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Original Research Paper

Inclusion and Argument Properties for a Certain Class of Analytic Functions Based on Fractional Derivatives

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Abstract. In this work, we introduce and investigate an interesting operator Q_{λ}^{ν} based on fractional derivative which is introduced by Owa and Srivastava in [10]. We consider a new technique to prove our results and then, we introduce two subclasses of analytic functions in the open unit disk $\mathcal{U}$ concerning with this operator. Some results such as inclusion relations, subordination properties, integral preserving properties and argument estimate are investigated.

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1 Introduction

The study of fractional operators plays a vital and essential role in mathematical analysis. Recently, there is a growing interest to define generalized differential operators and development their basic properties in a loosely defined area of holomorphic analytic functions in open unit disk. Many authors generalized fractional integral and generalized fractional differential operators on well known classes of analytic and univalent functions to discover and modify new classes and to investigate multi-various interesting properties of new classes, for example (see [1, 2, 13, 14, 15].
Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Definition 1.1. [12] Fractional derivative of order λ $(0 \leq \lambda < 1)$ is defined as

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt,$$

where $f(z)$ is the analytic function in a simply-connected region of the z-plane, containing the origin, and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

S.Owa and H.M.Srivastava [10] introduced the operator Ω^{λ}, for the function f of the form (1) as

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k.$$

Considering $\Psi(z)$ as

$$\Psi(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k$$

and positive real number ν, we define $\Psi^{\perp}(z)$ with

$$\Psi(z) * \Psi^{\perp}(z) = \frac{z}{(1-z)^{\nu}}, \tag{2}$$

where $*$ stand for the Hadamard product (convolution) which for $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ and $h(z) = z + \sum_{k=2}^{\infty} c_k z^k$ is given by

$$(g * h)(z) = z + \sum_{k=2}^{\infty} b_k c_k z^k.$$

In view of (2) we define the operator $Q_{\lambda}^{\nu} f$, for the function f of the form (1) as

$$Q_{\lambda}^{\nu} f(z) = \Psi^{\perp}(z) * f(z),$$

then it is easy to see that

$$Q_{\lambda}^{\nu} f(z) = z + \sum_{k=2}^{\infty} \frac{(\nu)_{k-1}\Gamma(k+1-\lambda)}{k(\Gamma(k))^2\Gamma(2-\lambda)} a_k z^k. \tag{3}$$

Now from (3) we obtain the easily verified following identities

$$z(Q_{\lambda}^{\nu} f(z))' = \nu Q_{\lambda}^{\nu+1} f(z) + (1-\nu) Q_{\lambda}^{\nu} f(z), \tag{4}$$

$$z(Q_{\lambda+1}^{\nu} f(z))' = (1-\lambda) Q_{\lambda}^{\nu} f(z) + \lambda Q_{\lambda+1}^{\nu} f(z).$$

Let $\mathcal{P}$ be the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

which are analytic and convex in $\mathcal{U}$ satisfying the condition $Re(p(z)) > 0, z \in \mathcal{U}$. For two functions f and g , analytic in $\mathcal{U}$, we say the f is subordinate to g in $\mathcal{U}$ and write $f \prec g$, if there exists a Schwarz function w such that

$$f(z) = g(w(z)), \quad z \in \mathcal{U}.$$

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $S\varphi(\nu, \lambda, \eta)$ if it satisfies the following subordination relation

$$\frac{1}{1-\eta}\left(\frac{z(Q_{\lambda}^{\nu} f(z))'}{Q_{\lambda}^{\nu} f(z)} - \eta\right) \prec \varphi(z),$$

where $\nu > 0$,$0 \leq \lambda < 1$, $\eta \geq 0$ and $\varphi \in \mathcal{P}$.

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $C\varphi(\nu, \lambda, \eta)$ if $zf' \in S\varphi(\nu, \lambda, \eta)$, that is

$$\frac{1}{1-\eta}\left(1 + \frac{z(Q_\lambda^\nu f(z))''}{(Q_\lambda^\nu f(z))'} - \eta\right) \prec \varphi(z).$$

In particular both of $S\varphi(\nu, \lambda, \eta)$ and $C\varphi(\nu, \lambda, \eta)$ contain well-known subclasses of analytic function which are studied by many authors see [3, 4, 5]. The main aim of this paper is to study some properties of the classes $S\varphi(\nu, \lambda, \eta)$ and $C\varphi(\nu, \lambda, \eta)$ such as inclusion, argument and subordination results.

2 main results

The following lemmas, studied by different authors (as cited), will be required to prove our main results.

Lemma 2.1. *(see [6]) Let $\zeta, \vartheta \in \mathbb{C}$. Suppose that m is convex and univalent in $\mathcal{U}$ with*

$$m(0) = 1 \text{ and } Re(\zeta m(z) + \vartheta) > 0, \quad (z \in \mathcal{U}).$$

If u is analytic in $\mathcal{U}$ with $u(0) = 1$, then the following subordination

$$u(z) + \frac{zu'(z)}{\zeta u(z) + \vartheta} \prec m(z), \quad (z \in \mathcal{U}),$$

implies that

$$u(z) \prec m(z), \quad (z \in \mathcal{U}).$$

Lemma 2.2. *(see [7]) Let h be convex univalent in $\mathcal{U}$ and w be analytic in $\mathcal{U}$ with*

$$Re(w(z)) \geq 0, \quad (z \in \mathcal{U}).$$

If q is analytic in $\mathcal{U}$ and $q(0) = h(0)$, then the subordination

$$q(z) + w(z)zq'(z) \prec h(z), \quad (z \in \mathcal{U}),$$

implies that

$$q(z) \prec h(z), \quad (z \in \mathcal{U}).$$

Lemma 2.3. *(see [9]) Let q be analytic in $\mathcal{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathcal{U}$. If there exist two points $z_1, z_2 \in \mathcal{U}$ such that*

$$-\frac{\pi}{2}\alpha_1 = \arg(q(z_1)) < \arg(q(z)) < \arg(q(z_2)) = \frac{\pi}{2}\alpha_2,$$

for some α_1 and α_2 $(\alpha_1, \alpha_2 > 0)$ and for all z $(|z| < |z_1 = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1 + \alpha_2}{2}\right)m,$$

and

$$\frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1 + \alpha_2}{2}\right)m,$$

where

$$m \geq \frac{1 - |b|}{1 + |b|} \quad and \quad , b = i \tan \frac{\pi}{4}\left(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}\right).$$

Lemma 2.4. *(see [11]) The function*

$$(1 - z)^{\gamma} \equiv \exp(\gamma \log(1 - z)), \quad (\gamma \neq 0),$$

is univalent if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.

Lemma 2.5. *(see [8]) Let $q(z)$ be univalent in $\mathcal{U}$ and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $q(\mathcal{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathcal{U})$.set*

$$Q(z) = zq'(z)\varphi(q(z)), \quad h(z) = \theta(q(z)) + Q(z),$$

and suppose that

1.$Q(z)$ is starlike (univalent) in $\mathcal{U}$;

2. $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0, \quad (z \in \mathcal{U})$.

If p is analytic in $\mathcal{U}$ with $p(0) = q(0)$ and $p(\mathcal{U}) \subset \mathcal{D}$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z) + zq'(z)\varphi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

Theorem 2.6. *Let $f \in S\varphi(\nu+1, \lambda, \eta)$ with $Re((1-\eta)\varphi(z)+\nu+\eta-1) > 0$, then*

$$S\varphi(\nu+1, \lambda, \eta) \subseteq S\varphi(\nu, \lambda, \eta).$$

Proof. Let $f \in S\varphi(\nu+1, \lambda, \eta)$ and suppose that

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(Q_\lambda^\nu f(z))'}{Q_\lambda^\nu f(z)} - \eta\right). \tag{5}$$

Then q is analytic in $\mathcal{U}$ with $q(0) = 1$. Using (4) in (5) we obtain

$$(1-\eta)q(z) + \nu + \eta - 1 = \nu \frac{Q_\lambda^{\nu+1} f(z)}{Q_\lambda^\nu f(z)}. \tag{6}$$

Differentiating logarithmically both sides of (6) are using (5) we get

$$q(z) + \frac{zq'(z)}{(1-\eta)q(z) + \nu + \eta - 1} = \frac{1}{1-\eta}\left(\frac{z(Q_\lambda^{\nu+1} f(z))'}{Q_\lambda^{\nu+1} f(z)} - \eta\right) \prec \varphi(z),$$

an application of Lemma 2.1 gives our desired result. $\square$

Theorem 2.7. *Let $1 < \xi < 2$ and $\varepsilon \neq 0$. be a real number satisfying either*

$$|2\varepsilon\nu(\xi - 1) - 1| \leq 1,$$

or

$$|2\varepsilon\nu(\xi - 1) + 1| \leq 1.$$

If $f \in \mathcal{A}$ satisfies the condition

$$Re\left\{\frac{Q_\lambda^{\nu+1} f(z)}{Q_\lambda^\nu f(z)}\right\} > 3 - \xi, \tag{7}$$

then

$$\left(\frac{Q_\lambda^\nu f(z)}{z}\right)^\varepsilon \prec (1-z)^{2\varepsilon\nu(1-\xi)}.$$

Proof. Let us assume that $p(z) = \left(\frac{Q_\lambda^\nu f(z)}{z}\right)^\varepsilon$, then by differentiating logarithmically and using (4) we obtain

$$1 + \frac{zp'(z)}{\varepsilon\nu p(z)} = \frac{Q_\lambda^{\nu+1} f(z)}{Q_\lambda^\nu f(z)}.$$

Now from (7) we find that

$$1+\frac{zp'(z)}{\varepsilon\nu p(z)} \prec \frac{1+(2\xi-3)z}{1-z}, \tag{8}$$

if we choose

$$\theta(z)=1, q(z)=(1-z)^{2\varepsilon\nu(1-\xi)}, \varphi(z)=\frac{1}{\varepsilon\nu z},$$

then under the assumptions of the theorem and making use of Lemma 2.4, it is clear that q is univalent in $\mathcal{U}$ and $p(0)=q(0)$. Also $Q(z)=zq'(z)\varphi(q(z))=\frac{2(\xi-1)z}{1-z}$, is starlike in $\mathcal{U}$.
Since, $h(z)=\theta(q(z))+Q(z)=\frac{1-(2\xi-3)z}{1-z}$.
Therefore we have

$$Re\frac{zh'(z)}{Q(z)}=Re\frac{1}{1-z}>0.$$

Now (8) implies that

$$\theta(p(z))+zp'(z)\varphi(p(z)) \prec \theta(q(z))+zq'(z)\varphi(q(z)).$$

Thus Lemma 2.5 shows that our assertion holds true. □

Theorem 2.8. *Let $f \in S\varphi(\nu,\lambda,\eta)$ with*

$$Re((1-\eta)\varphi(z)+\mu+\eta)>0, \qquad z\in\mathcal{U},$$

then the integral operator F defined by

$$F(z)=\frac{\mu+1}{z^{\mu}}\int_0^z t^{\mu-1}f(z)dt, \qquad z\in\mathcal{U}, \mu>-\infty, \tag{9}$$

belongs to the class $S\varphi(\nu,\lambda,\eta)$.

Proof. Let $f\in S\varphi(\nu,\lambda,\eta)$. Then from (9) we find that

$$z(Q_\lambda^\nu F(z))'=(1+\mu)Q_\lambda^\nu f(z)-\mu Q_\lambda^\nu F(z). \tag{10}$$

By setting

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(Q_\lambda^\nu F(z))'}{Q_\lambda^\nu F(z)}-\eta\right), \tag{11}$$

clearly $q(z)$ is analytic in $\mathcal{U}$ with $q(0) = 1$. It follows from (10) and (11) that

$$\mu + \eta + (1-\eta)q(z) = (1+\mu)\frac{Q_\lambda^\nu f(z)}{Q_\lambda^\nu F(z)}. \tag{12}$$

Differentiating both sides of (12) logarithmically with respect to z and using (11), we obtain

$$q(z) + \frac{zq'(z)}{\mu + \eta + (1-\eta)q(z)} = \frac{1}{1-\eta}\left(\frac{z(Q_\lambda^\nu f(z))'}{Q_\lambda^\nu f(z)} - \eta\right) \prec \varphi(z).$$

An application of Lemma 2.1 gives the result. $\square$

Theorem 2.9. *If $f \in C\varphi(\nu, \lambda, \eta)$ then $F(z)$ defined by (9) belongs to $C\varphi(\nu, \lambda, \eta)$.*

Proof. By applying Theorem 2.8 it follows that

$$\begin{aligned} f(z) \in C\varphi(\nu,\lambda,\eta) &\Longleftrightarrow zf'(z) \in S\varphi(\nu,\lambda,\eta) \\ &\Longrightarrow F(zf'(z)) \in S\varphi(\nu,\lambda,\eta) \\ &\Longleftrightarrow z(F(z))' \in S\varphi(\nu,\lambda,\eta) \\ &\Longleftrightarrow F(z) \in C\varphi(\nu,\lambda,\eta), \end{aligned}$$

which proves our desired result. $\square$

Theorem 2.10. *Let $f \in \mathcal{A}$, $0 < \delta_1, \delta_2 \leq 1$ and $0 \leq \eta < 1$. If*

$$\frac{-\pi}{2}\delta_1 < \arg\left(\frac{z(Q_\lambda^{\nu+1} f(z))'}{Q_\lambda^{\nu+1} g(z)} - \eta\right) < \frac{\pi}{2}\delta_2,$$

for some $g \in S\varphi(\nu+1, \lambda, \eta)$ with

$$\varphi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1,$$

then

$$\frac{-\pi}{2}\alpha_1 < \arg\left(\frac{z(Q_\lambda^\nu f(z))'}{Q_\lambda^\nu g(z)} - \eta\right) < \frac{\pi}{2}\alpha_2,$$

where α_1 *and* α_2 $(0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the following equations*

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{R(A,B)}\right), & (B \neq -1), \\ \alpha_1, & (B = -1) \end{cases}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{R(A,B)}\right), & (B \neq -1), \\ \alpha_2, & (B = -1). \end{cases}$$

with

$$b = \tan\frac{\pi}{4}\left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1}\right),$$

$$R(A,B) := 2(1+|b|)\left(\eta + \nu - 1 + \frac{(1-\eta)(1+A)}{1+B}\right) + (1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t,$$

and

$$t = \frac{2}{\pi}\sin^{-1}\left(\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB) - (\eta+\nu-1)(1-B^2)}\right).$$

Proof. Let us define $q_1(z)$ by

$$q_1(z) = \frac{1}{1-\eta}\left(\frac{z(Q_\lambda^\nu f(z))'}{Q_\lambda^\nu g(z)} - \eta\right), \tag{13}$$

then $q_1(z)$ is analytic in $\mathcal{U}$ with $q_1(z) = 1$. It follows from (4) and (13) that

$$[\eta + (1-\eta)q_1(z)]Q_\lambda^\nu g(z) = \nu Q_\lambda^{\nu+1} f(z) + (1-\nu)Q_\lambda^\nu f(z). \tag{14}$$

Differentiating both sides of (14) and multiplying the resulting equation by z we get

$$\begin{aligned}(1-\eta)zq_1'(z)(Q_\lambda^\nu g(z)) &+ [\eta + (1-\eta)q_1(z)]z(Q_\lambda^\nu g(z))' \\ &= \nu z(Q_\lambda^{\nu+1} f(z))' + (1-\nu)z(Q_\lambda^\nu f(z))'. \end{aligned} \tag{15}$$

Theorem 2.6 implies that $g \in S\varphi(\nu,\lambda,\eta)$ with $\varphi(z)=\frac{1+Az}{1+Bz}$. If we set

$$q_2(z)=\frac{1}{1-\eta}\left(\frac{z(Q_\lambda^\nu g(z))'}{Q_\lambda^\nu g(z)}-\eta\right), \tag{16}$$

then we easily obtain

$$\frac{Q_\lambda^\nu g(z)}{Q_\lambda^{\nu+1} g(z)}=\frac{\nu}{\eta+\nu-1+(1-\eta)q_2(z)}, \tag{17}$$

and

$$q_2(z)\prec\frac{1+Az}{1+Bz}\qquad -1\le B<A\le 1. \tag{18}$$

Now from (13), (15), (16) and (17) after some long algebraic calculations we find that

$$\frac{1}{1-\eta}\left(\frac{z(Q_\lambda^{\nu+1} f(z))'}{Q_\lambda^{\nu+1} g(z)}-\eta\right)=q_1(z)+\frac{zq_1'(z)}{\eta+\nu-1+(1-\eta)q_2(z)}. \tag{19}$$

It follows from (18) that

$$\left|q_2(z)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2},\quad z\in\mathcal{U},\quad \mathcal{B}(\neq-\infty)$$

and

$$Re q_2(z)>\frac{1-A}{2},\quad z\in\mathcal{U},\quad (\mathcal{B}=-\infty),$$

which are equivalent respectively with

$$\left|\eta+\nu-1+(1-\eta)q_2(z)-\frac{(\eta+\nu-1)(1-B^2)+(1-\eta)(1-AB)}{1-B^2}\right|$$
$$<\frac{(1-\eta)(A-B)}{1-B^2},\quad (B\neq-1)$$

and

$$Re\left(\eta+\nu-1+(1-\eta)q_2(z)\right)>\frac{(1-\eta)(1-A)}{2}+\eta+\nu-1.\quad (B=-1).$$

Now let us assume that

$$\eta+\nu-1+(1-\eta)q_2(z)=r\exp(i\frac{\pi}{2}\theta),$$

then

$$-\theta_1 < \theta < \theta_1, \quad (B \neq -1)$$

and

$$-1 < \theta < 1, \quad (B = -1)$$

where

$$\theta_1 = \tan^{-1}\left(\frac{(1-\eta)(A-B)}{(\eta+\nu-1)(1-B^2)+(1-\eta)(1-AB)}\right).$$

Therefore

$$\eta+\nu-1+\frac{(1-\eta)(1-A)}{1-B} < r < \eta+\nu-1+\frac{(1-\eta)(1+A)}{1+B}, \ (B \neq -1)$$

and

$$r > \eta+\nu-1+\frac{(1-\eta)(1-A)}{2}. \quad (B = -1)$$

By the assumption of the theorem we have

$$Req_1(z) > 0. \quad (z \in \mathcal{U})$$

Since $\varphi(z) = \frac{1+Az}{1+Bz}$ is convex univalent in $\mathcal{U}$, therefore by applying the assertion of Lemma 2.2 with

$$w(z) = \eta+\nu-1+(1-\eta)q_2(z)$$

for (19) we obtain

$$q_1(z) \prec \frac{1+Az}{1+Bz}.$$

Now it is clear that q_1 is analytic in $\mathcal{U}$ with $q_1(0) = 1$ and $q_1(z) \neq 0$ for all $z \in \mathcal{U}$. If we suppose that

$$G(z) = \frac{1}{1-\eta}\left(\frac{z(Q_\lambda^{\nu+1}f(z))'}{Q_\lambda^{\nu+1}g(z)} - \eta\right),$$

then from (19) we have

$$\arg G(z) = \arg q_1(z) + \arg\left(1+\frac{zq_1'(z)}{[\eta+\nu-1+(1-\eta)q_2(z)]q_1(z)}\right). \quad (20)$$

If there exist two points $z_1, z_2 \in \mathcal{U}$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg q_1(z_1) < \arg q_1(z) < \arg q_1(z_2) = \frac{\pi}{2}\alpha_2,$$

for some α_1 and α_2 $(\alpha_1, \alpha_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$ then Lemma 2.3 implies that

$$\frac{z_1 q_1'(z_1)}{q_1(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right)m,$$

and

$$\frac{z_2 q_1'(z_2)}{q_1(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right)m,$$

where

$$m \geq \frac{1-|b|}{1+|b|} \quad and \quad b = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right).$$

If $B \neq -1$ then from (20) we find that

$$\begin{aligned}
\arg G(z_1) &= \frac{-\pi}{2}\alpha_1 + \arg(1 - i\left(\frac{\alpha_1+\alpha_2}{2}\right)mr^{-1}\exp\left(-i\frac{\pi}{2}\theta)\right) \\
&\leq \frac{-\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{m(\alpha_1+\alpha_2)\sin\frac{\pi}{2}(1-\theta)}{2r + m(\alpha_1+\alpha_2)\cos\frac{\pi}{2}(1-\theta)}\right) \\
&\leq \frac{-\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{R(A,B)}\right) \\
&= \frac{-\pi}{2}\delta_1,
\end{aligned}$$

and

$$\begin{aligned}
\arg G(z_2) &= \arg q_1(z_2) + q_1(z_2) \\
&\quad + \arg\left(1 + \frac{z_2 q_1'(z_2)}{[\eta+\nu-1+(1-\eta)q_2(z_2)]q_1(z_2)}\right) \\
&\geq \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{R(A,B)}\right) \\
&= \frac{\pi}{2}\delta_2.
\end{aligned}$$

Similarly for the case $B=-1$ we obtain

$$\arg G(z_1) = \arg\left(q_1(z_1) + \frac{z_1 q_1'(z_1)}{\eta+\nu-1+(1-\eta)q_2(z_1)}\right) \le \frac{-\pi}{2}\alpha_1,$$

and

$$\arg G(z_2) = \arg\left(q_1(z_2) + \frac{z_1 q_1'(z_2)}{\eta+\nu-1+(1-\eta)q_2(z_2)}\right) \ge \frac{-\pi}{2}\alpha_2.$$

The above two cases obviously contradict the assertion of Theorem 2.10. The proof now is completed. □

Following we state Theorems 2.11 and 2.12 which their proofs are similar to Theorems 2.6 and 2.9 respectively so we omit the details.

Theorem 2.11. *Let $f \in S\varphi(\nu,\lambda,\eta)$ with $Re\,((1-\eta)\varphi(z)+\eta-\lambda)>0$ then*

$$S\varphi(\nu,\lambda,\eta) \subseteq S\varphi(\nu,\lambda+1,\eta)$$

Theorem 2.12. *Let $f \in \mathcal{A}$, $0<\delta_1,\delta_2\le 1$ and $0\le\eta<1$. If*

$$\frac{-\pi}{2}\delta_1 < \arg\left(\frac{z(Q_\lambda^\nu f(z))'}{Q_\lambda^\nu g(z)} - \eta\right) < \frac{\pi}{2}\delta_2,$$

for some $g \in S\varphi(\nu,\lambda,\eta)$ with

$$\varphi(z) = \frac{1+Az}{1+Bz}, -1\le B<A\le 1,$$

then

$$\frac{-\pi}{2}\alpha_1 < \arg\left(\frac{z(Q_{\lambda+1}^\nu f(z))'}{Q_{\lambda+1}^\nu g(z)} - \eta\right) < \frac{\pi}{2}\alpha_2,$$

where α_1 and α_2 $(0<\alpha_1,\alpha_2\le 1)$ are the solutions of the following equations

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{S(A,B)}\right), & (B\ne -1),\\ \alpha_1, & (B=-1),\end{cases}$$

and

$$\delta_2 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{S(A,B)}\right), & (B\ne -1),\\ \alpha_2, & (B=-1),\end{cases}$$

with

$$b = \tan \frac{\pi}{4} \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right),$$

$$S(A,B) := 2(1+|b|) \left(\eta - \lambda + \frac{(1-\eta)(1+A)}{1+B}) + (1-|b|\right) (\alpha_1+\alpha_2) \sin \frac{\pi}{2} t,$$

and

$$t = \frac{2}{\pi} \sin^{-1} \left(\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB) - (\eta-\lambda)(1-B^2)} \right).$$

Remark 2.13. Generally, all bounds in Theorem 2.9 and 2.11 are not sharp, the sharpness is still an open problem.

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