2\mathrm{cov}(\sum_{i=1}^{m} v_i\tilde{x}_{ij}, \sum_{d=1}^{D} w_d\tilde{z}_{dj})$$

Therefore, $\frac{s_1 - (\sum_{r=1}^{s} u_r y_{ro} - k)}{\sigma^o(k,u)} = \Phi^{-1}(\alpha)$. By applying the same approach for other constraints, the deterministic equivalent form of model (9) will be as follows:

$$\begin{aligned}
&\tilde{E}_o^{s'} = \max\ k \\
&\text{s.t.}\quad \sum_{r=1}^{s} u_r y_{ro} - k + \Phi^{-1}(\alpha)\sigma^o(k,u) = s_1 \\
&\qquad \sum_{i=1}^{m} v_i x_{io} - \Phi^{-1}(\alpha)\sigma^I(k,v) + s_2 = 1 \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma_j(w,u) + s_{3j} = 0 \quad j=1,\ldots,n \\
&\qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\sigma'_j(v,w) + s_{4j} = 0 \quad j=1,\ldots,n \\
&\qquad u_r, w_d, v_i \geq 0 \quad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m \\
&\qquad s_{3j}, s_{4j} \geq 0 \;\; j=1,\ldots,n \\
&\qquad s_1, s_2 \geq 0
\end{aligned} \tag{11}$$

Note that this model is a nonlinear programming. Thus, by following the Cooper et al. [5] model, we transform this model to a quadratic programming problem. For this purpose we use the non-negative variables

λ, λ', λ_j, λ'_j and obtain the quadratic programming problem as follows:

$$
\begin{aligned}
&\tilde{E}_o^{s'} = \max\ k \\
&\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{ro} - k + \Phi^{-1}(\alpha)\lambda - s_1 = 0 \\
&\qquad \sum_{i=1}^{m} v_i x_{io} - \Phi^{-1}(\alpha)\lambda' + s_2 = 1 \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\lambda_j + s_{3j} = 0 \quad j = 1, \dots, n \\
&\qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\lambda'_j + s_{4j} = 0 \quad j = 1, \dots, n \\
&\lambda^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) \\
&{\lambda'}^2 = \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{io}, \tilde{x}_{i'o}) \qquad\qquad (12) \\
&\lambda_j^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) \\
&\qquad - 2\mathrm{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj}) \\
&{\lambda'_j}^2 = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j}) \\
&\qquad - 2\mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj}) \\
&\qquad u_r, w_d, v_i \geq 0 \quad r = 1, \dots, s \quad d = 1, \dots, D \quad i = 1, \dots, m \\
&\qquad \lambda, \lambda', \lambda_j, \lambda'_j, s_{3j}, s_{4j} \geq 0 \ \ j = 1, \dots, n \\
&\qquad s_1, s_2 \geq 0
\end{aligned}
$$

Theorem 3.2. *For* $\alpha \in (0, 0.5]$ *if* $(u_r^*, w_d^*, v_i^*, \lambda^*, \lambda'^*, \lambda_j^*, \lambda_j'^*, s_1^*, s_2^*, s_{3j}^*, s_{4j}^*)$ *is an optimal solution, we have* $0 < \tilde{E}^{s'} \leq 1$.

Proof. If $\alpha \in (0, 0.5]$, then $\Phi^{-1}(\alpha) \leq 0$. In each optimal solution, we have:

$$\begin{cases} \sum_{r=1}^{s} u_r^* y_{rj} - \sum_{d=1}^{D} w_d^* z_{dj} \leq 0 \\ \sum_{d=1}^{D} w_d^* z_{dj} - \sum_{i=1}^{m} v_i^* x_{ij} \leq 0 \end{cases} \Rightarrow \sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} \leq 0$$

And also, based on the constraints $\sum_{r=1}^{s} u_r^* y_{ro} \geq k$, $\sum_{i=1}^{m} v_i^* x_{io} \leq 1$ of model (12) the proof is complete. □

Now for $\alpha \in (0, 0.5]$ and any optimal solution $(u_r^*, w_d^*, v_i^*, \lambda^*, \lambda'^*, \lambda_j^*, \lambda_j'^*, s_1^*, s_2^*, s_{3j}^*, s_{4j}^*)$ of model (12), the overall efficiency as well as the efficiency of the first and the second stages are defined as:

$$\tilde{E}_o^{s'} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad \tilde{E}_o^{I} = \frac{\sum_{d=1}^{D} w_d^* z_{do}}{\sum_{i=1}^{m} v_i^* x_{io}}, \quad \tilde{E}_o^{II} = \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{D} w_d^* z_{do}}.$$

Thus, we have: $\tilde{E}_o^{s'} = \tilde{E}_o^{I} \cdot \tilde{E}_o^{II}$.

Lemma 3.3. *For* $\alpha \in (0, 0.5]$ *and each* DMU_o*, we have:* $0 < \tilde{E}_o^{I} \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$.

Proof. In any optimal solution $(u_r^*, w_d^*, v_i^*, \lambda^*, \lambda'^*, \lambda_j^*, \lambda_j'^*, s_1^*, s_2^*, s_{3j}^*, s_{4j}^*)$ of the model (12), for $j = o$, we have

$$\sum_{r=1}^{s} u_r^* y_{ro} - \sum_{d=1}^{D} w_d^* z_{do} - \Phi^{(-1)}(\alpha)\lambda_o^* + s_{3o}^* = 0,$$
$$\sum_{d=1}^{D} w_d^* z_{do} - \sum_{i=1}^{m} v_i^* x_{io} - \Phi^{(-1)}(\alpha)\lambda_o'^* + s_{4o}^* = 0.$$

And also, we know that

$$\Phi^{(-1)}(\alpha), \lambda_o^*, \lambda_o'^* \geq 0, S_{3o}^*, s_{4o}^* \geq 0$$

Thus, in any optimal solution, we have:

$$\sum_{r=1}^{s} u_r^* y_{ro} - \sum_{d=1}^{D} w_d^* z_{do} \leq 0, \quad \sum_{d=1}^{D} w_d^* z_{do} - \sum_{i=1}^{m} v_i^* x_{io} \leq 0$$

These constraints mean that $0 < \tilde{E}_o^I \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$ and the proof is complete. □

Lemma 3.4. *For $\alpha \in (0, 0.5]$, DMU_o, is stochastic overall efficient under the model (12) if and only if the first and the second stages are stochastic efficient, i.e. $\tilde{E}_o^{s'} = 1$ if and only if $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$.*

Proof. Suppose $\tilde{E}_o^{s'} = 1$, i.e $\sum_{r=1}^{s} u_r^* y_{ro} = \sum_{i=1}^{m} v_i^* x_{io}$. And also, for $\alpha \in (0, 0.5]$, we have

$$\sum_{r=1}^{s} u_r^* y_{ro} - \sum_{d=1}^{D} w_d^* z_{do} \leq 0, \quad \sum_{d=1}^{D} w_d^* z_{do} - \sum_{i=1}^{m} v_i^* x_{io} \leq 0$$

Therefore, we conclude that $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$. Conversely, if $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$, the proof is obvious. □

3.2 Stochastic efficiency of additive models

In this subsection, the stochastic versions of the additive models will be presented in the presence of stochastic data. Then, the deterministic equivalent forms of these stochastic models are obtained. The proposed model of the previous section are unable to calculate the efficiency of a two-stage system under VRS assumption. Thereby, by following the Chen et al. [3] model we provide the stochastic models that measure the efficiency of the two-stage systems under CRS and VRS assumptions respectively. Our proposed models are as follows:

$$\begin{aligned}
\tilde{E}_o^{(chen-CRS)s} &= \max \sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \\
\text{s.t.} \quad & P\left\{ \sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} = 1 \right\} \geq (1-\alpha) \\
& P\left\{ \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0 \right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& P\left\{ \sum_{d=1}^{D} w_s \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{13}$$

$$\begin{aligned}
\tilde{E}_o^{(chen-VRS)s} &= \max \sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} + u_{01} + u_{02} \\
\text{s.t.} \quad & P\left\{ \sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} = 1 \right\} \geq (1-\alpha) \\
& P\left\{ \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} + u_{02} \leq 0 \right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& P\left\{ \sum_{d=1}^{d} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} + u_{01} \leq 0 \right\} \geq (1-\alpha), \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& u_{01}, u_{02} \ free
\end{aligned} \tag{14}$$

Note that in these models, p means probability. The amount of α is predetermined that determines the level of error. In the objective functions of models (13) and (14) where there is a random variable and we also have the following expression which is wrong

$$p\left\{\sum_{i=1}^{m} v_i^* x_{io} + \sum_{d=1}^{D} w_d^* z_{do} = 1\right\} = 0 \geq (1-\alpha) \Rightarrow \alpha \geq 1$$

Thus, similar to the approach of the subsection (3-1), we presented the following models:

$$\begin{aligned}
E_o^{(chen-CRS)s} &= \max\ k \\
\text{s.t.}\quad & \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} \geq k \\
& \sum_{i=1}^{m} v_i x_{io} - \sum_{d=1}^{D} w_d z_{do} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned} \tag{15}$$

$$\begin{aligned}
E_o^{(chen-VRS)s} &= \max\ k \\
\text{s.t.}\quad & \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} \geq k \\
& \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} \leq 0, \quad j = 1, \ldots, n \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} \leq 0, \quad j = 1, \ldots, n \\
& u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& u_{01}, u_{02}\ \ free
\end{aligned} \tag{16}$$

Now, the alternative form of the models (3) and (5), can be written as follows:

$$
\begin{aligned}
& E_o^{(chen-CRS)s'} = \max \ k \\
& \text{s.t.} \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} \geq k \\
& \qquad \sum_{i=1}^{m} v_i x_{io} - \sum_{d=1}^{D} w_d z_{do} \leq 1 \\
& \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, \quad j = 1, \ldots, n \\
& \qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\
& \qquad u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
& E_o^{(chen-VRS)s'} = \max \ k \\
& \text{s.t.} \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} \geq k \\
& \qquad \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} \leq 1 \\
& \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} \leq 0, \quad j = 1, \ldots, n \\
& \qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} \leq 0, \quad j = 1, \ldots, n \\
& \qquad u_r, w_d, v_i \geq 0 \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
& \qquad u_{01}, u_{02} \ \mathit{free}
\end{aligned}
\tag{18}
$$

Theorem 3.5. *For models* (15) *and* (17) *(and also, for models* (16) *and (18)), we have* $E_o^{(chen-CRS)s} = E_o^{(chen-CRS)s'}$ *(and* $E_o^{(chen-VRS)s} = E_o^{(chen-VRS)s'}$ *).*

Proof. The proof is similar to the proof of theorem 3.1. □

Hence, the stochastic versions of the models (17) and (18) are as follows:

$$
\begin{aligned}
&\tilde{E}_o^{s(chen-CRS)s'} = \max \ k \\
&\text{s.t.} \quad P\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \geq k\right\} \geq (1-\alpha) \\
&\qquad P\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \leq 1\right\} \geq (1-\alpha) \\
&\qquad P\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} \leq 0\right\} \geq (1-\alpha), \quad j=1,\ldots,n \\
&\qquad P\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0\right\} \geq (1-\alpha), \quad j=1,\ldots,n \\
&\qquad u_r, w_d, v_i \geq 0, \quad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\tilde{E}_o^{s(chen-VRS)s'} = \max \ k \\
&\text{s.t.} \quad P\left\{\sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} + u_{01} + u_{02} \geq k\right\} \geq (1-\alpha) \\
&\qquad P\left\{\sum_{i=1}^{m} v_i \tilde{x}_{io} + \sum_{d=1}^{D} w_d \tilde{z}_{do} \leq 1\right\} \geq (1-\alpha) \\
&\qquad P\left\{\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{d=1}^{D} w_d \tilde{z}_{dj} + u_{02} \leq 0\right\} \geq (1-\alpha), \quad j=1,\ldots,n \\
&\qquad P\left\{\sum_{d=1}^{D} w_d \tilde{z}_{dj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} + u_{01} \leq 0\right\} \geq (1-\alpha), \quad j=1,\ldots,n \\
&\qquad u_r, w_d, v_i \geq 0, \quad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m \\
&\qquad u_{01}, u_{02} \ free
\end{aligned}
\tag{20}
$$

3.2.1 Deterministic equivalent of model (19)

In this subsection, we obtain the deterministic equivalent of model (19). By using the similar procedure, the deterministic equivalent form of model (19) is as follows:

$$
\begin{aligned}
&\tilde{E}_o^{(chen-CRS)s'} = \max\ k \\
\text{s.t.}\quad &\sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{dj} - k + \Phi^{-1}(\alpha)\sigma^o(k,u,w) = s'_1 \\
&\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma^I(k,v,w) + s'_2 = 1 \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma_j(w,u) + s'_{3j} = 0 \quad j=1,\ldots,n \\
&\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\sigma'_j(v,w) + s'_{4j} = 0 \quad j=1,\ldots,n \\
&u_r, w_d, v_i \geq 0, \quad r=1,\ldots,s \quad d=1,\ldots,D \quad i=1,\ldots,m \\
&s'_{3j}, s'_{4j} \geq 0 \ \ j=1,\ldots,n \\
&s'_1, s'_2 \geq 0
\end{aligned}
\tag{21}
$$

That:

$$
\begin{aligned}
(\sigma^o(k,u,w))^2 &= \mathrm{var}\Big(\sum_{r=1}^{s} u_r \tilde{y}_{ro} + \sum_{d=1}^{D} w_d \tilde{z}_{do} - k\Big) \\
&= \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o}) \\
&\quad + 2\mathrm{cov}\Big(\Big(\sum_{r=1}^{s} u_r \tilde{y}_{ro}\Big), \Big(\sum_{d=1}^{D} w_d \tilde{z}_{do} - k\Big)\Big)
\end{aligned}
$$

$$(\sigma^I(k,v,w))^2 = \mathrm{var}(1-(\sum_{i=1}^{m} v_i\tilde{x}_{io} + \sum_{d=1}^{D} w_d\tilde{z}_{do}))$$

$$= \mathrm{var}(\sum_{i=1}^{m} v_i\tilde{x}_{io} + \sum_{d=1}^{D} w_d\tilde{z}_{do}) = \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'}\mathrm{cov}(\tilde{x}_{io},\tilde{x}_{i'o})$$

$$+ \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\mathrm{cov}(\tilde{z}_{do},\tilde{z}_{d'o}) + \mathrm{cov}(\sum_{i=1}^{m} v_i\tilde{x}_{io}, \sum_{d=1}^{D} w_d\tilde{z}_{do})$$

$$(\sigma_j(w,u))^2 = \mathrm{var}(\sum_{d=1}^{D} w_d\tilde{z}_{dj} - \sum_{r=1}^{s} u_r y_{rj}) = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'}\mathrm{cov}(\tilde{y}_{rj},\tilde{y}_{r'j})$$

$$+ \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\mathrm{cov}(\tilde{z}_{dj},\tilde{z}_{d'j}) - 2\mathrm{cov}(\sum_{d=1}^{D} w_d\tilde{z}_{dj}, \sum_{r=1}^{s} u_r\tilde{y}_{ro})$$

$$(\sigma'_j(w,u))^2 = \mathrm{var}(\sum_{i=1}^{m} v_i\tilde{x}_{ij} - \sum_{d=1}^{D} w_d\tilde{z}_{dj}) = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'}\mathrm{cov}(\tilde{z}_{dj},\tilde{z}_{d'j})$$

$$+ \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'}\mathrm{cov}(\tilde{x}_{ij},\tilde{x}_{i'j}) - 2\mathrm{cov}(\sum_{i=1}^{m} v_i\tilde{x}_{ij}, \sum_{d=1}^{D} w_d\tilde{z}_{dj})$$

This model is a nonlinear programming. In order to convert this model to a quadratic programming problem, the non-negative variables are γ, γ', γ_j, γ'_j. Therefore Therefore the following quadratic programming problem is obtained:

$$
\begin{aligned}
&\tilde{E}_o^{(chen-CRS)s'} = \max\ k \\
&\text{s.t.}\quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} - k + \Phi^{-1}(\alpha)\gamma - s'_1 = 0 \\
&\qquad \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} - \Phi^{-1}(\alpha)\gamma' + s'_2 = 1 \\
&\qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\gamma_j + s'_{3j} = 0 \quad j = 1, \dots, n \\
&\qquad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} - \Phi^{-1}(\alpha)\gamma'_j + s'_{4j} = 0 \quad j = 1, \dots, n \\
&\gamma^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o}) \\
&\qquad + 2\mathrm{cov}((\sum_{r=1}^{s} u_r \tilde{y}_{ro}), (\sum_{d=1}^{D} w_d \tilde{z}_{do} - k)) \\
&{\gamma'}^2 = \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{io}, \tilde{x}_{i'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{do}, \tilde{z}_{d'o}) \\
&\qquad + \mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{io}, \sum_{d=1}^{D} w_d \tilde{z}_{do}) \\
&\gamma_j^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \mathrm{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) \\
&\qquad - 2\mathrm{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj}) \\
&{\gamma'_j}^2 = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \mathrm{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \mathrm{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j}) \\
&\qquad - 2\mathrm{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj}) \\
&\qquad u_r, w_d, v_i \geq 0 \quad r = 1, \dots, s \quad d = 1, \dots, D \quad i = 1, \dots, m \\
&\qquad \gamma_j, \gamma'_j, s'_{3j}, s'_{4j} \geq 0 \ \ j = 1, \dots, n \\
&\qquad s'_1, s'_2, \gamma, \gamma' \geq 0
\end{aligned}
\tag{22}
$$

Theorem 3.6. *For* $\alpha \in (0, 0.5]$, *if* $(u_r^*, w_d^*, v_i^*, \gamma^*, {\gamma'}^*, \gamma_j^*, {\gamma'_j}^*, {s'_1}^*, {s'_2}^*, {s'_{3j}}^*, {s'_{4j}}^*)$ *be an optimal solution of model* (22), *we have* $0 < \tilde{E}_o^{(chen-CRS)s'} \leq 1$.

Proof. The proof is similar to the proof of Theorem 3.2. □
Now, if $(u_r^*, w_d^*, v_i^*, \gamma^*, {\gamma'}^*, \gamma_j^*, {\gamma'_j}^*, {s'_1}^*, {s'_2}^*, {s'_{3j}}^*, {s'_{4j}}^*)$ is an optimal solution of model (22), for $j = o$, the efficiency of the first and the second stages are defined:

$$\tilde{E}_o^I = \sum_{d=1}^{D} w_d^* z_{do} / \sum_{i=1}^{m} v_i^* x_{io}, \tilde{E}_o^{II} = \sum_{r=1}^{s} u_r^* y_{ro} / \sum_{d=1}^{D} w_d^* z_{do}.$$

Therefore, there is $\lambda \in (0, 1)$ that $\tilde{E}_o^{(chen-CRS)s'} = \lambda \tilde{E}_o^I + (1 - \lambda)\tilde{E}_o^{II}$.

Lemma 3.7. *For* $\alpha \in (0, 0.5]$ *and each* DMU_o, *we have:* $0 < \tilde{E}_o^I \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$.

Proof. The proof is similar to the proof of lemma 3.3. □

Lemma 3.8. *For* $\alpha \in (0, 0.5]$, DMU_o *is stochastic overall efficient under the model* (18) *if and only if, the first and the second stages are stochastic efficient, i.e.* $\tilde{E}_o^{(chen-VRS)s'} = 1$ *if and only if,* $\tilde{E}_o^I = \tilde{E}_o^{II} = 1$.

Proof. The proof is similar to the proof of lemma 3.4. □
By applying the aforementioned manner to the model (14), the deterministic equivalent form of this model can be obtained as follows that is

a nonlinear programming:

$$
\begin{aligned}
&\tilde{E}_o^{(chen-VRS)s'} = \max \ k \\
&\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} - k \\
&\qquad\qquad\qquad + \Phi^{-1}(\alpha)\sigma^o(k, u, w, u_{01}, u_{02}) = s''_1 \\
&\quad \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} - \Phi^{-1}(\alpha)\sigma^I(k, v, w) + s''_2 = 1 \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} - \Phi^{-1}(\alpha)\sigma_j(w, u, u_{02}) + s''_{3j} = 0, \qquad (23)\\
&\qquad\qquad\qquad j = 1, \ldots, n \\
&\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} - \Phi^{-1}(\alpha)\sigma'_j(v, w, u_{01}) + s''_{4j} = 0, \\
&\qquad\qquad\qquad j = 1, \ldots, n \\
&\quad u_r, w_d, v_i \geq 0, \quad r = 1, \ldots, s \quad d = 1, \ldots, D \quad i = 1, \ldots, m \\
&\quad s''_{3j}, s''_{4j} \geq 0, \ \ j = 1, \ldots, n \\
&\quad s''_1, s''_2 \geq 0 \\
&\quad u_{01}, u_{02} \ free
\end{aligned}
$$

We use the non-negative variables $\eta, \eta', \eta_j, \eta'_j \geq 0$ in order to achieve a quadratic programming problem. Therefore the quadratic programming problem is as follows:

$$
\begin{aligned}
&\tilde{E}_o^{(chen-VRS)s'} = \max\ k\\
&\text{s.t.}\quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{d=1}^{D} w_d z_{do} + u_{01} + u_{02} - k + \Phi^{-1}(\alpha)\eta = s_1''\\
&\quad \sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} w_d z_{do} - \Phi^{-1}(\alpha)\eta' + s_2'' = 1\\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} + u_{02} - \Phi^{-1}(\alpha)\eta_j + s_{3j}'' = 0 \quad j = 1,\dots,n\\
&\quad \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} + u_{01} - \Phi^{-1}(\alpha)\eta_j' + s_{4j}'' = 0 \quad j = 1,\dots,n\\
&\eta^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \operatorname{cov}(\tilde{y}_{ro}, \tilde{y}_{r'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \operatorname{cov}(\tilde{z}_{do}, \tilde{z}_{d'o})\\
&\qquad + 2\operatorname{cov}((\sum_{r=1}^{s} u_r \tilde{y}_{ro} + u_{01} + u_{02}), (\sum_{d=1}^{D} w_d \tilde{z}_{do} - k))\\
&\eta'^2 = \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \operatorname{cov}(\tilde{x}_{io}, \tilde{x}_{i'o}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \operatorname{cov}(\tilde{z}_{do}, \tilde{z}_{d'o})\\
&\quad + \operatorname{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{io}, \sum_{d=1}^{D} w_d \tilde{z}_{do})\\
&\eta_j^2 = \sum_{r=1}^{s}\sum_{r'=1}^{s} u_r u_{r'} \operatorname{cov}(\tilde{y}_{rj}, \tilde{y}_{r'j}) + \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \operatorname{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j})\\
&\quad - 2\operatorname{cov}(\sum_{d=1}^{D} w_d \tilde{z}_{dj}, \sum_{r=1}^{s} u_r \tilde{y}_{rj} + u_{02})\\
&\eta_j'^2 = \sum_{d=1}^{D}\sum_{d'=1}^{D} w_d w_{d'} \operatorname{cov}(\tilde{z}_{dj}, \tilde{z}_{d'j}) + \sum_{i=1}^{m}\sum_{i'=1}^{m} v_i v_{i'} \operatorname{cov}(\tilde{x}_{ij}, \tilde{x}_{i'j})\\
&\quad - 2\operatorname{cov}(\sum_{i=1}^{m} v_i \tilde{x}_{ij}, \sum_{d=1}^{D} w_d \tilde{z}_{dj} - u_{01})\\
&\quad u_r, w_d, v_i \geq 0 \quad r = 1,\dots,s \quad d = 1,\dots,D \quad i = 1,\dots,m\\
&\quad \eta, \eta', \eta_j, \eta_j', s_{3j}'', s_{4j}'' \geq 0 \ \ j = 1,\dots,n\\
&\quad s_1'', s_2'', \geq 0\\
&\quad u_{01}, u_{02}\ free
\end{aligned}
\tag{24}
$$

After solving this model for $\alpha \in (0, 0.5]$, we define the efficiencies of the system as follows:

$$\tilde{E}_o^{(chen-VRS)s'} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro} + \sum\limits_{d=1}^{D} w_d^* z_{do} + u_{01}^* + u_{02}^*}{\sum\limits_{i=1}^{m} v_i^* x_{io} + \sum\limits_{d=1}^{D} w_d^* z_{do}}$$

$$\tilde{E}_o^{I} = \frac{\sum\limits_{d=1}^{D} w_d^* z_{do} + u_{02}^*}{\sum\limits_{i=1}^{m} v_i^* x_{io}}, \quad \tilde{E}_o^{II} = \frac{\sum\limits_{r=1}^{s} u_r^* y_{ro} + u_{01}^*}{\sum\limits_{d=1}^{D} w_d^* z_{do}}.$$

Wherein $\tilde{E}_o^{(chen-VRS)s'}$, $\tilde{E}_o^{I}$, $\tilde{E}_o^{II}$ indicate the stochastic overall efficiency and the stochastic efficiency of stage 1, 2 respectively. Therefore, there is $\lambda \in (0, 1)$ that $\tilde{E}_o^{(chen-VRS)s'} = \lambda \tilde{E}_o^{I} + (1 - \lambda)\tilde{E}_o^{II}$.

Lemma 3.9. *For $\alpha \in (0, 0.5]$ and each DMU_o, we have: $0 < \tilde{E}_o^{I} \leq 1$, $0 < \tilde{E}_o^{II} \leq 1$.*

Proof. The proof is similar to the proof of lemma 3.3. □

Lemma 3.10. *For $\alpha \in (0, 0.5]$, DMU_o is stochastic overall efficient under the model (22) if and only if the first and the second stages are stochastic efficient, i.e. $\tilde{E}_o^{(chen-VRS)s'} = 1$ if and only if $\tilde{E}_o^{I} = \tilde{E}_o^{II} = 1$.*

Proof. The proof is similar to the proof of lemma 3.10. □

Finally, we note that the results of the efficiency of the system and stages are in range $(0, 1]$ in all of the proposed stochastic models for $\alpha \in (0, 0.5]$. If $\alpha \in (0.5, 1)$, it is probable that efficiencies are negative or greater than 1. It must be noted that in many cases models (12), (22) and (24) have multiple optimal solutions. Thus, in these models, the overall efficiency decomposition will not be distinctive. Hence, we are unable to compare the efficient stages of different DMUs together in each model. Therefore, by following the Kao and Hwang [9] approach, we presume that the efficiency of stage1, is the most important stage from the point of view of the decision maker (DM) and compute the maximum efficiency of stage 1, while the overall efficiency of system is unchanged. Then, we calculate the maximum efficiency of stage 2, while the efficiency of stage 1 and the overall efficiency of system are unchanged.

4 Case Study

In this section, we will illustrate the deterministic equivalent form of the proposed stochastic model (9) for 10 Taiwanese non-life insurance companies with data for the years 2000, 2001 and 2002. (Extracted from Kao and Hwang [10]). Each company has a two-stage structure. Table 1, shows the inputs, intermediate measures and outputs which we utilized to illustrate the proposed models. And also, the expected values, variance and covariance of inputs and outputs and intermediate products of 10 Taiwanese non-life insurance companies over 3 years (2000, 2001, 2002) are reported in the Tables 2, 3, 4.

Table 1. Case-Study Data

Inputs	Intermediate measures	outputs
Operating expenses (X1)	Direct written premiums (Z1)	Underwriting profit (Y1)
Insurance expenses (X2)	Reinsurance premiums (Z2)	Investment profit (Y2)

4.1 Results of model (12)

Table 5 shows the obtained efficiencies of model (12). The results computed by GAMS software and have been summarized in Table 5, by assuming $\alpha = 0.45$. In Table 5, first column renders the digit of each. The stochastic efficiency of stages 1 and 2 including the overall efficiency

Table 2. Expected values of inputs and outputs and intermediate products

DMU	x_1	x_2	z_1	z_2	y_1	y_2
1	609.144	331.84067	3555.58433	434.71933	497.68233	308.54633
2	692.59867	685.96133	4836.93887	932.00267	1604.134	525.65567
3	685.96133	1746.196	17973.607	955.91233	1785.40433	227329567
4	1302.60233	431.97567	4748.43367	691.696	1250.30433	245.712
5	1882.52467	893.47933	8437.897	461.48067	1527.24733	314.55
6	643.68967	623.28767	3913.41267	266.801	832.955	254.061
7	1242.916	303.727	4489.91633	535.016	1126.69433	486.38433
8	1074.35133	310.51767	4990.15433	405.27133	1343.47533	190.75167
9	596.62933	329.95067	2580.06167	223.407	716.62667	84.879
10	687.394	557.275	3790.80967	220.83467	1201.766	128.18767

Table 3. Variance of DMUs

DMU	$V(x_1)$	$V(x_2)$	$V(y_1)$	$V(y_2)$	$V(z_1)$	$V(z_2)$
1	1784.26994	300.79361	671.52486	4936.03845	202064.75133	2099.73017
2	758.33795	1515.54017	51205.23591	41687.23591	125265.48223	2242.89558
3	9351.92953	2555.45268	216152.29449	549621.14553	3209601.78055	18868.09631
4	525.14956	29565.33416	72211.44903	18234.64829	179858.90393	141630.42004
5	21.50215	2141.8848	34081.27578	2457.04398	219670.39097	20606.32184
6	472.97451	11829.12608	127696.23112	5485.46206	189680.52738	2492.09949
7	3573.21137	25569.13235	116044.055	36413.60875	612679.9989	14489.4648
8	259	2028.876	48031.09782	7807.63465	120665.5085	1242.68069
9	954.6386	350.10067	21321.09776	507.10401	51477.83656	4398.47299
10	1242.916	303.727	36299.7866	3550.70277	612679.9989	2339.72772

Table 4. Covariance of DMUs

DMU	$C(x_1, z_1)$	$C(x_2, z_1)$	$C(x_1, z_2)$	$C(x_2, z_2)$	$C(y_1, z_1)$	$C(y_2, z_1)$	$C(y_1, z_2)$	$C(y_2, z_2)$
1	−1142.123	5087.865	984.4877	386.31060	−7343.44307	20612.57991	−641.112633	−18576.79194
2	−39770.723	−1107737.842	867.58313	885.19702	−51455.15378	−47494.62574	7130.72663	107891.8135
3	114936.28	60335.36	−7245.75694	−3392.15646	3209601.78055	−884800.1054	−18314.19532	53085.30383
4	5751.3461	−17373.886	−5588.3328	41217.20638	−50161.00695	−38165.55747	67314.5018	190560.3478
5	1388.28367	12031.667	−26409.12516	−3602.5339	33647.48797	−4924.38931	9771.87462	−81285.01944
6	2961.5767	43358.164	−380.79257	1927.14580	144285.4787	−26263.10335	9080.46352	291571.539
7	−14675.617	−36341.814	953.167494	12713.63501	65148.36857	−6119.64696	−5523.15373	−138352.2662
8	3613.9758	10156.61	−92.181387	−704.81939	21500.52459	14275.4333	3053.65983	17504.65016
9	−3385.12503	5067.92167	1668.0860	74583.117	33195.77192	−3275.72559	1463.72892	−26429.05463
10	71786.02378	32579.81545	4262.26277	1860.79293	36804.03399	−24800.49433	909473.1086	189365.5732

are listed in the columns 2 and 3 and 4 of Table 5, all of DMUs are inefficient. Between the inefficient, with scores of 0.54, 0.11 have the best and the lowest overall efficiency. It is efficient in stage 2. The highest efficiency belongs to stage 1 and in stage 2 with scores of 0.94 and 0.65,

Table 5. Stochastic efficiency obtained from model (12)

DMU	Efficiency of stage 1	Efficiency of stage 2	Overall efficiency
1	0.94	0.4	0.37
2	0.72	0.59	0.43
3	0.54	1	0.54
4	0.8	0.24	0.19
5	0.49	0.23	0.11
6	0.37	0.59	0.22
7	0.46	0.26	0.12
8	0.38	0.65	0.25
9	0.47	0.34	0.16
10	0.55	0.46	0.25

respectively. In stage 1 and 2, showing efficiency scores of 0.37 and 0.23 have the lowest efficiency.

5 Conclusion

In practice, there are many systems with internal structures such as network systems. NDEA is employed to evaluate the performance of the network systems in presence of deterministic data. A special distinction of network systems are their two-stage systems denoting the first stage which consumes the inputs to produce the intermediate measures, then these intermediate measures deploy to generate the outputs of the second stage. In practice, the observations of inputs, intermediate measures, and outputs are imprecise and they can be considered as stochastic data. Hence, SDEA is a useful method for measuring the efficiency of black box systems with stochastic data. Mirbolouki et al. [15] proposed a stochastic model that evaluates the efficiency of a black box system based on a multiplier form of DEA. Therefore, in this paper, by using the non-compensatory property of the multiplication operator and the compensatory property of the additive operator, we extended NDEA and SDEA models and proposed the SNDEA models for computing the stochastic efficiencies of the two-stage systems, in presence of stochastic data, based on multiplicative and additive models. Then, for our proposed stochastic models, we obtained the deterministic equivalent forms and converted these deterministic forms into the quadratic programming problems. Likewise, we showed that the obtained efficiencies of these models are positive for $\alpha \in (0, 0.5]$. The proposed model (12) is illustrated on a set of data for 10 Taiwanese non-life insurance companies in the years 2000, 2001 and 2002, which were studied by Kao and Hwang [9] utilizing the GAMS software. For future study, this work can be extended to non- radial DEA models for measuring the efficiency of a two-stage system in presence of stochastic data and ranking them in cases where weakness of efficiencies for $\alpha \in (0.5, 1)$ are not witnessed.

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