Journal of Mathematical Extension
Vol. 15, No. 2, (2021) (9)1-14
URL: https://doi.org/10.30495/JME.2021.1314
ISSN: 1735-8299
Original Research Paper

Existence and Uniqueness of Solutions of a Class of Quantum Stochastic Evolution Equations

S. A. Bishop*
University of Lagos

E. O. Ayoola
University of Ibadan

Abstract. We study the existence and uniqueness of solutions of a class of Quantum Stochastic Evolution Equations (QSEEs) defined on a locally convex space whose topology is generated by a family of seminorms defined via the norm of the range space of the operator processes. These solutions are called strong solutions in comparison with the solutions of similar equations defined on the space of operator processes where the topology is generated by the family of seminorms defined via the inner product of the range space. The evolution operator generates a bounded semigroup. We show that under some more general conditions, the unique solution is stable. These results extend some existing results in the literature concerning strong solutions of quantum stochastic differential equations.

AMS Subject Classification: 58J65, 81S25, 60H10
Keywords and Phrases: Strong solutions, Stability, Bounded semigroup, General Lipschitz condition.

1 Introduction

Several results on weak forms of solutions of the following quantum stochastic differential equation have been studied. See [1, 3-7] and the

Received: June 2019; Accepted: April 2020
*Corresponding Author

references therein. The motivations for studying this class of equations have been discussed in the references.

$$\begin{aligned} dz(t) &= U(t,z(t))d\wedge_{\pi}(t) + V(t,z(t))dA_g(t) \\ &\quad + W(t,z(t))dA_{f+}(t) + H(t,z(t))dt, \\ z(t_0) &= z_0,\ t \in I = [t_0, T] \end{aligned} \tag{1}$$

In Eq. (1), the coefficients U, V, W, and H lie in a certain class of stochastic processes defined in [1], while the gauge, creation, annihilation processes $\Lambda_{\Pi}, A_{f+}, A_g$ and the Lebesgue measure t are well defined in [2] and the references therein. $z \in \tilde{\mathcal{B}}$ is a locally convex space.
Quantum stochastic differential equation (QSDE) (1) is understood in the framework of the Hudson and Parthasarathy [9] quantum stochastic calculus. It has found applications in many physical systems, especially those that have to do with quantum optics, quantum measure theory, quantum open systems and quantum dynamical systems (see [1-7]).
In [6], some properties of solutions of Eq. (1) were studied. Results on the existence and uniqueness of solutions of this class of equations were established in the space of the operator processes endowed with the weak topologies. In [7], quantum stochastic differential inclusions of hypermaximal monotone type were studied under some general conditions and the existence of solution of an evolution operator connected with these inclusions were established. Also, see [4] for some results on evolution inclusions where the multivalued map P_1 is of hypermaximal monotone type. Further studies were carried out by [5] on properties of solution sets of quantum stochastic differential inclusions of Eq. (1) under the weak topologies. However, Ayoola in [1] investigated some existence properties on the space when endowed with the strong topology under a more general Lipschitz condition on the coefficients (U, V, W, H). Some new results including stability results were obtained. The results in [1, 2] generalized some similar results in the classical setting. This paper is concerned with the study of the properties of solutions of an evolution equation defined on the space with the strong topology. In [3, 12] existence of mild solutions of evolution QSDEs was studied under the weak topologies. Evolution problems have found practical applications in virtually all fields of sciences. See the references [11, 13-15] for some applications of evolution problems. The results in the present work

extend some existing results on strong solutions of Eq. (1) and extend the solution space for which QSDE will be applicable. We will consider some applications in our subsequent work.

2 Preliminaries

In what follows, the following evolution equation is considered.

$$\begin{aligned} dz(t) &= A(t)z(t) + U(t,z(t))d\wedge_{\pi}(t) + V(t,z(t))dA_g(t) \\ &\quad + W(t,z(t))dA_{f^+}(t) + H(t,z(t))dt, \\ z(t_0) &= z_0,\ t \in I \end{aligned} \tag{2}$$

where A generates a bounded semigroup $\{S(t) : t \geq 0\}$. For details on semigroup and their applications , see the references [8,10]. We adopt in most cases the definitions and notations of the spaces used in this paper from the references [1-3]. $\tilde{\mathcal{B}}$ is the completion of the topological space $(\tilde{\mathcal{B}}, \tau)$, and τ is the topology generated by the family of seminorms $||\phi||_{\xi} = ||\phi\xi||, \xi \in I\!\!D\underline{\otimes}I\!\!E$, where $||.||$ is the norm of the space $\mathcal{R} \otimes \Gamma(L^2_{\gamma}(\mathbb{R}_+))$. The space $\mathcal{B}$ is the linear space of all linear operators on $\mathcal{R} \otimes \Gamma(L^2_{\gamma}(\mathbb{R}_+))$. $I\!\!D, I\!\!E$, and $\mathcal{R}$ are well defined in [1]. The notations and structures of the following spaces are from the references [1, 2]. $\mathcal{R} \otimes \Gamma(L^2_{\gamma}(\mathbb{R}_+))$, $Ad(\tilde{\mathcal{B}})_{ac}$, $L^p_{loc}(\tilde{\mathcal{B}})$, $L^2_{\gamma}(\mathbb{R}_+)$, $L(\tilde{\mathcal{B}})$, $I\!\!D\underline{\otimes}I\!\!E$, Fin$(I\!\!D\underline{\otimes}I\!\!E)$. $I\!\!D, I\!\!E$, and $\mathcal{R}$ is well defined in [1].

Definition 2.1.
(i) $\phi : I \to \tilde{\mathcal{B}}$ is a stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$.
(ii) If $\phi(t) \in \tilde{\mathcal{B}}_t$, $t \in I$, then ϕ is said to be adapted and we denote the set of all such stochastic processes by $Ad(\tilde{\mathcal{B}})$.
(iii) $\phi(t) \in Ad(\tilde{\mathcal{B}})_{ac}$ is said to be adapted, absolutely continuous.
(iv) $\phi(t) \in L^p_{loc}(\tilde{\mathcal{B}})$ is said to be locally, absolutely p-integrable, where $p \in (0, \infty)$.
(v) Since the evolution operator A generates a bounded semigroup $\{S(t)\}_{t\geq 0}$, for each $t \geq 0$, there exists a constant $M > 0$ such that $\|S(t)\|_{\xi} \leq M$.
(vi) Let $\theta \in Fin(I\!\!D\underline{\otimes}I\!\!E)$ and $z \in \tilde{\mathcal{B}}$ then, $||z||_{\theta} = max_{\xi\in\theta}||z||_{\xi}$, where the set $\{\|.\|_{\theta} : \theta \in Fin(I\!\!D\underline{\otimes}I\!\!E)\}$ is a family of seminorms on $\tilde{\mathcal{B}}$ and $Fin(I\!\!D\underline{\otimes}I\!\!E)$ denote the set of all finite subsets of $I\!\!D\underline{\otimes}I\!\!E$. Also see Definitions 2.5 and 2.6 in [2].

Definition 2.2.
A stochastic process $\phi \in L^2_{loc}(\tilde{\mathcal{B}})$ is called a strong solution of the problem (2) on I if it is absolutely continuous and satisfies

$$\begin{aligned} \phi(t) &= S(t)\phi_0 + \int_{t_0}^t S(t-s)[U(s,\phi(s))d\wedge_\pi(s) + V(s,\phi(s))dA_g(s) \\ &\quad + W(s,\phi(s))dA_{f^+}(s) + H(s,\phi(s))ds], \\ \phi(t_0) &= \phi_0,\ t \in I \end{aligned} \tag{3}$$

Definition 2.3.
$\Phi : I \times \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ is Lipschitzian if

$$\|\Phi(t,y) - \Phi(t,z)\|_\xi \le K_\xi^\Phi(t)\|y-z\|_{\theta_{\Phi(\xi)}}$$

is satisfied for each $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, where $y,z \in \tilde{\mathcal{B}}$, $\theta \in (\mathbb{D}\underline{\otimes}\mathbb{E}, Fin(\mathbb{D}\underline{\otimes}\mathbb{E}))$ and $K_\xi^\Phi : I \to (0,\infty)$ is a Lipschitz function lying in $L^1_{loc}(I)$. $I = [0,T] \subseteq \mathbb{R}_+$.

Remark 2.4. Theorem 2.2 and Remarks (a) - (c) in [1] hold in this case.

For the remaining part of this paper, $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ is arbitrary, except otherwise stated, the following result established in [1] will be used to establish the major results.

Theorem 2.5. (a) Let $p,q,u,v \in L^2_{loc}(\tilde{\mathcal{B}})$ and let $\mathbf{M}$ be their stochastic integral. If $\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ where $\xi = d \otimes e(\beta), \alpha, \beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $t \ge 0$, then

$$\begin{aligned} <\eta, \mathbf{M}(t)\xi> &= \int_0^t <\eta, \{\alpha(s), \pi(s)\beta(s)>_\gamma p(s) \\ &+ <f(s), \beta(s)>_\gamma q(s) \\ &+ <\alpha(s), g(s)>_\gamma u(s) + v(s)\}\xi> ds. \end{aligned} \tag{4}$$

(b) Let
$K(T) = \sup_{0 \le s \le T} \max\{|\langle\beta(s), \pi(s)\beta(s)\rangle|, |\langle f(s), \beta(s)\rangle|, |\langle\beta(s), g(s)\rangle|, ||\pi(s)\beta(s)||^2, ||g(s)||^2\}$.
Then for $T > 0$ and $0 \le t \le T$,

$$\begin{aligned} ||\mathbf{M}(t)\xi||^2 &\le 6K(T)^2 \int_0^t e^{t-s}\{||p(s)\xi||^2 + ||q(s)\xi||^2 + ||u(s)\xi||^2 \\ &+ ||v(s)\xi||^2\}ds. \end{aligned} \tag{5}$$

(c) Let $0 \leq s \leq t \leq T$. Then

$$\begin{aligned} ||(\mathbf{M}(t) - \mathbf{M}(s))\xi||^2 &\leq 6K(T)^2 \int_0^t e^{t-\tau}\{||p(\tau)\xi||^2 + ||q(\tau)\xi||^2 \\ &+ ||u(\tau)\xi||^2 + ||v(\tau)\xi||^2\} d\tau. \end{aligned} \quad (6)$$

Note 2.6. $\mathbf{M}$ is absolutely continuous, hence, $\mathbf{M} \in L^2_{loc}(\tilde{\mathcal{B}})$.

3 Main Results

This section is dedicated to the main results on existence and uniqueness of strong solutions of (2). Subsequently, except otherwise stated, $t \in I = [t_0, T] \subseteq \mathbb{R}_+$ and $\xi \in I\!\!D \underline{\otimes} I\!\!E$ is arbitrary.

Theorem 3.1.
Suppose that the coefficients $U, V, W, H \in L^2_{loc}(I \times \tilde{\mathcal{B}})$ are Lipschitzian. Then for $(t_0, z_0) \in I \times \tilde{\mathcal{A}}$ there exists a unique strong solution φ of equation (2) satisfying $\varphi(t_0) = z_0$.

Proof. To prove the theorem, we make the following assumptions:
H_1. Let $\{\varphi_n(t)\}_{n \geq 0}$ be a sequence of successive approximations of $\varphi \in \tilde{\mathcal{B}}$ and
H_2. $\varphi_n(t),\ n \geq 1$ define an absolutely continuous process in $L^2_{loc}(\tilde{\mathcal{A}})$.
Let $T > t_0,\ t \in I$ be fixed. Then, we prove $H_1 - H_2$ as follows.
For $n \geq 0$, we have

$$\begin{aligned} \varphi_{n+1}(t) &= S(t)z_0 + \int_{t_0}^t S(t-s)[U(s, \varphi_n(s))d\wedge_\pi(s) \\ &+ V(s, \varphi(s))dA_g^+(s) + W(s, \varphi_n(s))dA_f(s) + H(s, \varphi_n(s))ds]. \end{aligned}$$

By hypothesis, $U(s, z_0), V(s, z_0), W(s, z_0), H(s, z_0) \in \tilde{\mathcal{B}}_s$ for $s \in [t_0, T]$ while
$U(., z_0), V(., z_0), W(., z_0), H(., z_0) \in L^2_{loc}(\tilde{\mathcal{B}})$.
Therefore, the quantum stochastic integral which defines $\varphi_1(t)$ exists for $t \in [t_0, T]$. By Theorem 2.5, $\varphi_1(t) \in L^2_{loc}(\tilde{\mathcal{B}})$.
Hence, it implies that each
$U(s, \varphi_n(s)), V(s, \varphi(s)), W(s, \varphi_n(s))$ and $H(s, \varphi_n(s) \in L^2_{loc}(\tilde{\mathcal{B}})$.
This proves assumptions $H_1 - H_2$.

Next, we show that the sequence of successive approximations converges as follows:

$$\begin{aligned} \| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi &= \| \int_{t_0}^t S(t-s)[(U(s,\varphi_n(s)) \\ &\quad -U(s,\varphi_{n-1}(s)))d\wedge_\pi(s) \\ &\quad +(V(s,\varphi(s)) - V(s,\varphi_{n-1}(s)))dA_g^+(s) \\ &\quad +(W(s,\varphi_n(s)) - W(s,\varphi_{n-1}(s)))dA_f(s) \\ &\quad +(H(s,\varphi_n(s)) - H(s,\varphi_{n-1}(s)))ds]\|_\xi. \end{aligned}$$

By Theorem 2.5 and (v) of Definition 2.1, we get

$$\begin{aligned} \| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi^2 &\leq 6M^2K(T)^2 \int_{t_0}^t e^{t-s}\{\|U(s,\varphi_n(s)) \\ &\quad -U(s,\varphi_{n-1}(s))\|_\xi^2 \\ &\quad +\|V(s,\varphi(s)) - V(s,\varphi_{n-1}(s))\|_\xi^2 \\ &\quad +\|W(s,\varphi_n(s)) - W(s,\varphi_{n-1}(s))\|_\xi^2 \\ &\quad +\|H(s,\varphi_n(s)) - H(s,\varphi_{n-1}(s))\|_\xi^2\}ds. \qquad (7) \end{aligned}$$

By definition 2.3, we have

$$\| \mathbf{M}(s,\varphi_n(s)) - \mathbf{M}(s,\varphi_{n-1}(s)) \|_\xi \leq K_\xi^{\mathbf{M}}(s) \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\theta_{\mathbf{M}\xi}},$$

for each $\mathbf{M} \in \{U, V.W, H\}$. Thus, there exists $\xi_{\mathbf{M}}^1 \in \theta_{\mathbf{M}}(\xi)$ satisfying

$$\| \varphi_n(s) - \varphi_{n-1}(s) \|_{\theta_{\mathbf{M}\xi}} = \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_{\mathbf{M}}^1}.$$

Using (7), we obtain

$$\begin{aligned} \| \varphi_{n+1}(t) - \varphi_n(t) \|_\xi^2 &\leq NC(T)L_\xi \int_{t_0}^t e^{t-s} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1}^2 \, ds \\ &= NC(T)L_\xi e^t \\ &\quad \times \int_{t_0}^t e^{-s} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1}^2 \, ds. \qquad (8) \end{aligned}$$

Where

$$\| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_1} = \max_{\mathbf{M} \in \{U,V.W,H\}} \| \varphi_n(s) - \varphi_{n-1}(s) \|_{\xi_{\mathbf{M}}^1}. \qquad (9)$$

and $N = M^2,\ C(T) = 6K(T)^2$,

$$L_\xi = \text{ess}\sup_{s\in[0,T]}\left[K_\xi(s) = \sum_{\mathbf{M}\in\{U,V.W,H\}} K_\xi^{\mathbf{M}}(s)^2\right].$$

Continuing the iteration and replacing ξ_2 with ξ_1 in (9), yields

$$\begin{aligned}\parallel \varphi_{n+1}(t)-\varphi_n(t)\parallel_\xi^2 \leq{}& N^2C(T)^2L_\xi L_{\xi_1}e^t\\ &\times\int_{t_0}^t\int_{t_0}^s e^{-s'}\parallel \varphi_{n-1}(e^{-s'})-\varphi_{n-2}(e^{-s'})\parallel_{\xi_2}^2 ds'ds\\ &\leq N^nC(T)^n\mathbf{M}(\xi)^ne^t\int_{t_0}^t ds_1\int^{s_1} ds_2...\int_{t_0}^{s_{n-2}} ds_{n-1}\\ &\times\int_{t_0}^{s_{n-1}} e^{-s_n}\parallel \varphi_1(s_n)-\varphi_0(s_n)\parallel_\xi^2 ds_n, \qquad (10)\end{aligned}$$

where $\mathbf{M}_n(\xi) = \max\{L_{\xi,j}, j = 0,1,...,n-1\}$, $\mathbf{M}(\xi) = \sup_{n\in\mathbb{N}}\{\mathbf{M}_n(\xi)\}$, and $L_{\xi,j}, j = 0,1,...,n-1$ are positive real numbers.
Since the map $s \longrightarrow \|\varphi_1(s)-z_0\|_\xi$ is continuous on I, we obtain,
$R_{\xi_n} = \sup_{s\in I}\|\varphi_1(s)-z_0\|_{\xi_n} < \infty$ and put $R_\xi = \sup_{n\in\mathbb{N}}\{R_{\xi_n}\}$ in (10) to get

$$\parallel \varphi_{n+1}(t)-\varphi_n(t)\parallel_\xi^2 \leq [NC(T)\mathbf{M}(\xi)]^n e^T\frac{T^n}{n!}R_\xi^2, n = 0,1,2,....$$

For $n > k$ we get,

$$\begin{aligned}\parallel \varphi_{n+1}(t)-\varphi_{k+1}(t)\parallel_\xi &= \|\Sigma_{m=k+1}^n(\varphi_{m+1}(t)-\varphi_m(t))\|_\xi\\ &\leq \Sigma_{m=k+1}^n\|\varphi_{m+1}(t)-\varphi_m(t)\|_\xi\\ &\leq e^{\frac{T}{2}}R_\xi\sum_{m=k+1}^n\left(\frac{[NC(T)\mathbf{M}(\xi)]^mT^m}{m!}\right)^{\frac12} < \infty.\end{aligned}$$

Showing that $\varphi_n(t)$ is a Cauchy sequence in $\tilde{\mathcal{B}}$ and converges uniformly to $\varphi(t)$.
Now since $\varphi_n(t)$ is adapted and absolutely continuous, the same is true for $\varphi(t)$.

Next, we show that $\varphi(t)$ satisfies Eq. (2).
Let $\varphi(t_0) = z_0$ and by (8), there exists $\xi \in I\!D \underline{\otimes} I\!E$ such that

$$\begin{aligned}
&|| \int_{t_0}^{t} S(t-s)[U(s,\varphi_n(s))d\wedge_\pi(s) + V(s,\varphi_n(s))dA_g^+(s) \\
&\quad + W(s,\varphi_n(s))dA_f(s) + H(s,\varphi_n(s))ds]||_\xi^2 \\
&- || \int_{t_0}^{t} S(t-s)[U(s,\varphi(s))d\wedge_\pi(s) \\
&\quad + V(s,\varphi(s))dA_g^+(s) + W(s,\varphi(s))dA_f(s) \\
&+ \quad H(s,\varphi(s))ds]||_\xi^2 \\
&= || \int_{t_0}^{t} S(t-s)(P(s,\varphi_n(s)) - P(s,\varphi(s)))ds||_\xi^2 \\
&\leq NC(T)L_\xi e^t \times \int_{t_0}^{t} e^{-s} \parallel \varphi_n(s) - \varphi(s) \parallel_\xi^2 ds \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $\varphi_n(s) \to \varphi(s)$ in $\tilde{\mathcal{B}}$ uniformly on $[t_0, T]$, we have

$$\begin{aligned}
\varphi(t) &= \lim_{n\to\infty} \varphi_{n+1}(t) \\
&= S(t)z_0 + \lim_{n\to\infty}(\int_{t_0}^{t} S(t-s)(U(s,\varphi_n(s))d\wedge_\pi(s) \\
&+ V(s,\varphi_n(s))dA_g^+(s) \\
&+ W(s,\varphi_n(s))dA_f(s) + H(s,\varphi_n(s))ds) \\
&= S(t)z_0 + \int_{t_0}^{t} S(t-s)(U(s,\varphi(s))d\wedge_\pi(s) \\
&+ V(s,\varphi(s))dA_g^+(s) \\
&+ W(s,\varphi(s))dA_f(s) + H(s,\varphi(s))ds),, t \in I.
\end{aligned}$$

This shows that $\varphi(t)$ is a solution of Eq. (2).

Uniqueness

Suppose that $y(t), t \in [t_0, T]$ is another adapted absolutely continuous solution with $y(t_0) = z_0$, then just as we established the above result, we obtain

$$\parallel \varphi(t) - y(t) \parallel_\xi^2 \leq [NC(T)\mathbf{M}(\xi)]^n e^T \frac{T}{n!} \sup_{t\in I} \parallel \varphi(t) - y(t) \parallel_\xi^2 < \infty. \quad (11)$$

By the right hand side of Eq. (11), we conclude that for $n \in \mathbb{N}$, $\| \varphi(t) - y(t) \|_{\xi} = 0$ and $\varphi(t) = y(t)$ on $I\!D \underline{\otimes} I\!E$, $t \in I$. Hence the solution is unique.

4 Stability

In this section, we show that under the condition (v) of Definition 2.1, the solution of Eq.(2) is stable.
(a) let the coefficients U, V, W, H satisfy the conditions of Theorem 3.1 and let $z(t), y(t)$, $t \in [t_0, T]$ be solutions of Eq. (2) such that $z(t_0) = z_0$ and $y(t_0) = y_0$, $z_0, y_0 \in \tilde{\mathcal{B}}$. The solution z(t) is stable under the changes in the initial condition over a finite time interval as follows:
(b) Let L_ξ, N and $C(T)$ be constants such that

$$L_\xi = \operatorname*{ess\,sup}_{s \in I} K_\xi(s),\ C(T) = 12K(T)^2 \text{ and } N = M^2$$

where $K(T)$ is as defined in Theorem 2.3 and $||S(t)||_\xi$ by (v) of Definition 2.1.
(c) Define the function $K_\xi(s)$ as

$$K_\xi(s) = \sum_{\mathbf{M} \in \{U,V.W,H\}} (K_\xi^{\mathbf{M}}(s))^2$$

Theorem 4.1. Let the conditions of Definition 2.1 hold and let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that if $\|z_0 - y_0\|_\xi < \delta$, then $\|z(t) - y(t)\|_\xi < \epsilon$, $\forall\, t \in [0, T]$.

Proof:
Let $z_n(t), y_n(t),\ n = 0, 1, ...$ be the iterates corresponding to z_0, y_0 respectively. Let $z_0(t) = z_0$ and $y_0(t) = y_0$, $0 \le t \le T$. Then we get

$$\begin{aligned}\| z_{n+1}(t) - y_{n+1}(t) \|_\xi \le{} & ||S(t-s)(z_0 - y_0)||_\xi \\ & + || \int_{t_0}^{t} S(t-s)[(U(s, z_n(s)) - U(s, y_n(s)))d\wedge_\pi(s) \\ & + (V(s, z(s)) - V(s, y_n(s)))dA_g^+(s) \\ & + (W(s, z_n(s)) - W(s, y_n(s)))dA_f(s) \\ & + (H(s, z_n(s)) - H(s, y_n(s)))ds]||_\xi\end{aligned}$$

So that by applying Theorem 2.5 and condition (v) of Definition 2.1, we obtain

$$\begin{aligned}\| z_{n+1}(t)-y_{n+1}(t) \|_\xi^2 \leq& 2M^2||z_0-y_0||_\xi^2\\ &+2M^2||\int_{t_0}^t S(t-s)[(U(s,z_n(s))-U(s,y_n(s)))d\wedge_\pi(s)\\ &+(V(s,z(s))-V(s,y_n(s)))dA_g^+(s)\\ &+(W(s,z_n(s))-W(s,y_n(s)))dA_f(s)\\ &+(H(s,z_n(s))-H(s,y_n(s)))ds]||_\xi^2\end{aligned}$$

$$\begin{aligned}\| z_{n+1}(t)-y_{n+1}(t) \|_\xi^2 \leq& 2N||z_0-y_0||_\xi^2\\ &+ NC(T)\int_{t_0}^t e^{s-t}\{||U(s,z_n(s))-U(s,y_n(s))||_\xi^2\\ &+ ||V(s,z(s))-V(s,y_n(s))||_\xi^2\\ &+ ||(W(s,z_n(s))-W(s,y_n(s))||_\xi^2\\ &+ (H(s,z_n(s))-H(s,y_n(s)))||_\xi^2\}ds.\end{aligned}$$

Since Definition 2.3 also holds for the coefficients U, V, W, H, we find elements $\xi_{\mathbf{M},1} \in \theta_{\mathbf{M}}(\xi) \in Fin(I\!\!D \underline{\otimes} I\!\!E)$, $\mathbf{M} \in \{U, V, W, H\}$ such that

$$\begin{aligned}\| z_{n+1}(t)-y_{n+1}(t) \|_\xi^2 \leq& 2N||z_0-y_0||_\xi^2 + NC(T)\\ &\times\int_{t_0}^t e^{t-s_1}[\sum_{\mathbf{M}\in\{U,V,W,H\}} K_\xi^{\mathbf{M}}(s_1)^2||z_n(s_1))-y_n(s_1))||_{\xi_{\mathbf{M},1}}^2]ds_1\\ &\leq 2N||z_0-y_0||_\xi^2\\ &+NC(T)L_\xi e^t\int_{t_0}^t e^{-s_1}||z_n(s_1)-y_n(s_1)||_\xi^2 ds_1. \qquad (12)\end{aligned}$$

where $\xi_1 \in \xi_{\mathbf{M},1} : \mathbf{M} \in \{U, V.W, H\}\}$ satisfies

$$\| \varphi_n(s)-\varphi_{n-1}(s) \|_{\xi_1}^2 = \max_{\mathbf{M}\in\{U,V.W,H\}} \| \varphi_n(s)-\varphi_{n-1}(s) \|_{\xi_{\mathbf{M},1}}^2,$$

where $s \in I$.
Also, if we have $\xi_2 \in I\!\!D \underline{\otimes} I\!\!E$ then,

$$\begin{aligned}\| z_n(s_1) - y_n(s_1) \|_\xi^2 &\leq 2N \|(z_0 - y_0)\|_{\xi_1}^2 \\ &+ NC(T)L_{\xi_1} \int_{s_1}^{t} e^{s_1 - s_2} ||z_{n-1}(s_2) - y_{n-1}(s_2)||_{\xi_2}^2 ds_2.\end{aligned}$$

By (12), we obtain for $t \in [0, T]$,

$$\begin{aligned}\| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2N \|(z_0 - y_0)\|_\xi^2 \\ &+ 2NC(T) \|z_0 - y_0\|_\xi^2 L_\xi e^t \int_0^t e^{-s_1} ds_1 \\ &+ N^2 C(T)^2 L_\xi L_{\xi_1} e^t \\ &\times \int_0^t \int_0^{s_1} e^{-s_2} ||z_{n-1}(s_2) - y_{n-1}(s_2)||_{\xi_2}^2 ds_2 ds_1.\end{aligned}$$

Continuous iterations yields,

$$\begin{aligned}\| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2N \|z_0 - y_0\|_\xi^2 + 2NC(T) \|z_0 - y_0\|_{\xi_1}^2 L_\xi e^T t \\ &+ 2N^2 C(T)^2 \|z_0 - y_0\|_{\xi_2}^2 L_\xi L_{\xi_1} e^T \int_0^t \int_0^{s_1} ds_2 ds_1 \\ &+ 2N^3 C(T)^3 \|z_0 - y_0\|_{\xi_2}^2 L_\xi L_{\xi_1} L_{\xi_2} e^T \int_0^t \int_0^{s_1} \\ &\times \int_0^{s_2} \int_0^{s_1} ds_3 ds_2 ds_1 \\ &+ ... + N^{n+1} C(T)^{(n+1)} e^T L_\xi L_{\xi_1} L_{\xi_2} ... L_{\xi_n} \int_0^t \int_0^{s_1} ... \\ &\times \int_0^{s_n} ||z_0(s_{n+1}) - y_0(s_{n+1})||_{\xi_{n+1}}^2 ds_1 ds_2 ds_3 ... ds_{n+1}.\end{aligned}$$

Now, by letting $\mathbf{K}(\xi) = \sup_{n \in \mathbb{N}} \{L_\xi, L_{\xi_1}, L_{\xi_2}, ..., L_{\xi_n}\}$,
$\eta_n \in \{\xi, \xi_1, \xi_2, ..., \xi_n, \xi_{n+1}\}$ so that if

$$\|z_0 - y_0\|_{\eta_n} = \max\{\|z_0 - y_0\|_{\xi_j}, j = 0, 1, ..., n+1\},$$

we obtain

$$\begin{aligned}\| z_{n+1}(t) - y_{n+1}(t) \|_\xi^2 &\leq 2e^T \|z_0 - y_0\|_{\eta_n}^2 \sum_{m=0}^{n+1} [NC(T)\mathbf{K}(\xi)]^m \frac{T^m}{m!} \\ &\leq 2\|z_0 - y_0\|_{\eta_n}^2 e^{(NC(T)\mathbf{K}(\xi)+T)}. \qquad (13)\end{aligned}$$

Thus, by taking the square root of both sides of (13) and letting $n \to \infty$, we obtain $\|z(t) - y(t)\|_\xi \leq \epsilon$.
Take $\delta = \epsilon[2e^{(NC(T)\mathbf{K}(\xi)T+T)}]^{-\frac{1}{2}}$, for all $t \in [0, T]$, and the desired result is obtained.
Remark 4.2 If $N < 1$, we obtain the results in [1].

Acknowledgments
The authors are grateful to the anonymous referee(s) for their useful comments.

References

[1] E. O. Ayoola, A. W. Gbolagade, Further Results on Existence, Uniqueness and Stability of Strong Solutions of Quantum Stochastic Differential Equations, *Applied mathematics letters* 18(2005) 219-227.

[2] E. O. Ayoola, Existence and stability results for strong solutions of Lipschitzian quantum stochastic differential equations. *Stoch. Anal. and Appl.*, 20(2)(2002), 263-281.

[3] S. A. Bishop, E. O. Ayoola and G. J. Oghonyon, Existence of mild solution of impulsive quantum stochastic differential equation with nonlocal conditions, *Anal. Math. Phys.* 7(3)(2017), 255-265.

[4] S. A. Bishop and E. O. Ayoola, On topological properties of solution sets of non Lipschitzian quantum stochastic differential inclusions, *Anal. Math. Phys.* 2015, DOI 10.1007/s13324-015-0109-1.

[5] S. A. Bishop, On continuous selection sets of non Lipschitzian quantum stochastic evolution inclusions, *Int. J. of Stoch. Anal.* Vol. 2015, Article ID 834194, pp. 5, http://dx.doi.org/10.1155/2015/834194.

[6] G.O.S. Ekhaguere, Topological solutions of non commutative stochastic differential equations *Stoch. Anal. and Appl.* 25(9)(2007), 961-993.

[7] G.O.S. Ekhaguere, Quantum stochastic evolutions, *Int. J. of Theor. Phys.*, 35(9)(1996), 1909 - 1946.

[8] E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc., 1949, N.Y.

[9] R. L. Hudson and K. R. Parthasarathy, Quantum Ito's formulae and stochastic evolutions. *Comm. Math. Phys.*, 93(1984), 301-3249.

[10] S. V. Lototsky, B. L. Rozovskii, Wiener chaos solutions of linear stochastic evolution equations, *The Annals of Prob.*, 34(2)(2006), 638-662.

[11] M. McKibben, *Discovering evolution equations with applications: Vol. 1-Deterministic equations* (Chapman & Hall/CRC Applied Mathematics and Nonlinear Science) 1st Edition, 2010 London N.Y.

[12] M. O. Ogundiran, On the existence and uniqueness of solution of impulsive quantum stochastic differential equation, *Differential equations and control processes*, N 2(2013), 63-73.

[13] B. L. Rozovskii, *Stochastic differential equations: theory and applications, interdisciplinary mathematical sciences*, A Volume in Honor of Professor Boris L. Rozovskii, April 2007, pp. 420.

[14] M. C. Veraar, Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations, *J. Evol. Equ.*, 10 (2010), 85-127.

[15] S. Zheng, *Nonlinear evolution equations, monograhps and surveys in pure and applied mathematics*, Chapman and Hall/CRC, 2004 pp. 304, London N.Y.

Sheila Amina Bishop
Department of Mathematics
Assistant Professor of Mathematics

University of Lagos
Akoka, Lagos Sate, Nigeria
E-mail:asbishop@unilag.edu.ng

Ezekiel Olusola Ayoola
Department of Mathematics
Professor of Mathematics
University of Ibadan
Ibadan, Oyo Sate, Nigeria
E-mail:eoayoola@gmail.com