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Original Research Paper

Signed Complete Graphs with Negative Paths

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Abstract. Let $\Gamma = (G, \sigma)$ be a signed graph, where G is the underlying simple graph and $\sigma : E(G) \longrightarrow \{-, +\}$ is the sign function on the edges of G. The adjacency matrix of a signed graph has -1 or $+1$ for adjacent vertices, depending on the sign of the connecting edges. Let $\Gamma = (K_n, \bigcup_{i=1}^{m} P_{r_i}^{-})$ be a signed complete graph whose negative edges induce a subgraph which is the disjoint union of m distinct paths. In this paper, by a constructive method, we obtain $n-1+\Sigma_{i=1}^{m}\left(\lfloor \frac{r_i}{2} \rfloor - r_i\right)$ eigenvalues of Γ, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of G is defined $|V(G)|$. Let K_n denote the

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complete graph of order n. We denote the path of order r, by P_r. The matrix $J_{r\times s}$ is all-one matrix of size $r \times s$.

A *signed graph* Γ is an ordered pair (G, σ), where $G = (V(G), E(G))$ is a simple graph (called the *underlying graph*), and let $\sigma : E(G) \longrightarrow \{-, +\}$ be a mapping defined on the edge set of G. Signed graphs were introduced by Harary [5] in connection with the study of theory of social balance in social psychology proposed by Heider [6]. The *adjacency matrix* of a signed graph $\Gamma = (G, \sigma)$ is a square matrix $A(\Gamma) = A(G, \sigma) = (a^{\sigma}_{ij})$, where $a^{\sigma}_{ij} = \sigma(v_i v_j)a_{ij}$ and $A(G) = (a_{ij})$ is the adjacency matrix of G. The characteristic polynomial of a matrix A is denoted by $\varphi(A)$. If Γ is a signed graph, $\varphi(\Gamma)$ denotes $\varphi(A(\Gamma))$. The eigenvalues of the adjacency matrix of a graph are often referred to as the eigenvalues of the graph. The spectrum of a signed graph Γ is the set of all eigenvalues of Γ along with their multiplicities. Let $m(\lambda)$ denote the multiplicity of the eigenvalue λ. The spectrum of graphs, in particular, signed graphs has been studied extensively by many authors, for instance see [1, 3, 4].

Let (K_n, H^-) be a signed complete graph whose negative edges induce a subgraph H. In this paper, by a constructive method, we obtain $n - r - 1 + \lfloor \frac{r}{2} \rfloor$ eigenvalues of (K_n, P_r^-), where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Next, we determine the characteristic polynomial of (K_n, P_r^-), for $2 \leqslant r \leqslant 8$. Let $(K_n, \bigcup_{i=1}^m P_{r_i}^-)$ be a signed complete graph whose negative edges induce a subgraph which is the disjoint union of m distinct paths. In the sequel, we find $n - 1 + \Sigma_{i=1}^m \big(\lfloor \frac{r_i}{2} \rfloor - r_i\big)$ eigenvalues of $(K_n, \bigcup_{i=1}^m P_{r_i}^-)$.

2. Eigenvalues of $(K_n, \bigcup_{i=1}^m P_{r_i}^-)$

In this section, we study the spectrum of $(K_n, \bigcup_{i=1}^m P_{r_i}^-)$. Before stating the main theorem, we need the following results.

Theorem 2.1. [7, Theorem 2.2] *Let $T_n(a, b, c)$ be an $n \times n$ tridiagonal matrix defined by*

$$T_n(a,b,c) = \begin{bmatrix} a & c & & \mathbf{0} \\ b & \ddots & \ddots & \\ & \ddots & \ddots & c \\ \mathbf{0} & & b & a \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$*. Then the eigenvalues of* $T_n(a, b, c)$ *are*

$$\lambda_k = a - 2\sqrt{bc}\, cos\, \frac{k\pi}{n+1},\ for\ k = 1, \ldots, n.$$

Theorem 2.2. [8, Theorem 2] *Let* $A_n(a, b, c)$ *be an* $n \times n$ *special tridiagonal matrix defined by*

$$A_n(a,b,c) = \begin{bmatrix} \sqrt{bc}+a & c & & & & \\ b & a & \ddots & & \mathbf{0} & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & \ddots & a & c \\ & & & & b & a \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$*. Then the eigenvalues of* $A_n(a, b, c)$ *are*

$$\lambda_k = a + 2\sqrt{bc}\, cos\, \frac{(2k-1)\pi}{2n+1},\ for\ k = 1, \ldots, n.$$

Corollary 2.3. [2, Corollary 1] *Let* (K_n, H^-) *be a signed complete graph whose negative edges induce a subgraph* H *of order* $k < n$*. Then*

$$\varphi(K_n, H^-) = (\lambda+1)^{n-k-1}\varphi\left(\begin{bmatrix} A(K_k, H^-) & (n-k)J_{k\times 1} \\ J_{1\times k} & n-k-1 \end{bmatrix}\right),$$

and so $m(-1) \geqslant n - k - 1$*.*

Now, we prove the main results.

Theorem 2.4. *Let* $\Gamma = (K_n, P_r^-)$ *be a signed complete graph. Then the following statements hold:*

(a) -1 *is an eigenvalue of* Γ *with the multiplicity at least* $n - r - 1$*.*

(b) If r *is odd, then* $\frac{r-1}{2}$ *eigenvalues of* Γ *are*

$$\lambda_k = -1 - 4\, cos\, \frac{2k\pi}{r+1},\ for\, k = 1, \ldots, \frac{r-1}{2}.$$

(c) If r is even, then $\frac{r}{2}$ eigenvalues of Γ are

$$\lambda_k = -1 + 4\,cos\,\frac{(2k-1)\pi}{r+1},\ \ for\,k = 1, \ldots, \frac{r}{2}.$$

Proof. (a) If $r < n$, by Corollary 2.3, we have $m(-1) \geqslant n - r - 1$. If $r = n$, there is nothing to proof.

For Parts (b) and (c), we assume that $r < n$. The proof for the case $r = n$ is similar. By Corollary 2.3, we have

$$\varphi(K_n, P_r^-) = (\lambda + 1)^{n-r-1} \det D,$$

where

$$D = \begin{bmatrix} \lambda I_r - A(K_r, P_r^-) & (r-n)J_{r\times 1} \\ -J_{1\times r} & \lambda + 1 + r - n \end{bmatrix}.$$

Let

$$\lambda I_r - A(K_r, P_r^-) = \begin{bmatrix} \lambda & 1 & & \mathbf{-1} \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ \mathbf{-1} & & 1 & \lambda \end{bmatrix}.$$

We apply finitely many elementary row and column operations on the matrix D to obtain the matrix $\begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$, where C is a tridiagonal matrix.

(b) First, suppose that $r \geqslant 7$. Consider the matrix D and add the last r columns to the first column. This leads to the following matrix,

$$D_1 = \begin{bmatrix} \lambda+3-n & 1 & & & & & r-n \\ \lambda+5-n & \lambda & 1 & & \mathbf{-1} & & r-n \\ \vdots & 1 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \lambda+5-n & & & \ddots & \ddots & 1 & r-n \\ \lambda+3-n & & \mathbf{-1} & & 1 & \lambda & r-n \\ \lambda+1-n & & & & & -1 & \lambda+1+r-n \end{bmatrix}.$$

Next, subtract the 2th row from all the lower rows except the last two rows and then subtract the first row from the rth row to obtain the following matrix,

$$D_2 = \begin{bmatrix} \lambda+3-n & 1 & -1 & -1 & \cdots & \cdots & \cdots & -1 & r-n \\ \lambda+5-n & \lambda & 1 & -1 & \cdots & \cdots & \cdots & -1 & r-n \\ 0 & -\lambda+1 & \lambda-1 & 2 & & & & & 0 \\ \vdots & -\lambda-1 & 0 & \lambda+1 & \ddots & & \mathbf{0} & & \vdots \\ \vdots & \vdots & -2 & 2 & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & -\lambda-1 & -2 & & \mathbf{0} & \ddots & \ddots & 2 & \vdots \\ 0 & -2 & 0 & & & & 2 & \lambda+1 & 0 \\ \lambda+1-n & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & \lambda+1+r-n \end{bmatrix}.$$

Now, add the $(r-i)$th column to the $(i+1)$th column, $i = 1, \ldots, \frac{r-3}{2}$. Next, subtract the ith row from the $(r+1-i)$th row, for $i = 3, \ldots, \frac{r-1}{2}$. Hence one can obtain the following matrix,

$$D_3 = \begin{bmatrix} \lambda+3-n & X_1 & -J_{1\times\frac{r-1}{2}} & r-n \\ \lambda+5-n & X_2 & -J_{1\times\frac{r-1}{2}} & r-n \\ \mathbf{0}_{\frac{r-5}{2}\times 1} & Y_{\frac{r-5}{2}\times\frac{r-1}{2}} & \mathbf{0}_{\frac{r-5}{2}\times\frac{r-1}{2}} & \mathbf{0}_{\frac{r-5}{2}\times 1} \\ 0 & X_3 & X_4 & 0 \\ \mathbf{0}_{\frac{r-1}{2}\times 1} & \mathbf{0}_{\frac{r-1}{2}} & \lambda I - T_{\frac{r-1}{2}} & \mathbf{0}_{\frac{r-1}{2}\times 1} \\ \lambda+1-n & X_5 & -J_{1\times\frac{r-1}{2}} & \lambda+1+r-n \end{bmatrix},$$

where

$$Y = \begin{bmatrix} -\lambda+1 & \lambda-1 & 2 & & & & \\ -\lambda-1 & 0 & \lambda+1 & \ddots & & \mathbf{0} & \\ \vdots & -2 & 2 & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \\ -\lambda-1 & -2 & \mathbf{0} & & 2 & \lambda+1 & 2 \end{bmatrix},$$

and

$$X_1 = \begin{bmatrix} 0 & -2 & \cdots & -2 & -1 \end{bmatrix}_{1\times\frac{r-1}{2}},$$
$$X_2 = \begin{bmatrix} \lambda-1 & 0 & -2 & \cdots & -2 & -1 \end{bmatrix}_{1\times\frac{r-1}{2}},$$
$$X_3 = \begin{bmatrix} -\lambda-1 & -2 & 0 & \cdots & 0 & 4 & \lambda+1 \end{bmatrix}_{1\times\frac{r-1}{2}},$$
$$X_4 = \begin{bmatrix} 2 & 0 & \cdots & 0 \end{bmatrix}_{1\times\frac{r-1}{2}},$$
$$X_5 = \begin{bmatrix} -2 & \cdots & -2 & -1 \end{bmatrix}_{1\times\frac{r-1}{2}}.$$

Also, $T_{\frac{r-1}{2}} = T_{\frac{r-1}{2}}(-1,-2,-2)$, see Theorem 2.1. Note that if $r = 7$, then $X_2 = \begin{bmatrix} \lambda-1 & 0 & -1 \end{bmatrix}$, $X_3 = \begin{bmatrix} -\lambda-1 & 2 & \lambda+1 \end{bmatrix}$ and $Y = \begin{bmatrix} -\lambda+1 & \lambda-1 & 2 \end{bmatrix}$. If $r = 9$, then we have $X_3 = \begin{bmatrix} -\lambda-1 & -2 & 4 & \lambda+1 \end{bmatrix}$. Now, apply the cyclic permutation $(1, 2, \dots, r+1)$ on the index of columns and rows of D_3. This leads to the following matrix,

$$D_4 = \begin{bmatrix} \lambda+1+r-n & \lambda+1-n & X_5 & -J_{1\times\frac{r-1}{2}} \\ r-n & \lambda+3-n & X_1 & -J_{1\times\frac{r-1}{2}} \\ r-n & \lambda+5-n & X_2 & -J_{1\times\frac{r-1}{2}} \\ \mathbf{0}_{\frac{r-5}{2}\times 1} & \mathbf{0}_{\frac{r-5}{2}\times 1} & Y_{\frac{r-5}{2}\times\frac{r-1}{2}} & \mathbf{0}_{\frac{r-5}{2}\times\frac{r-1}{2}} \\ 0 & 0 & X_3 & X_4 \\ \mathbf{0}_{\frac{r-1}{2}\times 1} & \mathbf{0}_{\frac{r-1}{2}\times 1} & \mathbf{0}_{\frac{r-1}{2}} & \lambda I - T_{\frac{r-1}{2}} \end{bmatrix}.$$

Let D_5 be the principle submatrix of D_4 over the rows and columns $1, 2, \dots, \frac{r+3}{2}$. Therefore the following holds:

$$\varphi(K_n, P_r^-) = (\lambda+1)^{n-r-1} \det(\lambda I - T_{\frac{r-1}{2}}) \det D_5.$$

If $r < 7$, then we apply elementary row and column operations on the matrix D as we did above, one can see that

$$\varphi(K_n, P_5^-) = (\lambda+1)^{n-6} \det(\lambda I - T_2) \det \begin{bmatrix} \lambda+6-n & \lambda+1-n & -2 & -1 \\ 5-n & \lambda+3-n & 0 & -1 \\ 5-n & \lambda+5-n & \lambda-1 & 1 \\ 0 & 0 & 3-\lambda & \lambda-1 \end{bmatrix}, \tag{1}$$

and

$$\varphi(K_n, P_3^-) = (\lambda+1)^{n-4}(\lambda+1)\det\begin{bmatrix} \lambda+4-n & \lambda+1-n & -1 \\ 3-n & \lambda+3-n & 1 \\ 3-n & \lambda+5-n & \lambda \end{bmatrix}, \quad (2)$$

where $T_2 = T_2(-1,-2,-2)$. So, by Theorem 2.1, the proof of Part (b) is complete.

(c) By a similar argument as we did in Part (b), one can obtain the following matrix which is equivalent to the matrix D, when $r \geqslant 6$.

$$D'_4 = \begin{bmatrix} \lambda+1+r-n & \lambda+1-n & -2J_{1\times\frac{r-2}{2}} & -J_{1\times\frac{r}{2}} \\ r-n & \lambda+3-n & X'_1 & -J_{1\times\frac{r}{2}} \\ r-n & \lambda+5-n & X'_2 & -J_{1\times\frac{r}{2}} \\ \\ \mathbf{0}_{\frac{r-6}{2}\times 1} & \mathbf{0}_{\frac{r-6}{2}\times 1} & Y_{\frac{r-6}{2}\times\frac{r-2}{2}} & \mathbf{0}_{\frac{r-6}{2}\times\frac{r}{2}} \\ \\ 0 & 0 & X'_3 & X'_4 \\ \\ \mathbf{0}_{\frac{r}{2}\times 1} & \mathbf{0}_{\frac{r}{2}\times 1} & \mathbf{0}_{\frac{r}{2}\times\frac{r-2}{2}} & \lambda I - A_{\frac{r}{2}} \end{bmatrix},$$

where $A_{\frac{r}{2}} = A_{\frac{r}{2}}(-1,-2,-2)$, see Theorem 2.2. Also

$$X'_1 = \begin{bmatrix} 0 & -2 & \cdots & -2 \end{bmatrix}_{1\times\frac{r-2}{2}},$$

$$X'_2 = \begin{bmatrix} \lambda-1 & 0 & -2 & \cdots & -2 \end{bmatrix}_{1\times\frac{r-2}{2}},$$

$$X'_3 = \begin{bmatrix} -\lambda-1 & -2 & 0 & \cdots & 0 & 2 & \lambda+3 \end{bmatrix}_{1\times\frac{r-2}{2}},$$

$$X'_4 = \begin{bmatrix} 2 & 0 & \cdots & 0 & \end{bmatrix}_{1\times\frac{r}{2}}.$$

Note that if $r=6$, then we have $X'_2 = \begin{bmatrix} \lambda-1 & 0 \end{bmatrix}$, $X'_3 = \begin{bmatrix} 1-\lambda & \lambda+1 \end{bmatrix}$ and also the matrices $\mathbf{0}_{\frac{r-6}{2}\times 1}$, $Y_{\frac{r-6}{2}\times\frac{r-2}{2}}$, $\mathbf{0}_{\frac{r-6}{2}\times\frac{r}{2}}$ are removed. If $r=8$, then $X'_3 = \begin{bmatrix} -\lambda-1 & 0 & \lambda+3 \end{bmatrix}$, and if $r=10$, then $X'_3 = \begin{bmatrix} -\lambda-1 & -2 & 2 & \lambda+3 \end{bmatrix}$. Let D'_5 be the principle submatrix of D'_4 over the rows and columns $1,2,\ldots,\frac{r+2}{2}$. Then the following holds:

$$\varphi(K_n, P_r^-) = (\lambda+1)^{n-r-1}\det(\lambda I - A_{\frac{r}{2}})\det D'_5.$$

Similarly, if $r < 6$, then one can see that

$$\varphi(K_n, P_4^-) = (\lambda{+}1)^{n-5} \det(\lambda I{-}A_2) \det \begin{bmatrix} \lambda+5-n & \lambda+1-n & -2 \\ 4-n & \lambda+3-n & 0 \\ 4-n & \lambda+5-n & \lambda+1 \end{bmatrix}, \tag{3}$$

and

$$\varphi(K_n, P_2^-) = (\lambda+1)^{n-3}(\lambda-1) \det \begin{bmatrix} \lambda+3-n & \lambda+1-n \\ 2-n & \lambda+3-n \end{bmatrix}, \tag{4}$$

where $A_2 = A_2(-1,-2,-2)$. Hence by Theorem 2.2, the proof of Part (c) is complete. $\square$

By a similar argument as we did in the proof of Theorem 2.4, we can find $n-1+\Sigma_{i=1}^{m}\big(\lfloor \frac{r_i}{2} \rfloor - r_i\big)$ eigenvalues of $(K_n, \bigcup_{i=1}^{m} P_{r_i}^-)$.

Corollary 2.5. *Let* $\Gamma = (K_n, \bigcup_{i=1}^{m} P_{r_i}^-)$ *be a signed complete graph. Then the following statements hold:*

(a) -1 *is an eigenvalue of* Γ *with the multiplicity at least* $n-1-\Sigma_{i=1}^{m} r_i$.

(b) *If* r_i *is odd* $(1 \leqslant i \leqslant m)$, *then* $\frac{r_i-1}{2}$ *eigenvalues of* Γ *are*

$$\lambda_k = -1 - 4\,cos\,\frac{2k\pi}{r_i+1},\ \ for\, k = 1, \ldots, \frac{r_i-1}{2}.$$

(c) *If* r_i *is even* $(1 \leqslant i \leqslant m)$, *then* $\frac{r_i}{2}$ *eigenvalues of* Γ *are*

$$\lambda_k = -1 + 4\,cos\,\frac{(2k-1)\pi}{r_i+1},\ \ for\, k = 1, \ldots, \frac{r_i}{2}.$$

In the sequel we would like to determine $\varphi(K_n, P_r^-)$, for $2 \leqslant r \leqslant 8$. By Equations (1), (2), (3), (4) and what we did in the proof of Theorem 2.4, we have the following result.

Corollary 2.6. *The characteristic polynomials of signed complete graphs* (K_n, P_r^-), *for* $r = 2,3,4,5,6,7,8$ *are as follows:*

$$\varphi(K_n, P_2^-) = (\lambda+1)^{n-3}(\lambda-1)\Big(\lambda^2 + (4-n)\lambda + 7 - 3n\Big),$$

$$\varphi(K_n, P_3^-) = (\lambda+1)^{n-3}\Big(\lambda^3 + (3-n)\lambda^2 + (3-2n)\lambda + 7n - 23\Big),$$

$$\varphi(K_n, P_4^-) = (\lambda+1)^{n-5}(\lambda^2-5)\Big(\lambda^3+(5-n)\lambda^2+(15-4n)\lambda+n-5\Big),$$

$$\varphi(K_n, P_5^-) = (\lambda+1)^{n-5}(\lambda-1)(\lambda+3)\Big(\lambda^3+(3-n)\lambda^2+(7-2n)\lambda+11n-51\Big),$$

$$\varphi(K_n, P_6^-) = (\lambda+1)^{n-7}\prod_{k=1}^{3}(\lambda-\lambda_k)\Big(\lambda^4+(6-n)\lambda^3+(24-5n)\lambda^2+(n-6)\lambda+13n-73\Big),$$

where $\lambda_k = -1 + 4\,cos\,\dfrac{(2k-1)\pi}{7}$.

$$\varphi(K_n, P_7^-) = (\lambda+1)^{n-8}\prod_{k=1}^{3}(\lambda-\lambda_k)f(\lambda),$$

where $\lambda_k = -1 - 4\,cos\,\frac{2k\pi}{8}$ *and*

$$f(\lambda) = \lambda^5+(5-n)\lambda^4+(18-4n)\lambda^3+(10n-54)\lambda^2+(28n-179)\lambda+113-17n.$$

$$\varphi(K_n, P_8^-) = (\lambda+1)^{n-9}\prod_{k=1}^{4}(\lambda-\lambda_k)g(\lambda),$$

where $\lambda_k = -1 + 4\,cos\,\frac{(2k-1)\pi}{9}$ *and*

$$g(\lambda) = \lambda^5+(7-n)\lambda^4+(34-6n)\lambda^3+6\lambda^2+(30n-211)\lambda+9n-61.$$

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