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Original Research Paper

Weighted Hermite-Hadamard and Simpson Type Inequalities for Double Integrals

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Abstract. In this paper, we first obtain two weighted identities for twice partially differentiable mappings. Moreover, utilizing these equalities, we establish the weighted Hermite-Hadamard type inequalities and weighted Simpson type inequalities for co-ordinated convex functions in a rectangle from the plane $\mathbb{R}^2$, respectively. The results given in this paper provide generalizations of some result established in earlier works.

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1. Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard see, e.g., [8], [24, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbb{R}$

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is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied.

The weighted version of the inequalities (1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejer in [13] as follow:

Theorem 1.1. *$f : [a, b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant \frac{f(a)+f(b)}{2} \int_a^b g(x)dx \tag{2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $g(x) = g(a + b - x)$).

On the other hand, the following inequality is well known in the literature as Simpson's inequality.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \operatorname{Spr}\left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([3], [9]-[11], [14], [15], [21], [25], [26], [28]-[31], [33], [34], [35], [37], [40]).

A formal defination for co-ordinated convex function may be stated as follows:

Definition 1.3. *A function $f:\Delta\to\mathbb{R}$ is called co-ordinated convex on Δ, for all $(x,u),(y,v)\in\Delta$ and $t,s\in[0,1]$, if it satifies the following inequality:*

$$f(tx+(1-t)\ y,su+(1-s)\ v) \tag{3}$$

$$\leqslant\ ts\ f(x,u)+t(1-s)f(x,v)+s(1-t)f(y,u)+(1-t)(1-s)f(y,v).$$

The mapping f is a co-ordinated concave on Δ if the inequality (3) holds in reversed direction for all $t,s\in[0,1]$ and $(x,u),(y,v)\in\Delta$.

In [7], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.4. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{4}$$

$$\leqslant\ \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]$$

$$\leqslant\ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

$$\leqslant\ \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx\right.$$

$$\left.+\ \frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right] \tag{5}$$

$$\leqslant\ \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

The above inequalities are sharp. The inequalities in (4) hold in reverse direction if the mapping f is a co-ordinated concave mapping.

Over the years, many papers are dedicated on the generalizations and new versions of the inequalities (4) using the different type convex functions. For the other Hermite-Hadamard type inequalities for co-ordinated convex functions, please refer to ([1], [2], [4] ,[6], [22], [27], [32], [22], [36], [38], [41])

In [23], Özdemir et al. gave the following identity and using the this identity, the authors established some Simpson type inequalities for double integrals:

Lemma 1.5. *$f:\Delta:=[a,b]\times[c,d]\rightarrow\mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then we have the following equality*

$$\begin{aligned}
&\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9}\\
&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}\\
&-\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx\\
&-\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy\\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
=\;&(b-a)(d-c)\int_0^1\int_0^1 q(t,s)\frac{\partial^2 f}{\partial t\partial s}(at+(1-t)b,cs+(1-s)d)\,dsdt
\end{aligned}$$

which the mapping $q:[0,1]\times[0,1]\rightarrow\mathbb{R}$ is defined by

$$q(t,s)=\begin{cases}
\left(t-\frac{1}{6}\right)\left(s-\frac{1}{6}\right) & 0\leqslant t\leqslant\frac{1}{2},\quad 0\leqslant s\leqslant\frac{1}{2}\\[2mm]
\left(t-\frac{1}{6}\right)\left(s-\frac{5}{6}\right) & 0\leqslant t\leqslant\frac{1}{2},\quad \frac{1}{2}\leqslant s\leqslant 1\\[2mm]
\left(t-\frac{5}{6}\right)\left(s-\frac{1}{6}\right) & \frac{1}{2}\leqslant t\leqslant 1,\quad 0\leqslant s\leqslant\frac{1}{2}\\[2mm]
\left(t-\frac{5}{6}\right)\left(s-\frac{5}{6}\right) & \frac{1}{2}\leqslant t\leqslant 1,\quad \frac{1}{2}\leqslant s\leqslant 1.
\end{cases}$$

Budak and Sarıkaya proved the following Hermite-Hadamard-Fejér inequalities for double integrals in [5]:

Theorem 1.6. *Let $p : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a positive, integrable and symmetric about $\frac{a+b}{2}$ and $\frac{c+d}{2}$. Let $f : \Delta \to \mathbb{R}$ be a co-ordinated convex on Δ, then we have the following Hermite-Hadamard-Fejer type inequality*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \int_a^b \int_c^d p(x,y)dydx \\
\leqslant\ & \frac{1}{2} \int_a^b \int_c^d \left[f\left(x, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, y\right)\right] p(x,y)dydx \\
\leqslant\ & \int_a^b \int_c^d f(x,y)p(x,y)dydx \\
\leqslant\ & \frac{1}{4} \int_a^b \int_c^d \left[f(x,c) + f(x,d) + f(a,y) + f(b,y)\right] p(x,y)dydx \\
\leqslant\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \int_a^b \int_c^d p(x,y)dydx.
\end{aligned}$$

Moreover, Farid et al. established a weighted version of the inequalities (4) in [12]. Please see ([16]-[20], [39]) for other papers focused on Hermite-Hadamard-Fejér inequalities for co-ordinated convex functions.

The aim of this paper is to establish some weighed generalizations of Hermite-Hadamard and Simpson type integral inequalities. The results presented in this paper provide extensions of those given in [23] and [27].

2. Weighted Hermite-Hadamard Type Inequalities

In this section, we first prove a weighted identity for twice partially differentiable mapping.

Then, using this identity, we established a weighted Hermite-Hadamard type inequality for co-ordinated convex mapping.

Lemma 2.1. *Let $p : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a positive and integrable function on Δ and let $f : \Delta \to \mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then we have the following equality*

$$\begin{aligned} & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx \qquad (6) \\ & -\frac{1}{4}\int_a^b\int_c^d [f(x,c)+f(x,d)+f(a,y)+f(b,y)]\,p(x,y)dydx \\ & +\int_a^b\int_c^d f(x,y)p(x,y)dydx \\ = \; & \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left[\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right]\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\,dsdt \end{aligned}$$

where $U_1(t) = (1-t)a + tb$, $U_2(t) = ta + (1-t)b$, $V_1(s) = (1-s)c + sd$ and $V_2(s) = sc + (1-s)d$.

Proof. Integrating the by parts we have,

$$\begin{aligned} & \int_0^1\int_0^1\left[\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right]\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\,dsdt \qquad (7) \\ = \; & \int_0^1\left\{\frac{1}{a-b}\left[\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right]\frac{\partial f}{\partial s}(U_2(t),V_2(s))\Bigg|_0^1\right. \\ & \left. -\int_0^1\left[\int_{V_1(s)}^{V_2(s)} p(U_2(t),v)dv + \int_{V_1(s)}^{V_2(s)} p(U_1(t),v)dv\right]\frac{\partial f}{\partial s}(U_2(t),V_2(s))\,dt\right\} ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left\{ \frac{1}{b-a} \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] \frac{\partial f}{\partial s}(a, V_2(s)) \right. \\
&\quad + \frac{1}{b-a} \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] \frac{\partial f}{\partial s}(b, V_2(s)) \\
&\quad - \int_0^1 \left[\int_{V_1(s)}^{V_2(s)} p(U_2(t),v)dv \right] \frac{\partial f}{\partial s}(U_2(t), V_2(s))\, dt \\
&\quad \left. - \int_0^1 \left[\int_{V_1(s)}^{V_2(s)} p(U_1(t),v)dv \right] \frac{\partial f}{\partial s}(U_2(t), V_2(s))\, dt \right\} ds \\
&= \frac{1}{b-a} \left\{ \int_0^1 \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] \frac{\partial f}{\partial s}(a, V_2(s))\, ds \right. \\
&\quad \left. + \int_0^1 \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] \frac{\partial f}{\partial s}(b, V_2(s))\, ds \right\} \\
&\quad - \int_0^1 \int_0^1 \left[\int_{V_1(s)}^{V_2(s)} p(U_2(t),v)dv \right] \frac{\partial f}{\partial s}(U_2(t), V_2(s))\, dtds \\
&\quad - \int_0^1 \int_0^1 \left[\int_{V_1(s)}^{V_2(s)} p(U_1(t),v)dv \right] \frac{\partial f}{\partial s}(U_2(t), V_2(s))\, dtds \\
&= \frac{1}{b-a} \{I_1 + I_2\} - I_3 - I_4.
\end{aligned}
$$

Using again the integration by parts, we obtain

$$\begin{aligned}
I_1 &= \int_0^1 \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] \frac{\partial f}{\partial s}(a, V_2(s))\, ds \qquad (8) \\
&= \frac{1}{c-d} \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] f(a, V_2(s)) \Bigg|_0^1 \\
&\quad - \int_0^1 \left[\int_a^b p(u, V_2(s))du + \int_a^b p(u, V_1(s))du \right] f(a, V_2(s))\, ds \\
&= \frac{1}{d-c} \left(\int_a^b \int_c^d p(u,v)dudv \right) [f(a,c) + f(a,d)] \\
&\quad - \int_0^1 \int_a^b p(u, V_2(s)) f(a, V_2(s))\, duds \\
&\quad - \int_0^1 \int_a^b p(u, V_1(s)) f(a, V_2(s))\, duds \\
&= \frac{1}{d-c} \left(\int_a^b \int_c^d p(u,v)dudv \right) [f(a,c) + f(a,d)] \\
&\quad - \frac{2}{d-c} \int_a^b \int_c^d p(u,v) f(a,v)dudv,
\end{aligned}$$

and similarly,

$$I_2 = \int_0^1 \left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv \right] \frac{\partial f}{\partial s}(b, V_2(s))\, ds \qquad (9)$$

$$
\begin{aligned}
&= \frac{1}{c-d}\left[\int_a^b \int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right] f\left(b, V_2(s)\right)\Bigg|_0^1 \\
&\quad -\int_0^1 \left[\int_a^b p(u,V_2(s))dudv + \int_a^b p(u,V_1(s))dudv\right] f\left(b, V_2(s)\right) ds \\
&= \frac{1}{d-c}\left(\int_a^b \int_c^d p(u,v)dudv\right)\left[f\left(b,c\right)+f(b,d)\right] \\
&\quad -\frac{2}{d-c}\int_a^b \int_c^d p(u,v)f(b,v)dudv.
\end{aligned}
$$

Onthe other hand, we get

$$
\begin{aligned}
I_3 &= \int_0^1 \int_0^1 \left[\int_{V_1(s)}^{V_2(s)} p(U_2(t),v)dv\right] \frac{\partial f}{\partial s}\left(U_2(t), V_2(s)\right) dtds \qquad (10)\\
&= \int_0^1 \Bigg\{ \frac{1}{c-d}\left[\int_{V_1(s)}^{V_2(s)} p(U_2(t),v)dv\right] f\left(U_2(t), V_2(s)\right)\Bigg|_0^1 \\
&\quad -\int_0^1 \left[p(U_2(t),V_2(s)) + p(U_2(t),V_1(s))\right] f\left(U_2(t), V_2(s)\right) ds \Bigg\} dt \\
&= \int_0^1 \Bigg\{ \frac{1}{d-c}\left(\int_c^d p(U_2(t),v)dv\right)\left[f\left(U_2(t),c\right)+f\left(U_2(t),d\right)\right] \\
&\quad -\int_0^1 p(U_2(t),V_2(s)) f\left(U_2(t), V_2(s)\right) ds \\
&\quad -\int_0^1 p(U_2(t),V_1(s)) f\left(U_2(t), V_2(s)\right) ds \Bigg\} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(u,v)\left[f(u,c)+f(u,d)\right] dvdu \\
&\quad - \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d p(u,v) f(u,v)\, dvdu
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
I_4 &= \int_0^1 \int_0^1 \left[\int_{V_1(s)}^{V_2(s)} p(U_1(t),v)dv\right] \frac{\partial f}{\partial s}\left(U_2(t),V_2(s)\right) dtds \qquad (11) \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(u,v)\left[f(u,c)+f(u,d)\right] dvdu \\
&\quad - \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d p(u,v) f(u,v)\, dvdu.
\end{aligned}
$$

If we substitute the equalities (8)-(11) in (7), we obtain

$$
\begin{aligned}
&\int_0^1 \int_0^1 \left[\int_{U_1(t)}^{U_2(t)} \int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right] \frac{\partial^2 f}{\partial t \partial s}\left(U_2(t),V_2(s)\right) dsdt \qquad (12) \\
&= \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d p(u,v)dudv\right) \left[f(a,c)+f(a,d)\right] \\
&\quad - \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d p(u,v) f(a,v)dudv \\
&\quad + \frac{1}{(b-a)(d-c)} \left(\int_a^b \int_c^d p(u,v)dudv\right) \left[f(b,c)+f(b,d)\right] \\
&\quad - \frac{2}{(b-a)(d-c)} \int_a^b \int_c^d p(u,v) f(b,v)dudv
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(u,v)\left[f(u,c)+f(u,d)\right]dvdu \\
&+\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d p(u,v)f(u,v)\,dvdu \\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p(u,v)\left[f(u,c)+f(u,d)\right]dvdu \\
&+\frac{2}{(b-a)(d-c)}\int_a^b\int_c^d p(u,v)f(u,v)\,dvdu \\
=\;& \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(b-a)(d-c)}\left(\int_a^b\int_c^d p(u,v)dudv\right) \\
-\;& \frac{2}{(b-a)(d-c)}\int_a^b\int_c^d p(u,v)\left[ff(u,c)+f(u,d)+(a,v)+f(b,v)\right]dvdu \\
+\;& \frac{4}{(b-a)(d-c)}\int_a^b\int_c^d p(u,v)f(u,v)\,dvdu.
\end{aligned}
$$

If we multiply the equality (12) by $\frac{(b-a)(d-c)}{4}$, then we obtain the desired result (6). □

Remark 2.2. *If we choose $p(x,y)=1$ in Lemma 2.1, then the Lemma 2.1 reduces to Lemma 1 in [27].*

Theorem 2.3. *Let $p:\Delta:=[a,b]\times[c,d]\to\mathbb{R}$ be a positive and integrable function on Δ and let $f:\Delta\to\mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a co-ordinated convex function on Δ, then we have the following inequality*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx\right. \tag{13}$$

$$-\frac{1}{4}\int_a^b\int_c^d [f(x,c)+f(x,d)+f(a,y)+f(b,y)]\,p(x,y)dydx$$

$$\left.+\int_a^b\int_c^d f(x,y)p(x,y)dydx\right|$$

$$= \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1 (H(t,s))^p\, dsdt\right)^{\frac{1}{p}}$$

$$\times \left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $H(t,s)$ *defined by*

$$H(t,s)=\left|\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right|$$

with $U_1(t)=(1-t)a+tb$, $U_2(t)=ta+(1-t)b$, $V_1(s)=(1-s)c+sd$ *and* $V_2(s)=sc+(1-s)d$.

Proof. Taking the modulus in Lemma 2.1 and by using the well-known Hölder's inequality, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_a^b\int_c^d p(x,y)dydx \right. \tag{14}$$

$$-\frac{1}{4}\int_a^b\int_c^d \left[f(x,c)+f(x,d)+f(a,y)+f(b,y)\right]p(x,y)dydx$$

$$\left.+\int_a^b\int_c^d f(x,y)p(x,y)dydx\right|$$

$$= \frac{(b-a)(d-c)}{4}\left|\int_0^1\int_0^1\left[\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right]\frac{\partial^2 f}{\partial t\partial s}\left(U_2(t),V_2(s)\right)dsdt\right|$$

$$\leqslant \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1\left|\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right|\left|\frac{\partial^2 f}{\partial t\partial s}\left(U_2(t),V_2(s)\right)\right|dsdt$$

$$\leqslant \frac{(b-a)(d-c)}{4}\left(\int_0^1\int_0^1\left|\int_{U_1(t)}^{U_2(t)}\int_{V_1(s)}^{V_2(s)} p(u,v)dudv\right|^p dsdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(U_2(t),V_2(s)\right)\right|^q dsdt\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a co-ordinated convex function on Δ, we obtain

$$\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}\left(U_2(t),V_2(s)\right)\right|^q dsdt$$

$$
\begin{aligned}
&= \int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sa+(1-s)d)\right|^q dsdt \qquad (15)\\
&\leqslant \int_0^1\int_0^1 \left[ts\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + t(1-s)\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.\\
&\quad \left.+(1-t)s\left|\tfrac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-t)(1-s)\left|\tfrac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right] dsdt\\
&= \frac{\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}.
\end{aligned}
$$

If we substitute the inequality (15) in (14), the we establish desired result. □

Remark 2.4. *If we choose* $p(x,y) = 1$, *then we obtain the following inequality*

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\
&-\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right.\\
&\left.+\frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|\\
\leqslant\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\\
&\times\left[\frac{\left|\tfrac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + \left|\tfrac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}
$$

which was proved by Sarikaya et al. in [27].

3. Weighted Simpson Type Inequalities

In this section, we first define the following mapping

$$\begin{aligned}
&\Theta(a,b;f,p)\\
=\;& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{1}{6}}^{\frac{5}{6}}\int_{\frac{1}{6}}^{\frac{5}{6}}p\left(u,v\right)dvdu\\
&+f\left(a,\frac{c+d}{2}\right)\int_{\frac{5}{6}}^{1}\int_{\frac{1}{6}}^{\frac{5}{6}}p\left(u,v\right)dvdu\\
&+f\left(b,\frac{c+d}{2}\right)\int_{0}^{\frac{1}{6}}\int_{\frac{1}{6}}^{\frac{5}{6}}p\left(u,v\right)dvdu\\
&+f\left(\frac{a+b}{2},c\right)\int_{\frac{1}{6}}^{\frac{5}{6}}\int_{\frac{5}{6}}^{1}p\left(u,v\right)dvdu\\
&+f\left(\frac{a+b}{2},d\right)\int_{\frac{1}{6}}^{\frac{5}{6}}\int_{0}^{\frac{1}{6}}p\left(u,v\right)dvdu\\
&+f\left(a,c\right)\int_{\frac{5}{6}}^{1}\int_{\frac{5}{6}}^{1}p\left(u,v\right)dvdu+f\left(b,c\right)\int_{0}^{\frac{1}{6}}\int_{\frac{5}{6}}^{1}p\left(u,v\right)dvdu\\
&+f\left(a,d\right)\int_{\frac{5}{6}}^{1}\int_{0}^{\frac{1}{6}}p\left(u,v\right)dvdu+f\left(b,d\right)\int_{0}^{\frac{1}{6}}\int_{0}^{\frac{1}{6}}p\left(u,v\right)dvdu\\
&-\frac{1}{b-a}\int_{a}^{b}\left(\int_{\frac{5}{6}}^{1}p\left(\frac{b-x}{b-a},v\right)dv\right)f\left(x,c\right)dx\\
&-\frac{1}{b-a}\int_{a}^{b}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}p\left(\frac{b-x}{b-a},v\right)dv\right)f\left(x,\frac{c+d}{2}\right)dx\\
&-\frac{1}{b-a}\int_{a}^{b}\left(\int_{0}^{\frac{1}{6}}p\left(\frac{b-x}{b-a},v\right)dv\right)f\left(x,d\right)dx\\
&-\frac{1}{d-c}\int_{c}^{d}\left(\int_{\frac{5}{6}}^{1}p\left(u,\frac{d-y}{d-c}\right)du\right)f\left(a,y\right)dy
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{d-c}\int_{c}^{d}\left(\int_{\frac{1}{2}}^{\frac{5}{6}}p\left(u,\frac{d-y}{d-c}\right)du\right)f\left(\frac{a+b}{2},y\right)dy\\
&-\frac{1}{d-c}\int_{c}^{d}\left(\int_{0}^{\frac{1}{6}}p\left(u,\frac{d-y}{d-c}\right)du\right)f\left(b,y\right)dy\\
&+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}p\left(\frac{b-x}{b-a},\frac{d-y}{d-c}\right)f\left(x,y\right)dydx.
\end{aligned}$$

Now, we give the following equality.

Lemma 3.1. *Let the mappings p, U_2 and V_2 be as in Lemma 2.1 and let $f : \Delta := [a,b] \times [c,d] \to \mathbb{R}$ be a twice partially differentiable mapping on Δ°. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then we have the following equality*

$$\Theta(a,b;f,p) = (b-a)(d-c)\int_0^1 \int_0^1 w(t,s) \frac{\partial^2 f}{\partial t \partial s}(U_2(t),V_2(s))\,dsdt$$

where the mapping $w : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$w(t,s) = \begin{cases} \int_{\frac{1}{6}}^{t} \int_{\frac{1}{6}}^{s} p(u,v)\,dudv & 0 \leqslant t \leqslant \frac{1}{2}, \quad 0 \leqslant s \leqslant \frac{1}{2} \\ \int_{\frac{1}{6}}^{t} \int_{\frac{5}{6}}^{s} p(u,v)\,dudv & 0 \leqslant t \leqslant \frac{1}{2}, \quad \frac{1}{2} \leqslant s \leqslant 1 \\ \int_{\frac{5}{6}}^{t} \int_{\frac{1}{6}}^{s} p(u,v)\,dudv & \frac{1}{2} \leqslant t \leqslant 1, \quad 0 \leqslant s \leqslant \frac{1}{2} \\ \int_{\frac{5}{6}}^{t} \int_{\frac{5}{6}}^{s} p(u,v)\,dudv & \frac{1}{2} \leqslant t \leqslant 1, \quad \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

Proof. From the definition of the mapping w, we have

$$\begin{aligned} & (b-a)(d-c)\int_0^1 \int_0^1 w(t,s) \frac{\partial^2 f}{\partial t \partial s}(U_2(t),V_2(s))\,dsdt \\ = \; & (b-a)(d-c)\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{6}}^{t} \int_{\frac{1}{6}}^{s} p(u,v)\,dvdu \right) \frac{\partial^2 f}{\partial t \partial s}(U_2(t),V_2(s))\,dtds \\ & + (b-a)(d-c)\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \left(\int_{\frac{1}{6}}^{t} \int_{\frac{5}{6}}^{s} p(u,v)\,dudv \right) \frac{\partial^2 f}{\partial t \partial s}(U_2(t),V_2(s))\,dtds \\ & + (b-a)(d-c)\int_{\frac{1}{2}}^{1} \int_0^{\frac{1}{2}} \left(\int_{\frac{5}{6}}^{t} \int_{\frac{1}{6}}^{s} p(u,v)\,dudv \right) \frac{\partial^2 f}{\partial t \partial s}(U_2(t),V_2(s))\,dtds \\ & + (b-a)(d-c)\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} \left(\int_{\frac{5}{6}}^{t} \int_{\frac{5}{6}}^{s} p(u,v)\,dudv \right) \frac{\partial^2 f}{\partial t \partial s}(U_2(t),V_2(s))\,dtds \\ = \; & (b-a)(d-c)\left[J_1 + J_2 + J_3 + J_4 \right]. \end{aligned}$$

Integration by parts, we obtain

$$
\begin{aligned}
J_1 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{6}}^{t} \int_{\frac{1}{6}}^{s} p(u,v)\, dv du \right) \frac{\partial^2 f}{\partial t \partial s} \left(U_2(t), V_2(s)\right) dt ds \\
&= \frac{1}{c-d} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{6}}^{t} \int_{\frac{1}{6}}^{\frac{1}{2}} p(u,v)\, dv du \right) \frac{\partial f}{\partial t} \left(U_2(t), \frac{c+d}{2}\right) dt \\
&\quad - \frac{1}{c-d} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{6}}^{t} \int_{\frac{1}{6}}^{0} p(u,v)\, dv du \right) \frac{\partial f}{\partial t} \left(U_2(t), d\right) dt \\
&\quad - \frac{1}{c-d} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{6}}^{t} p(s,v)\, dv \right) \frac{\partial f}{\partial t} \left(U_2(t), V_2(s)\right) dt ds \Bigg] \\
&= \frac{1}{c-d} \left[\frac{1}{a-b} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \int_{\frac{1}{6}}^{\frac{1}{2}} p(u,v)\, dv du \right) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\
&\quad - \frac{1}{a-b} \left(\int_{\frac{1}{6}}^{0} \int_{\frac{1}{6}}^{\frac{1}{2}} p(u,v)\, dv du \right) f\left(b, \frac{c+d}{2}\right) \\
&\quad - \frac{1}{a-b} \int_0^{\frac{1}{2}} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} p(t,v)\, dv \right) f\left(U_2(t), \frac{c+d}{2}\right) dt \Bigg] \\
&\quad - \frac{1}{c-d} \left[\frac{1}{a-b} \left(\int_{\frac{1}{6}}^{\frac{1}{2}} \int_{\frac{1}{6}}^{0} p(u,v)\, dv du \right) f\left(\frac{a+b}{2}, d\right) \right. \\
&\quad - \frac{1}{a-b} \left(\int_{\frac{1}{6}}^{0} \int_{\frac{1}{6}}^{0} p(u,v)\, dv du \right) f(b,d)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{a-b}\int_0^{\frac{1}{2}}\left(\int_{\frac{1}{6}}^{0} p(t,v)\,dv\right) f(U_2(t),d)\,dt\Bigg] \\
& -\Bigg[\frac{1}{c-d}\int_0^{\frac{1}{2}}\frac{1}{a-b}\left(\int_{\frac{1}{6}}^{\frac{1}{2}} p(s,v)\,dv\right) f\left(\frac{a+b}{2},V_2(s)\right) \\
& -\frac{1}{a-b}\left(\int_{\frac{1}{6}}^{0}\int_{\frac{1}{6}}^{0} p(u,v)\,dvdu\right) f(b,d) \\
& -\frac{1}{a-b}\int_0^{\frac{1}{2}} p(s,t)\, f(U_2(t),V_2(s))\,dt\Bigg] ds \\
= \; & \frac{1}{(b-a)(d-c)}\Bigg[f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{1}{6}}^{\frac{1}{2}}\int_{\frac{1}{6}}^{1} p(u,v)\,dvdu \\
& +f\left(b,\frac{c+d}{2}\right)\int_0^{\frac{1}{6}}\int_{\frac{5}{6}}^{1} p(u,v)\,dvdu \\
& -\int_0^{\frac{1}{2}}\left(\int_{\frac{5}{6}}^{\frac{1}{2}} p(t,v)\,dv\right) f\left(U_2(t),\frac{c+d}{2}\right) dt \\
& +f\left(\frac{a+b}{2},d\right)\int_{\frac{1}{6}}^{\frac{1}{2}}\int_0^{\frac{1}{6}} p(u,v)\,dudv \\
& +f(b,d)\int_0^{\frac{1}{6}}\int_{\frac{1}{2}}^{\frac{1}{6}} p(u,v)\,dudv \\
& -\int_0^{\frac{1}{2}}\left(\int_0^{\frac{1}{6}} p(t,v)\,dv\right) f(U_2(t),d)\,dt \\
& -\int_0^{\frac{1}{2}}\left(\int_{\frac{1}{6}}^{\frac{1}{2}} p(s,v)\,dv\right) f\left(\frac{a+b}{2},V_2(s)\right) ds \\
& -\int_0^{\frac{1}{2}}\left(\int_0^{\frac{1}{6}} p(s,v)\,dv\right) f(b,V_2(s))\,ds \\
& +\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} p(s,t)\, f(U_2(t),V_2(s))\,dtds\Bigg].
\end{aligned}
$$

Similarly, we get

$$\begin{aligned}
J_2 &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{6}}^{t}\int_{\frac{5}{6}}^{s} p(u,v)\,dudv\right)\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\,dtds \\
&= \frac{1}{(b-a)(d-c)}\left[f\left(\frac{a+b}{2},c\right)\int_{\frac{1}{6}}^{\frac{1}{2}}\int_{\frac{5}{6}}^{1} p(u,v)\,dudv\right. \\
&\quad +f(b,c)\int_0^{\frac{1}{6}}\int_{\frac{5}{6}}^{1} p(u,v)\,dudv \\
&\quad -\int_0^{\frac{1}{2}}\left(\int_{\frac{5}{6}}^{1} p(t,v)\,dv\right) f(at+(1-t)b,c)\,dt \\
&\quad +f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{1}{6}}^{\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{5}{6}} p(u,v)\,dudv \\
&\quad +f\left(b,\frac{c+d}{2}\right)\int_0^{\frac{1}{6}}\int_{\frac{1}{2}}^{\frac{5}{6}} p(u,v)\,dudv \\
&\quad -\int_0^{\frac{1}{2}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p(t,v)\,dv\right) f\left(U_2(t),\frac{c+d}{2}\right) dt \\
&\quad -\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{6}}^{\frac{1}{2}} p(u,s)\,ds\right) f\left(\frac{a+b}{2},V_2(s)\right) ds \\
&\quad -\int_{\frac{1}{2}}^{1}\left(\int_0^{\frac{1}{6}} p(u,s)\,ds\right) f(b,V_2(s))\,ds \\
&\quad \left.+\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} p(t,s)\,f(U_2(t),V_2(s))\,dtds\right],
\end{aligned}$$

$$\begin{aligned}
J_3 &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}\left(\int_{\frac{5}{6}}^{t}\int_{\frac{1}{6}}^{s} p(u,v)\,dudv\right)\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\,dtds \\
&= \frac{1}{(b-a)(d-c)} f\left[\left(a,\frac{c+d}{2}\right)\int_{\frac{5}{6}}^{1}\int_{\frac{1}{6}}^{\frac{1}{2}} p(u,v)\,dvdu\right. \\
&\quad +f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{1}{2}}^{\frac{5}{6}}\int_{\frac{1}{6}}^{\frac{1}{2}} p(u,v)\,dvdu
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{6}}^{\frac{1}{2}} p(t,v)\,dv\right) f\left(U_2(t),\frac{c+d}{2}\right)dt \\
&+f(a,d)\int_{\frac{5}{6}}^{1}\int_{0}^{\frac{1}{6}} p(u,v)\,dvdu \\
&+f\left(\frac{a+b}{2},d\right)\int_{\frac{1}{2}}^{\frac{5}{6}}\int_{0}^{\frac{1}{6}} p(u,v)\,dvdu \\
&-\int_{\frac{1}{2}}^{1}\left(\int_{0}^{\frac{1}{6}} p(t,v)\,dv\right) f(U_2(t),d)\,dt \\
&-\int_{0}^{\frac{1}{2}}\left(\int_{\frac{5}{6}}^{1} p(u,s)\,du\right) f(a,V_2(s))\,ds \\
&-\int_{0}^{\frac{1}{2}}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p(u,s)\,du\right) f\left(\frac{a+b}{2},V_2(s)\right)ds \\
&+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} p(t,s)\,f(U_2(t),V_2(s))\,dtds\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left(\int_{\frac{5}{6}}^{t}\int_{\frac{5}{6}}^{s} p(u,v)\,dudv\right)\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\,dtds \\
&= \frac{1}{(b-a)(d-c)}\Bigg[f(a,c)\int_{\frac{5}{6}}^{\frac{1}{2}}\int_{\frac{5}{6}}^{1} p(u,v)\,dvdu \\
&+f\left(\frac{a+b}{2},c\right)\int_{\frac{1}{2}}^{\frac{5}{6}}\int_{\frac{5}{6}}^{1} p(u,v)\,dvdu \\
&-\int_{\frac{1}{2}}^{1}\left(\int_{\frac{5}{6}}^{1} p(t,v)\,dv\right) f(U_2(t),c)\,dt \\
&+f\left(a,\frac{c+d}{2}\right)\int_{\frac{5}{6}}^{1}\int_{\frac{1}{2}}^{\frac{5}{6}} p(u,v)\,dvdu \\
&+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{1}{2}}^{\frac{5}{6}}\int_{\frac{1}{2}}^{\frac{5}{6}} p(u,v)\,dvdu
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p(t,v)\,dv\right) f\left(U_2(t),\frac{c+d}{2}\right) dt \\
&-\int_{\frac{1}{2}}^{1}\left(\int_{\frac{5}{6}}^{1} p(u,s)\,du\right) f\left(a,V_2(s)\right) ds \\
&-\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p(u,s)\,du\right) f\left(\frac{a+b}{2},V_2(s)\right) ds \\
&\left.+\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} p(t,s)\, f\left(U_2(t),V_2(s)\right) dtds\right].
\end{aligned}
$$

Using the change of variable, we obtain

$$
\begin{aligned}
&(b-a)(d-c)\left[J_1+J_2+J_3+J_4\right] \\
=\;& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\int_{\frac{1}{6}}^{\frac{5}{6}}\int_{\frac{1}{6}}^{\frac{5}{6}} p(u,v)\,dvdu \\
&+f\left(a,\frac{c+d}{2}\right)\int_{\frac{5}{6}}^{1}\int_{\frac{1}{6}}^{\frac{5}{6}} p(u,v)\,dvdu \\
&-\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p(t,v)\,dv\right) f\left(U_2(t),\frac{c+d}{2}\right) dt \\
&+f\left(b,\frac{c+d}{2}\right)\int_{0}^{\frac{1}{6}}\int_{\frac{1}{6}}^{\frac{5}{6}} p(u,v)\,dvdu \\
&+f\left(\frac{a+b}{2},c\right)\int_{\frac{1}{6}}^{\frac{5}{6}}\int_{\frac{5}{6}}^{1} p(u,v)\,dvdu \\
&+f\left(\frac{a+b}{2},d\right)\int_{\frac{1}{6}}^{\frac{5}{6}}\int_{0}^{\frac{1}{6}} p(u,v)\,dvdu \\
&+f(a,c)\int_{\frac{5}{6}}^{1}\int_{\frac{5}{6}}^{1} p(u,v)\,dvdu+f(b,c)\int_{0}^{\frac{1}{6}}\int_{\frac{5}{6}}^{1} p(u,v)\,dvdu \\
&+f(a,d)\int_{\frac{5}{6}}^{1}\int_{0}^{\frac{1}{6}} p(u,v)\,dvdu+f(b,d)\int_{0}^{\frac{1}{6}}\int_{0}^{\frac{1}{6}} p(u,v)\,dvdu
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{b-a}\int_a^b\left(\int_{\frac{5}{6}}^1 p\left(\frac{b-x}{b-a},v\right)dv\right)f(x,c)\,dx \\
&-\frac{1}{b-a}\int_a^b\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p\left(\frac{b-x}{b-a},v\right)dv\right)f\left(x,\frac{c+d}{2}\right)dx \\
&-\frac{1}{b-a}\int_a^b\left(\int_0^{\frac{1}{6}} p\left(\frac{b-x}{b-a},v\right)dv\right)f(x,d)\,dx \\
&-\frac{1}{d-c}\int_c^d\left(\int_{\frac{5}{6}}^1 p\left(u,\frac{d-y}{d-c}\right)du\right)f(a,y)\,dy \\
&-\frac{1}{d-c}\int_c^d\left(\int_{\frac{1}{2}}^{\frac{5}{6}} p\left(u,\frac{d-y}{d-c}\right)du\right)f\left(\frac{a+b}{2},y\right)dy \\
&-\frac{1}{d-c}\int_c^d\left(\int_0^{\frac{1}{6}} p\left(u,\frac{d-y}{d-c}\right)du\right)f(b,y)\,dy \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d p\left(\frac{b-x}{b-a},\frac{d-y}{d-c}\right)f(x,y)\,dydx
\end{aligned}
$$

which completes the proof. □

Remark 3.2. *If we choose $p(x,y)=1$ in Lemma 3.1, then Lemma 3.1 reduses to Lemma 1.5 which is proved by [23].*

Theorem 3.3. *Let the mappings p, U_2 and V_2 be as in Lemma 2.1. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a co-ordinated convex function on Δ, then we have the following inequality*

$$
\begin{aligned}
&|\Theta(a,b;f,p)| \\
\leqslant\ &(b-a)(d-c)\left(\int_0^1\int_0^1 |w(t,s)|^p\,dsdt\right)^{\frac{1}{p}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking the modulus in Lemma 3.1 and using the Hölders inequality we have

$$\begin{aligned}
&|\Theta(a,b;f,p)| \\
\leqslant\ & (b-a)(d-c)\int_0^1\int_0^1 |w(t,s)|\left|\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\right| dsdt \\
\leqslant\ & (b-a)(d-c)\left(\int_0^1\int_0^1 |w(t,s)|^p\, dsdt\right)^{\frac{1}{p}} \\
& \times\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial s}(U_2(t),V_2(s))\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\dfrac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a co-ordinated convex function on Δ, using the inequality (15), we obtain

$$\begin{aligned}
&|\Theta(a,b;f,p)| \\
\leqslant\ & (b-a)(d-c)\left(\int_0^1\int_0^1 |w(t,s)|^p\, dsdt\right)^{\frac{1}{p}} \\
& \times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof.

Corollary 3.4. *If we choose $p(x,y)=1$ in Theorem 3.3, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{9}\right. \\
&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36} \\
&-\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx \\
&-\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right| \\
\leqslant\ & \frac{(b-a)(d-c)}{36}\left[2^{p+1}+1\right]^{\frac{2}{p}}\left(\frac{1}{9(p+1)^2}\right)^{\frac{1}{p}} \\
& \times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3.5. *Let the mappings p, U_2 and V_2 be as in Lemma 2.1. If $\frac{\partial^2 f}{\partial t \partial s}$ is bounded on Δ, i.e.*

$$\left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty = \mathrm{Spr}_{(x,y)\in\Delta} \left| \frac{\partial^2 f}{\partial t \partial s}(x,y) \right| < \infty,$$

then we have the following inequality

$$|\Theta(a,b;f,p)| \leqslant (b-a)(d-c) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \int_0^1 \int_0^1 |w(t,s)|\, dsdt.$$

Proof. From Lemma 3.1, we have

$$|\Theta(a,b;f,p)| \leqslant (b-a)(d-c) \int_0^1 \int_0^1 |w(t,s)| \left| \frac{\partial^2 f}{\partial t \partial s}(U_2(t), V_2(s)) \right| dsdt.$$

Since $\frac{\partial^2 f}{\partial t \partial s}$ is bounded on Δ, we obtain

$$|\Theta(a,b;f,p)| \leqslant (b-a)(d-c) \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \int_0^1 \int_0^1 |w(t,s)|\, dsdt$$

which completes the proof. □

Remark 3.6. *If we choose $p(x,y) = 1$ in Theorem 3.5, then we have the following inequality*

$$\begin{aligned} &\left| \frac{f\left(a,\frac{c+d}{2}\right) + f\left(b,\frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},c\right) + f\left(\frac{a+b}{2},d\right)}{9} \right. \\ &+ \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36} \\ &- \frac{1}{6(b-a)} \int_a^b \left[f(x,c) + 4f\left(x,\frac{c+d}{2}\right) + f(x,d) \right] dx \\ &- \frac{1}{6(d-c)} \int_c^d \left[f(a,y) + 4f\left(\frac{a+b}{2},y\right) + f(b,y) \right] dy \\ &\left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \right| \\ \leqslant\ & \frac{25(b-a)(d-c)}{1296} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \end{aligned}$$

which was proved by Özdemir et al. in [23].

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