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Cross Efficiency Malmquist Index to Investigate the Productivity Change in DEA

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Abstract. In traditional data envelopment analysis (DEA), the efficiency and productivity changes computations are based on an optimistic perspective and an efficient unit may perform rather poorly when the realist weights are assigned to inputs and outputs. Hence, the results of productivity change of decision making units (DMUs) between two time periods may change from progress to regress or vice versa when the weights are modified. Because of the cross efficiency merits, we use it to obtain a common set of weights so called common set of cross weights. On the other hand, we need a base for comparing the productivity change of DMUs. To this end, the common set of cross weights are used to approximate the cross efficient frontier as a base for determining cross Malmquist (CM) index for evaluating the productivity change. This leads to introduce a new efficiency, weight efficiency, and the decomposition of the cross efficiency. Some DEA and cross efficiency models are modified to find the value of the proposed CM index and its components. An empirical example is used to compare the proposed method and the technical Malmquist index.

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1 Introduction

The basic idea of the cross efficiency is to use data envelopment analysis (DEA) [6] in a peer-evaluation instead of a self-evaluation [11]. The best relative efficiency can be achieved for each decision making unit (DMU) by the weights obtained in self evaluations, while in peer-evaluation each DMU to be evaluated with the weights determined by the other DMUs. These are some of the advantages of the cross efficiency method: The producer eliminates unrealistic weight schemes without requiring the elicitation of weight restrictions from application area experts as well as the cross efficiency discriminates effectively to differentiate between good and poor performance [1].

Doyle and Green [10] presented mathematical formulations for possible implementations of aggressive and benevolent cross efficiencies. The aggressive models are focused on obtaining the set of optimal weights where not only maximize the efficiency of the DMU under evaluation but also to minimize the average efficiency of the other DMUs. The idea in the benevolent method is to choose the set of optimal weights maximizing not only the efficiency of the DMU under evaluation but also the average efficiency of other DMUs. To avoid the difficulty of choosing between the aggressive and benevolent formulations, Wang and Chin [28] proposed the neutral DEA model to determine a set of input and output weights for each DMU without being aggressive or benevolent to the others which maximizes the relative efficiency of each output and reduces the number of zero weights of outputs. However, the neutral DEA model cannot reduce the number of zero weights among inputs. To resolve this problem, Wang et al. [29] proposed a simultaneously input-and output-oriented weight determination DEA model for the cross efficiency evaluation.

DEA allows individual DMUs to select the most advantageous weights in calculating their efficiency. Therefore, there is a high flexibility in DEA for selecting the weights and it leads to comparison of DMUs on uncommon bases. To this end, the Common Set of Weights (CSW) method in DEA was initially introduced by Cook et al. [8] and developed by Roll et al. [23]. In this paper, we use cross efficiency of DMUs to obtain a common set of weights (CSW) so called common set of cross weights. Accordingly, we need a common base to determine the productivity changes of DMUs from one periods to another.

To estimate productivity change of a DMU over time, Malmquist index (MI) that was first introduced in literature by Caves et al. [5] have been the most commonly used [7, 15]. Using DEA efficiency scores, Färe et al. [13] decomposed the constant returns to scale Malmquist index (MI) into technical efficiency change (TEC) and technical change (TC). Infeasibility can occur when linear programming (LP) techniques are used to compute and decompose of this index. Pastor and Lovell [20] demonstrated that the source of above problem is the specification of adjacent period technologies. A global index with technology formed from data of all DMUs in all periods. It is immune to LP infeasibility and it satisfies circularity. In other words, a common base technology can be used to make the circular Malmquist index [21]. Maniadakis and Thanassoulis [18] developed a productivity index that accounts not only for technical efficiency and technical changes but also for allocative efficiency and for the effects of input price change. The cost Malmquist index was extended by Tohidi et al. [27] into the profit Malmquist index. Also, Tohidi et al. [26] proposed a global cost Malmquist productivity index, that used the weighted average of the inputs' costs for different periods of time and obtained a global cost efficient frontier. To calculate the global Malmquist productivity index, Kao [17] proposed a common-weights DEA model for time-series evaluates. Tohidi and Razavyan [25] suggested a new index referred to as the circular global profit Malmquist productivity index to remove the difficulty caused by different frontiers in calculating profit efficiency changes and profit efficiency components changes. This index is applicable when the input costs and output prices are known.

Han et al. [16] used an improved DEA cross-model to analyze the performance of China ethylene plants. They used cross model to allocate weights for inputs and outputs, and determined the productivity change of China ethylene plants by using their improved DEA cross-model. However, there may be alternative optimal solutions in relation to their improved DEA cross-model, and so the optimal weights for the inputs and outputs may not be unique, while the current paper uses two modified cross efficiency models with unique optimal solution. This development leads to generate unique inputs and output weights for determining unique cross efficiencies under different periods of time.

In traditional DEA, a DMU receives the most favorable efficiency score relative to its peers [19]. This may lead to distorted view of performance and productivity change by producing scores for some DMUs that overestimated their efficiency. To this end, this paper uses cross efficiency of DMUs to obtain a CSW so called common set of cross weights. The proposed weights are used to approximate the cross efficient frontier as a same base to determine cross Malmquist (CM) index for evaluating the productivity change. This leads to introduce a new efficiency, weight efficiency (WE), and the decomposition of the cross efficiency. Then, some DEA and cross efficiency models are modified to find the value of the proposed CM index and its components. The modified cross efficiency models have unique optimal solution. This leads to generate unique input and output weights for determining cross efficiencies. Therefore, all of DMUs are compared with a unique point, which is considered as reference point.This is a fundamental matter to investigate the productivity changes. An empirical example is used to compare the proposed method and the technical Malmquist index.

The rest of this paper is organized as follows. The next section describes a background of DEA. The cross efficient frontier and its properties are presented in Section 3. The CM index and its components are provided in Section 4. In Section 5, we modify some DEA and cross efficiency models to obtain the CM index and its components. Section 6 presents an illustrative empirical example, and finally conclusions are made in Section 7.

2 Background

Let x_{ij} and y_{rj} be the positive levels of the $i^{th}(i = 1, \ldots, m)$ input and $r^{th}(r = 1, \ldots, s)$ output, respectively, of the $j^{th}(j = 1, \ldots, n)$ DMU. The efficiency scores which is called the technical efficiency (TE), of DMUs, say DMU_k, can usually be measured by the following CCR model [6]:

$$
\begin{aligned}
E_{kk} = \max \quad & \sum_{r=1}^{s} u_{rk} y_{rk} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_{ik} x_{ik} = 1 \\
& \sum_{r=1}^{s} u_{rk} y_{rj} - \sum_{i=1}^{m} v_{ik} x_{ij} \leq 0, j = 1, \ldots, n \\
& u_{rk}, v_{ik} \geq 0, r = 1, \ldots, s, i = 1, \ldots, m,
\end{aligned}
\tag{1}
$$

where $u_{rk}(r = 1, \ldots, s)$ and $v_{ik}(i = 1, \ldots, m)$ are decision variables which represent the relative importance of outputs and inputs of the under evaluation DMU, DMU_k, respectively.By using the optimal weights of model (1),$(v_k^*, u_k^*) = (v_{1k}^*, \ldots, v_{mk}^*, u_{1k}^*, \ldots, u_{sk}^*)$, the cross efficiency of $\mathrm{DMU}_j(j = 1, \ldots, n, j \neq k)$ is calculated as follows:

$$
E_{kj} = \frac{\sum_{r=1}^{s} u_{rk}^* y_{rj}}{\sum_{i=1}^{m} v_{ik}^* x_{ij}}.
$$

For $\mathrm{DMU}_j(j = 1, \ldots, n)$, the average of $E_{kj}(k = 1, \ldots, n)$, $C(x_j, y_j) = \frac{1}{n} \sum_{k=1}^{n} E_{kj}$, is treated as the cross efficiency (CE) score for DMU_j.

To solve the problem of non-uniqueness of the optimal weights, the aggressive and benevolent models were introduced [10]. The aggressive and benevolent cross efficiencies are usually not the same. To avoid the difficulty of choosing between the aggressive and benevolent formulations, and to reduce simultaneously the number of zero weights of inputs and outputs, Wang et al. [29] proposed the following neutral model to determine a set of input and output weights for each DMU.

$$
\begin{array}{ll}
\max & \delta+\gamma \\
\text{s.t.} & \sum_{r=1}^{s} u_{rk} y_{rk}=E_{kk} \\
& \sum_{i=1}^{m} v_{ik} x_{ik}=1 \\
& \sum_{r=1}^{s} u_{rk} y_{rj}-\sum_{i=1}^{m} v_{ik} x_{ij} \leq 0, j=1, \ldots, n, j \neq k \\
& u_{rk} y_{rk}-\delta \geq 0, r=1, \ldots, s \\
& v_{ik} x_{ik}-\gamma \geq 0, i=1, \ldots, m \\
& \delta, \gamma, u_{rk}, v_{ik} \geq 0, r=1, \ldots, s, i=1, \ldots, m.
\end{array} \tag{2}
$$

where δ and γ represent the minimum relative efficiency of the s outputs and m inputs of DMU_k, respectively. So, the economic meaning of model (2) can be interpreted as a set of input weights for DMU_k to choose its each output and each input as efficient, while keeping its CCR efficiency (E_{kk}) unchanged, such that each input can be sufficiently utilized and every output can produce sufficient efficiency as an individual.

3 Cross Efficient Frontier

To determine the productivity change and its components by using the cross efficiency, it is necessary to approximate the cross efficient frontier. To this end, this paper uses the cross efficiency scores of DMUs and generates a common set of cross weights. To obtain a common set of cross weights the following model is proposed:

$$
\begin{array}{ll}
\min & \delta \\
\text{s.t.} & \delta \geq z_j^{+}, j=1, \ldots, n \\
& \delta \geq z_j^{-}, j=1, \ldots, n \\
& \dfrac{\sum_{r=1}^{s} u_r y_{rj}+z_j^{+}}{\sum_{i=1}^{m} v_i x_{ij}-z_j^{-}}=C(x_j, y_j), j=1, \ldots, n \\
& u_r, v_i \geq \varepsilon>0, z_j^{-}, z_j^{+} \geq 0, r=1, \ldots, s, i=1, \ldots, m, j=1, \ldots, n,
\end{array} \tag{3}
$$

where ε is a non-Archimedean infinitesimal positive number. Let $(\overline{v}, \overline{u})$ is the optimal weights of model (3). The cross efficient frontier (CEF) is defined as $CEF = \{(x, y)|\overline{u}y - \overline{v}x = 0\}$. In other words, for DMU$_j (j = 1, \ldots, n)$ the term $\frac{\overline{u}y_j}{\overline{v}x_j}$ is considered as an approximation of its cross efficiency. In order to approximation of CEF, the fraction $\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}$ must be increased by the numerator increasing and/or the denominator decreasing. To implement this, the DMU must minimize the sum of the total virtual gaps to the CEF by adding z_j^+ to $\sum_{r=1}^{s} u_r y_{rj}$ and taking z_j^- away from $\sum_{i=1}^{m} v_i x_{ij}$. The objective function δ minimizes the maximum deviations from z_j^+ and z_j^-.

Indeed, model (3) determines $(\overline{v}, \overline{u})$ as a common set of cross weights to approximate $C(x_j, y_j)$ by $\frac{\overline{u}y_j}{\overline{v}x_j}$. Model (3) can be converted to the following linear programming model:

$$\begin{array}{ll} \min & \delta \\ \text{s.t.} & \delta \geq z_j^+, j = 1, \ldots, n \\ & \delta \geq z_j^-, j = 1, \ldots, n \\ & \displaystyle\sum_{r=1}^{s} u_r y_{rj} + z_j^+ - C(x_j, y_j)(\sum_{i=1}^{m} v_i x_{ij} - z_j^-) = 0, j = 1, \ldots, n \\ & u_r, v_i \geq \varepsilon > 0, z_j^-, z_j^+ \geq 0, r = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n. \end{array} \tag{4}$$

Model (4) is feasible and it has a finite optimal value. Hence, model (4) always provides a common set of weights to determine the cross efficient frontier as a base for finding the productivity change and its components by cross efficiency score.

Using $(v^*, u^*, z^{+*}, z^{-*}, \delta^*)$ as optimal solution of model (4) the CEF is defined as $CEF = \{(x, y)|u^*y - v^*x = 0\}$, where $z^{+*} = (z_1^{+*}, \ldots, z_n^{+*})$, $z^{-*} = (z_1^{-*}, \ldots, z_n^{-*})$, $v^* = (v_1^*, \ldots, v_m^*)$ and $u^* = (u_1^*, \ldots, u_s^*)$. The hyperplane $CEF = \{(x, y)|u^*y - v^*x = 0\}$ divides the space of R^{m+s} into two halfspaces where $T_c = \{(x, y)|x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda_j \geq 0, j = 1, \ldots n, \}$ is in one of its halfspaces. We assume a pair of non-negative input $x \in R^m$ and output $y \in R^s$ an activity and express them by the notation (x, y).

If DMU$_k$ is cross efficient, CEF is a supporting hyperplane of T_c at (x_k, y_k), otherwise CEF is not a supporting hyperplane of T_c. The following Theorems and Corollary demonstrate these properties.

Theorem 1: *Let* $(v^*, u^*, z^{+*}, z^{-*}, \delta^*)$ *be an optimal solution of model* (4). *Then,* $\frac{u^* y_j}{v^* x_j} \leq C(x_j, y_j)$ *for* $j = 1, \ldots, n$.
Proof: The proof is evident. $\square$
Theorem 2: *Let* $(v^*, u^*, z^{+*}, z^{-*}, \delta^*)$ *be an optimal solution of model* (4) *and* $H^- = \{(x, y) | u^* y - v^* x \leq 0\}$. *If* $(x, y) \in T_c$, *then* $(x, y) \in H^-$.
Proof: Since $z_j^{-*}, z_j^{+*} \geq 0, j = 1, \ldots, n,$ we have $1 \geq C(x_j, y_j) = \frac{u^* y_j + z_j^{+*}}{v^* x_j - z_j^{-*}} \geq \frac{u^* y_j}{v^* x_j}$ and so $u^* y_j - v^* x_j \leq 0, j = 1, \ldots, n$. Therefore, for each $j \in \{1, \ldots, n\}, (x_j, y_j) \in H^-$. On the other hand, for each $(x, y) \in T_c$ there exists $\widehat{\lambda}$ such that $x \geq \sum_{j=1}^n \widehat{\lambda}_j x_j$ and $y \leq \sum_{j=1}^n \widehat{\lambda}_j y_j$. Hence, using $v^* \geq 0$ and $u^* \geq 0$ we have $-v^* x \leq -\sum_{j=1}^n v^* \widehat{\lambda}_j x_j$ and $u^* y \leq \sum_{j=1}^n u^* \widehat{\lambda}_j y_j$, and therefore, $u^* y - v^* x \leq \sum_{j=1}^n \widehat{\lambda}_j (u^* y_j - v^* x_j) \leq 0$. Hence, $(x, y) \in H^-$. $\square$

Corollary: *If* $C(x_j, y_j) < 1, j = 1, \ldots, n,$ *then* $CEF \cap T_c = \phi$.

Theorem 3: *For one input and one output case, the technical frontier and the CEF are the same under the constant returns-to-scale (CRS) assumption.*
Proof: In the case of one input and one output, and under the CRS assumption, the optimal solutions of CCR model for all DMUs are proportional. Therefore, there is only one CCR efficient face. In this case, all rows of the cross matrix are the same and for each $j \in \{1, \ldots, n\}$, $C(x_j, y_j) = E_{jj}$, where E_{jj} is the optimal objective value of the model (1). This completes the proof. $\square$

3.1 Illustrative Example

Table 1 shows ten units with two inputs (x_1 and x_2) and 1 output (y).

The vector $(v^*, u^*) = (v_1^*, v_2^*, u^*) = (0.00010, 0.00010, 0.00042)$ is the optimal solution of model (4). Therefore, the CEF is as $CEF = \{(x_1, x_2, y) : 0.0001x_1 + 0.0001x_2 = 0.00042y\}$. For $y = 1$ we have $CEF = \{(x_1, x_2, 1) : x_1 + x_2 = 4.2\}$. Indeed, the CEF is a straight line with a slop of $-\frac{v_1^*}{v_2^*} = -\frac{1}{1} = -1$ as designated by the CEF in Fig. 1. The last row of the Table 1, indicates the cross efficiency of DMUs by the

Table 1: The data of example, the technical and the cross efficiencies

DMU	A	B	C	D	E	F	H	I	J	K
Input1	4	7	8	4	2	10	9	8.5	5.5	5
Input2	3	3	1	2	4	1	2	3.5	4	4.5
Output	1	1	1	1	1	1	1	1	1	1
E_{jj}	0.8571	0.6316	1	1	1	1	0.7059	0.5333	0.6316	0.6316
$C(x_j, y_j)$	0.6978	0.7224	0.7224	0.6978	0.6978	0.4675	0.7224	0.7224	0.6978	0.6978
CE	0.5981	0.4187	0.4652	0.6978	0.6978	0.3806	0.3806	0.3489	0.4407	0.4407

cross efficient frontier. For instance, for DMU$_A$, $CE_A = \frac{OA_1}{OA} = 0.5981 < TE_A = \frac{OA_2}{OA} = 0.8571$.

In Fig. 1, $OG \perp CEF$, $BG \perp OG$ and $AG_3 \perp OG$; "$\perp$" is a perpendicular symbol. Therefore, AG_3 and GB are parallel to A_1B_1 and hence, using the trigonometric properties, we have:

$$\begin{array}{l} CE_B = \frac{OB_1}{OB} = \frac{OB_1}{OB_2} \times \frac{OB_2}{OB} = \frac{OG_1}{OG}(BG \| B_1G_1), \\ CE_A = \frac{OA_1}{OA} = \frac{OA_1}{OA_2} \times \frac{OA_2}{OA} = \frac{OG_1}{OG_3}(AG_3 \| A_1G_1). \end{array}$$

It can be seen that the DMUs A and B are compared with the different points A_2 and B_2 on different faces of the technical efficient frontier. Therefore, the DMUs A and B cannot be compared with toghether, while all of the DMUs are compared with the point G_1, as a unique point, on the CEF. In other words, all of DMUs are compared with a unique point, which is considered as reference point. This is a fundamental matter to investigate the productivity changes [17]. For DMU$_A$, we consider the relative distance of A_1 and A_2 to obtain the ratio $0 \leq \frac{OA_1}{OA_2} \leq 1$, and we refer to the $\frac{OA_1}{OA_2}$ as weight efficiency (WE) score of DMU$_A$. When the most preferred weights, say the CCR model optimal weights, are assigned to the inputs and outputs, we have $\frac{OA_1}{OA_2} = 1$, otherwise $\frac{OA_1}{OA_2} < 1$.

Therefore, the weight efficiency (WE) provides a measure of effect the changing of the weights (v, u) from the most preferred weights to the common set of cross weights. Hence, the WE is defined as follows:

$$WE = \frac{CE}{TE}. \tag{5}$$

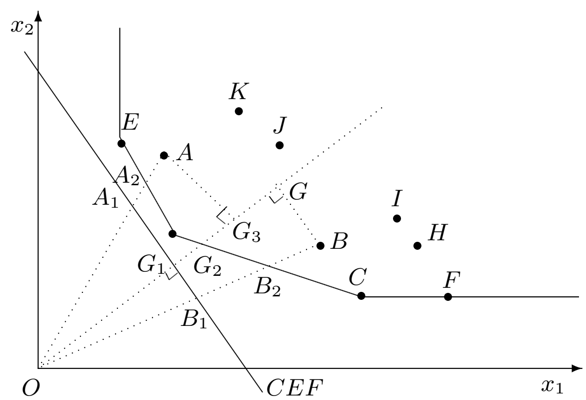

Figure 1. Illustration of the cross efficient frontier (CEF)

According to (5), the cross efficiency (CE) is decomposed as a product of the CCR efficiency score (TE) and the weight efficiency (WE).

4 The Cross Malmquist Productivity Index

This paper uses the cross efficient frontier at periods t and $t+1$, and proposes a new Malmquist index, the cross efficiency Malmquist index, to investigate the productivity change over time.

Let (v^t, u^t) and (v^{t+1}, u^{t+1}) be the common set of cross weights obtained by model (4) for the DMUs of the periods of time t and $t+1$, respectively. Using (v^t, u^t) and (v^{t+1}, u^{t+1}), the CEF at time periods t and $t+1$ are defined as follows, respectively:

$$CEF^t = \{(x, y) | u^t y - v^t x = 0\},$$
$$\text{and } CEF^{t+1} = \{(x, y) | u^{t+1} y - v^{t+1} x = 0\}.$$

Let t be the reference time period. Using $C^t(x_j^t, y_j^t)$ and $C^t(x_j^{t+1}, y_j^{t+1})$ respectively as the cross efficiencies of $\text{DMU}_j (j = 1, \ldots, n)$ at time periods t and $t+1$, the cross Malmquist (CM) index for $\text{DMU}_j (j = 1, \ldots, n)$ is defined as

$$CM^t(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) = \frac{C^t(x_j^{t+1}, y_j^{t+1})}{C^t(x_j^t, y_j^t)}. \tag{6}$$

Similarly, when $t+1$ is the reference time period, using $C^{t+1}(x_j^t, y_j^t)$ and $C^{t+1}(x_j^{t+1}, y_j^{t+1})$ the CM index for $\text{DMU}_j (j = 1, \dots, n)$ is defined as

$$CM^{t+1}(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) = \frac{C^{t+1}(x_j^{t+1}, y_j^{t+1})}{C^{t+1}(x_j^t, y_j^t)}. \tag{7}$$

Since the reference time periods are not the same, to avoid an arbitrary choice of a reference period, the geometric mean of $CM^t(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1})$ and $CM^{t+1}(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1})$ is considered as the CM index as shown in (8):

$$CM(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) = \left[CM^t(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) \times CM^{t+1}(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1})\right]^{1/2}. \tag{8}$$

$CM > 1$ indicates progress in the productivity of DMU_j from period t to $t+1$, while $CM = 1$ and $CM < 1$, respectively indicate statue quo and deterioration in productivity.

The indexes (6), (7) and (8) are similar to those used by Balk [3, 2] and Färe and Grosskopf [13]. Fig. 2 illustrates the $CM(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1})$ index, where units A_1, B_1, C_1 and D_1 observed in time period t use two inputs (x_1, x_2) to produce one output $y = 1$. These units move to A_2, B_2, C_2 and D_2, respectively, in time period $t+1$. The frontier technologies at periods t and $t+1$ are $L_1A_1B_1C_1M_1$ and $L_2A_2B_2C_2M_2$, respectively. In addition, CEF^t and CEF^{t+1} are the cross efficient frontiers at time periods t and $t+1$,respectively. By using CEF^t and CEF^{t+1} in Fig. 2, for unit D_1 we have:

$$CM^t(D_1, D_2) = \frac{OG/OD_2}{OK/OD_1} \text{ and } CM^{t+1}(D_1, D_2) = \frac{OE/OD_2}{OI/OD_1}.$$

Therefore,

$$CM(D_1, D_2) = \left[\frac{OG/OD_2}{OK/OD_1} \times \frac{OE/OD_2}{OI/OD_1}\right]^{1/2}.$$

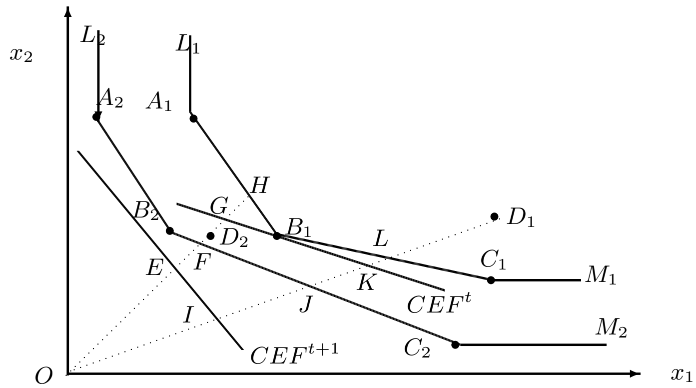

Figure 2. The interpretation of the CEF under the CRS assumption

4.1 Decomposition of the CM Index

4.1.1 The First Stage of Decomposition

The proposed index, $CM(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1})$, can be decomposed into the cross efficiency change (CEC) and the cross technical change (CTC) as follows:

$$\begin{aligned} CM(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) &= CEC(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) \times CTC(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) \\ &= \frac{c^{t+1}(x_j^{t+1}, y_j^{t+1})}{c^t(x_j^t, y_j^t)} \times \left[\frac{c^t(x_j^{t+1}, y_j^{t+1})}{c^{t+1}(x_j^{t+1}, y_j^{t+1})} \times \frac{c^t(x_j^t, y_j^t)}{c^{t+1}(x_j^t, y_j^t)}\right]^{1/2}, \end{aligned}$$

where the component outside the brackets (CEC) is the cross efficiency change and the term inside the brackets (CTC) is the cross technical change. The CEC provides a new measure of the cross technical change and shows the catch up effect from the period t to the period $t+1$. $CEC(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) > 1$ indicates that there is an increase in the cross efficiency score of DMU$_j$ from period t to $t+1$, while $CEC(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) < 1$ and $CEC(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) = 1$ respectively show regress and no change in the cross efficiency score of DMU$_j$.

By using CEF^t and CEF^{t+1} in Fig. 2, for units D_1 and D_2 we have:

$$\begin{aligned} CM(D_1,D_2) &= \frac{OE/OD_2}{OK/OD_1}\left[\frac{OG/OD_2}{OE/OD_2}\times\frac{OK/OD_1}{OI/OD_1}\right]^{1/2} \\ &= \frac{OE/OD_2}{OK/OD_1}\left[\frac{OG}{OE}\times\frac{OK}{OI}\right]^{1/2}. \end{aligned}$$

Using the cross efficiency change (CEC) and the technical efficiency change (TEC), the weight efficiency change (WEC) is defined as

$$WEC(x_j^t,y_j^t,x_j^{t+1},y_j^{t+1}) = \frac{CEC(x_j^t,y_j^t,x_j^{t+1},y_j^{t+1})}{TEC(x_j^t,y_j^t,x_j^{t+1},y_j^{t+1})} \tag{9}$$

$$= \frac{\frac{C^{t+1}(x_j^{t+1},y_j^{t+1})}{C^t(x_j^t,y_j^t)}}{\frac{D^{t+1}(x_j^{t+1},y_j^{t+1})}{D^t(x_j^t,y_j^t)}} = \frac{\frac{C^{t+1}(x_j^{t+1},y_j^{t+1})}{D^{t+1}(x_j^{t+1},y_j^{t+1})}}{\frac{C^t(x_j^t,y_j^t)}{D^t(x_j^t,y_j^t)}}$$

where $D^p(x_j^p,y_j^p)=\max\{\theta|(\frac{x_j^p}{\theta},y_j^p)\in T_c^p\}$ for $p=t,t+1$ is the distance function of DMU$_j$ in the periods of time t and $t+1$ [24]. Therefore, the weight change factor is the ratio of the weight effects of periods t and $t+1$. By using Fig. 2 the weight change factor, $WEC(D_1,D_2)$, is approximated as

$$WEC(D_1,D_2) = \frac{\frac{OE/OD_2}{OF/OD_2}}{\frac{OK/OD_1}{OL/OD_1}} = \frac{OE/OF}{OK/OL}.$$

4.1.2 The Second Stage of Decomposition

In the second stage of decomposition, CEC is separated into the effect of the common set of weights and the technical change when DMU$_j$ moves from the time period t to $t+1$. Hence, the CEC component is decomposed as

$$CEC_j(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) = \frac{C^{t+1}(x_j^{t+1}, y_j^{t+1})}{C^t(x_j^t, y_j^t)} \tag{10}$$

$$= \frac{D^{t+1}(x_j^{t+1}, y_j^{t+1})}{D^t(x_j^t, y_j^t)} \times \frac{\frac{C^{t+1}(x_j^{t+1}, y_j^{t+1})}{D^{t+1}(x_j^{t+1}, y_j^{t+1})}}{\frac{C^t(x_j^t, y_j^t)}{D^t(x_j^t, y_j^t)}}.$$

The first term on the right hand side of (10) is the catch-up component [9] and the second term refers to the weight change factor (see (9)) between time periods t and $t+1$.

The CEC decomposition in (10) indicates that part of the cross efficiency change occurs due to the change in the technical efficiency score and the other part occurs due to the change in the weight efficiency score. The first term on the right hand side of (10) is the technical efficiency change (TEC), and the second term refers to the weight efficiency change (WEC) between time periods t and t+1. Clearly, TEC component represents a change in technical efficiency score of DMU$_j$ due to changes in the most favorable input/output weights from period t to period t+1.

The numerator and the denominator of WEC component represent the weight efficiency scores of DMU$_j$ respectively in time periods t and t+1. As stated before, the weight efficiency score determines the rate of change of the efficiency of DMU$_j$ when the proposed common cross weights are assigned to the inputs and outputs instead of the most favorable weights for this DMU. Therefore, WEC>1 indicates that, in period t+1 compared to the period t, the proposed common input/output cross weights were closer to the most favorable input/output weights of DMU$_j$, while WEC<1 indicates that the difference between the common cross weights and the most favorable input/output weights of DMU$_j$ increases from period t to period t+1. Also, WEC=1 means the difference between the two sets of weights is the same in two periods t and t+1.

In Fig. 2, for units D_1 and D_2, the component $CEC(D_1, D_2)$ is computed by using CEF^t and CEF^{t+1} as follows:

$$CEC(D_1, D_2) = \frac{OF/OD_2}{OL/OD_1} \times \frac{\frac{OE/OD_2}{OF/OD_2}}{\frac{OK/OD_1}{OL/OD_1}} = \frac{OE/OD_2}{OK/OD_1}.$$

The second component of CM index, CTC, can be decomposed as

$$\begin{aligned} CTC(x_j^t, y_j^t, x_j^{t+1}, y_j^{t+1}) =& \Big[\frac{D^t(x_j^{t+1}, y_j^{t+1})}{D^{t+1}(x_j^{t+1}, y_j^{t+1})} \times \frac{D^t(x_j^t, y_j^t)}{D^{t+1}(x_j^t, y_j^t)}\Big]^{1/2} \\ & \times \left[\frac{\frac{C^t(x_j^{t+1}, y_j^{t+1})}{D^t(x_j^{t+1}, y_j^{t+1})}}{\frac{C^{t+1}(x_j^{t+1}, y_j^{t+1})}{D^{t+1}(x_j^{t+1}, y_j^{t+1})}} \times \frac{\frac{C^t(x_j^t, y_j^t)}{D^t(x_j^t, y_j^t)}}{\frac{C^{t+1}(x_j^t, y_j^t)}{D^{t+1}(x_j^t, y_j^t)}}\right]^{1/2}. \end{aligned} \tag{11}$$

The first term in the right hand side of (11) is the technical change (TC) component [12] that reflects the shift of the production boundary between periods t and t+1, measured along rays (x_j^{t+1}, y_j^{t+1}) and (x_j^t, y_j^t). The second square brackets in (11), refers to the residual impact (RI) component. It is clear that the numerator and the denominator of the first ratio in the RI component represent the weight efficiency scores of (x_j^{t+1}, y_j^{t+1}), which have been evaluated by considering the production technologies of periods t and t+1 as the reference time period, respectively. This means the first ratio of the IR component captures the change in the weight efficiency score of (x_j^{t+1}, y_j^{t+1}) that occurs due to the change in the best practice gap between the production boundary and the cross efficient boundary from period t to period t+1, along ray (x_j^{t+1}, y_j^{t+1}). The second ratio of the IR component can also be interpreted similarly, but along ray (x_j^t, y_j^t). It can be said that the two ratios that make the IR component measure the residual impact of relative common weight changes on the shift of the cross efficient boundary from period t to period t+1.

For units D_1 and D_2 in Fig. 2, the $CTC(D_1, D_2)$ is determined as

$$\begin{aligned} CTC(D_1, D_2) &= \left[\frac{OH/OD_2}{OF/OD_2} \times \frac{OL/OD_1}{OJ/OD_1}\right]^{1/2} \times \left[\frac{\frac{OG/OD_2}{OH/OD_2}}{\frac{OE/OD_2}{OF/OD_2}} \times \frac{\frac{OK/OD_1}{OL/OD_1}}{\frac{OI/OD_1}{OJ/OD_1}}\right]^{1/2} \\ &= \left[\frac{OG}{OE} \times \frac{OK}{OI}\right]^{1/2}. \end{aligned}$$

Further decompositions of the CM index by using the pure technical and the scale efficiency is possible [13]. For instance, by $CE = WE \times TE$ and three-factor decomposition of [22] a four-factor decomposition for CM index is obtained. Similarly, using $CE = WE \times TE$ and four-factor decomposition of [4] a five-factor decomposition for CM index can be generated.

5 Computation of the CM Index and Its Components

Assume that, in time period $t(t = 1, \ldots, T)$, $\text{DMU}_j(j = 1, \ldots, n)$ consumes the amount x_{ij}^t of input $i(i = 1, \ldots, m)$ to produce the amount y_{rj}^t of output $r(r = 1, \ldots, s)$. For DMU_j, the cross efficiency score is considered as

$$C^p(x_j^q, y_j^q) = \frac{1}{n}\sum_{k=1}^{n} E_{kj} = \frac{1}{n}\sum_{k=1}^{n} \frac{\sum_{r=1}^{s} u_{rk}^{p*} y_{rj}^{q}}{\sum_{i=1}^{m} v_{ik}^{p*} x_{ij}^{q}},\ p, q = t, t+1, j = 1, \ldots, n.$$

To determine $C^p(x_j^q, y_j^q)$ when $p = q = t, t+1$, the optimal weights, i.e. $(v_{1k}^{p*}, \ldots, v_{mk}^{p*}, u_{1k}^{p*}, \ \ldots, \ u_{sk}^{p*})$, can be determined by the following model:

$$\begin{array}{ll}
\max & \delta^p + \gamma^p \\
\text{s.t.} & \sum_{r=1}^{s} u_{rk}^p y_{rk}^p = E_{kk} \\
& \sum_{i=1}^{m} v_{ik}^p x_{ik}^p = 1 \\
& \sum_{r=1}^{s} u_{rk}^p y_{rj}^p - \sum_{i=1}^{m} v_{ik}^p x_{ij}^p \leq 0, j = 1, \ldots, n \\
& u_{rk}^p y_{rk}^p - \delta^p \geq 0, r = 1, \ldots, s \\
& v_{ik}^p x_{ik}^p - \gamma^p \geq 0, i = 1, \ldots, m \\
& \delta^p, \gamma^p, u_{rk}^p, v_{ik}^p \geq 0, r = 1, \ldots, s, i = 1, \ldots, m.
\end{array} \tag{12}$$

Theorem 4: *The optimal solution of model (12) is unique.*
Proof: In the case of $p = q, p, q = t, t+1$, the efficiency score E_{kk} is

determined by the corresponding CCR model. By summing the constraints $u^p_{rk}y^p_{rk} - \delta^p \geq 0, r = 1, \ldots, s$, we have $E_{kk} = \sum_{r=1}^{s} u^p_{rk}y^p_{rk} \geq s\delta^p$, and hence, $0 \leq \delta^p \leq \frac{E_{kk}}{s}$. Similarly, the constraints $v^p_{ik}x^p_{ik} - \gamma^p \geq 0, i = 1, \ldots, m$ leads to $1 = \sum_{i=1}^{m} v^p_{ik}x^p_{ik} \geq m\gamma^p$. Therefore, $0 \leq \gamma^p \leq \frac{1}{m}$. If we set $\delta^{p*} = \frac{E_{kk}}{s}$, and $\gamma^{p*} = \frac{1}{m}$, then we have $u^p_{rk} \geq \frac{\delta^{p*}}{y^p_{rk}} = \frac{E_{kk}}{sy^p_{rk}}, r = 1, \ldots, s$ and $v^p_{ik} \geq \frac{\gamma^{p*}}{x^p_{ik}} = \frac{1}{mx^p_{ik}}, i = 1, \ldots, m$. Let $\Psi^* = (u^{p*}_{1k}, \ldots, u^{p*}_{sk}, v^{p*}_{1k}, \ldots, v^{p*}_{mk}, \delta^{p*}, \gamma^{p*}) = (\frac{E_{kk}}{sy^p_{1k}}, \ldots, \frac{E_{kk}}{sy^p_{sk}}, \frac{1}{mx^p_{1k}}, \ldots, \frac{1}{mx^p_{mk}}, \frac{E_{kk}}{s}, \frac{1}{m})$. We prove that the Ψ^* is the unique optimal solution of model (12). Ψ^* satisfies the constraints $E_{kk} = \sum_{r=1}^{s} u^p_{rk}y^p_{rk}$ and $1 = \sum_{i=1}^{m} v^p_{ik}x^p_{ik}$. By inserting Ψ^* in the ratio of ratio form of the CCR model [9], we have $E_{kk} = \max\limits_{u^p_k \geq 0, v^p_k \geq 0} \frac{\frac{\sum_{r=1}^{s} u^p_{rk}y^p_{rk}}{\sum_{i=1}^{m} v^p_{ik}x^p_{ik}}}{\max\limits_{1 \leq j \leq n}\left\{\frac{\sum_{r=1}^{s} u^p_{rk}y^p_{rj}}{\sum_{i=1}^{m} v^p_{ik}x^p_{ij}}\right\}} = \frac{E_{kk}}{\max\limits_{1 \leq j \leq n}\left\{\frac{\sum_{r=1}^{s} u^{p*}_{rk}y^p_{rj}}{\sum_{i=1}^{m} v^{p*}_{ik}x^p_{ij}}\right\}}$. Therefore, $1 = \max\limits_{1 \leq j \leq n}\left\{\frac{\sum_{r=1}^{s} u^{p*}_{rk}y^p_{rj}}{\sum_{i=1}^{m} v^{p*}_{ik}x^p_{ij}}\right\} = \max\limits_{1 \leq j \leq n}\left\{\frac{\sum_{r=1}^{s} \frac{E_{kk}y^p_{rj}}{sy^p_{rk}}}{\sum_{i=1}^{m} \frac{x^p_{ij}}{mx^p_{ik}}}\right\}$. and hence, $\frac{\sum_{r=1}^{s} \frac{E_{kk}y^p_{rj}}{sy^p_{rk}}}{\sum_{i=1}^{m} \frac{x^p_{ij}}{mx^p_{ik}}} \leq 1$ and then $\sum_{r=1}^{s} \frac{E_{kk}y^p_{rj}}{sy^p_{rk}} - \sum_{i=1}^{m} \frac{x^p_{ij}}{mx^p_{ik}} \leq 0$ and $\sum_{r=1}^{s} u^{p*}_{rj}y^p_{rj} - \sum_{i=1}^{m} v^{p*}_{ij}x^p_{ij} = \sum_{r=1}^{s} \frac{E_{kk}y^p_{rj}}{sy^p_{rk}} - \sum_{i=1}^{m} \frac{x^p_{ij}}{mx^p_{ik}} \leq 0, j = 1, \ldots, n$. Therefore, Ψ^* satisfies all of the constraints of model (12) and is its feasible solution. To complete the proof, let there is another optimal solution, say $\hat{\Psi} = (\hat{u}^p_{1k}, \ldots, \hat{u}^p_{sk}, \hat{v}^p_{1k}, \ldots, \hat{v}^p_{mk}, \hat{\delta}^p, \hat{\gamma}^p) \neq \Psi^*$. Since $0 \leq \delta^p, \hat{\delta}^p \leq \frac{E_{kk}}{s}$, $0 \leq \gamma^p, \hat{\gamma}^p \leq \frac{1}{m}$, $\delta^{p*} = \frac{E_{kk}}{s}$ and $\gamma^{p*} = \frac{1}{m}$, we must have $\delta^{p*} = \hat{\delta}^p = \frac{E_{kk}}{s}$ and $\delta^{p*} = \hat{\delta}^p = \frac{1}{m}$. Now, let there is $f \in \{1, \ldots, s\}$ such that $\hat{u}^p_{fk} \neq u^{p*}_{fk}$. There are two cases:

Case 1: $\hat{u}^p_{fk} < u^{p*}_{fk}$. Since $\delta^{p*} = \hat{\delta}^p = \frac{E_{kk}}{s}$ and $\hat{u}^p_{fk} \geq \frac{\hat{\delta}^p}{y^p_{fk}} = \frac{E_{kk}}{sy^p_{fk}}$, hence the case $\hat{u}^p_{fk} < u^{p*}_{fk}$ is impossible.

Case 2: $\hat{u}^p_{fk} > u^{p*}_{fk}$. Since $E_{kk} = \sum_{r=1}^{s} u^{p*}_{rk}y^p_{rk} = \sum_{r=1}^{s} \hat{u}^p_{rk}y^p_{rk}$ and $u^{p*}_{rk}, y^p_{rk}, \hat{u}^p_{rk} \geq 0$, in this case we must have $e \in \{1, \ldots, s\}$ such that $\hat{u}^p_{ek} < u^{p*}_{ek}$. According to case 1, $\hat{u}^p_{ek} < u^{p*}_{ek}$ is impossible. Therefore, $(u^{p*}_{1k}, \ldots, u^{p*}_{sk}) = (\hat{u}^p_{1k}, \ldots, \hat{u}^p_{sk})$. Since $\sum_{i=1}^{m} v^{p*}_{ik}x^p_{ik} = \sum_{i=1}^{m} \hat{v}^p_{ik}x^p_{ik} = 1$, similarly, it can be proven that $(v^{p*}_{1k}, \ldots, v^{p*}_{sk}) = (\hat{v}^p_{1k}, \ldots, \hat{v}^p_{sk})$, and hence $\hat{\Psi} = \Psi^*$. This concludes the proof. □

In calculating $C^p(x^q_j, y^q_j)$ when $p, q = t, t+1, p \neq q$ the optimal weights are the optimal solution of the following modified model:

$$\begin{array}{ll} \max & \delta^p + \gamma^p \\ \text{s.t.} & \sum_{r=1}^{s} u_{rk}^p y_{rk}^q = E_{kk} \\ & \sum_{i=1}^{m} v_{ik}^p x_{ik}^q = 1 \\ & \sum_{r=1}^{s} u_{rk}^p y_{rj}^p - \sum_{i=1}^{m} v_{ik}^p x_{ij}^p \leq 0, j = 1, \ldots, n \\ & u_{rk}^p y_{rk}^q - \delta^p \geq 0, r = 1, \ldots, s \\ & v_{ik}^p x_{ik}^q - \gamma^p \geq 0, i = 1, \ldots, m \\ & \delta^p, \gamma^p, u_{rk}^p, v_{ik}^p \geq 0, r = 1, \ldots, s, i = 1, \ldots, m. \end{array} \tag{13}$$

Theorem 5: *The optimal solution of model (13) is unique.*
Proof: The proof of this theorem is similar to the proof of Theorem 4. □

According to Theorem 5, the unique optimal solution of model (13) is as follows:

$$\begin{aligned} \Delta^* &= (u_{1k}^{p*}, \ldots, u_{sk}^{p*}, v_{1k}^{p*}, \ldots, v_{mk}^{p*}, \delta^{p*}, \gamma^{p*}) \\ &= (\frac{E_{kk}}{sy_{1k}^q}, \ldots, \frac{E_{kk}}{sy_{sk}^q}, \frac{1}{mx_{1k}^q}, \ldots, \frac{1}{mx_{mk}^q}, \frac{E_{kk}}{s}, \frac{1}{m}). \end{aligned}$$

Therefore, by using Theorems 4 and 5, the $C^p(x_j^q, y_j^q)$, $p, q = t, t+1$, are uniquely determined by Δ^* and Ψ^*. Based on the CRS assumption the term $D^p(x_j^q, y_j^q)$ can be computed using $D^p(x_j^q, y_j^q) = \min\{\theta_j | (\theta_j x_j^q, y_j^q) \in T_c^p\}$, where

$$T_c^p = \{(x, y) | x \geq \sum_{j=1}^{n} \lambda_j x_j^p, y \leq \sum_{j=1}^{n} \lambda_j y_j^p, \lambda_j \geq 0, j = 1, \ldots, n\}$$

and $p, q = t, t+1$.

6 Numerical Example

This paper applies the proposed CM index to analyze the productivity changes in semiconductor packaging and testing firms in Taiwan between the years 2000 and 2003 ($t = 1, \ldots, 4$) [14]. There are 15 companies with one input, x_1=Liability ratio (%), and four outputs: (i) y_1 = Growth rate (%), (ii) y_2 = Net profit after tax ($100 million NT dollars), (iii) y_3 = Profitability ratio (%), and (iv) y_4 = Output value by employee ($million/people). The input and outputs of the DMUs over four years is listed in Table 2.

Applying model (4) to the data in Table 2, the u^{*t}, v^{*t} and δ^{*t} for the different periods of time ($t = 1, \ldots, 4$) are obtained. Columns 2, 3 and 4 of Table 3 show the u^{*t}, v^{*t} and δ^{*t} of model (4), respectively, for the time period t ($t = 1, \ldots, 4$). The last column of Table 3 shows the CEF for the time periods $t = 1, \ldots, 4$. For instance, for $t = 1$ we have $(u^{*t}, v^{*t}, \delta^{*t}) = (0.001114, 0.001438, 0.000100, 0.005368, 0.011271,\ 0.000021)$. Hence, the CEF for the first time period is as $0.01114y_1 + 0.01438y_2 + 0.001y_3 + 0.05368y_4 = 0.11271x_1$.

The columns 3, 4 and 5 of the Table 4 show the Malmquist index (MI$(t, t+1)$) and the columns 6, 7 and 8 of Table 4 present the cross Malmquist index (CM$(t, t{+}1)$), for all of the DMUs from the time period t to $t + 1$ for $t = 1, 2$ and 3. As can be seen from Table 4, for DMU$_{10}$ the CM$(1, 2) = 1.0322 > 1$ indicates the progress from period 1 to 2 in the productivity, while CM$(1, 2)$ for other DMUs is less than 1 and it indicates the deterioration in productivity from period time 1 to 2.

It can be seen that, in some periods of time the productivity change of some DMUs using cross and technical efficient frontiers are not the same. For instance, for DMU$_9$ the MI$(1, 2) = 1.035 > 1$ and it indicates the progress in the productivity from the time period 1 to 2, while by using the cross Malmquist index we have CM$(1, 2) = 0.8493 < 1$. It indicates the deterioration in the productivity from the time period 1 to 2. This case shows that the Malmquist index at different periods using different frontier facets may produce misleading results.

Looking at the average values in Table 4, for example, we can see that the average cross Malmquist index of all of the DMUs from the first period to the second period (0.6231) is less than their average cross Malmquist index from the second period to the third period (1.1474). The average of columns 7 and 8 show that the average growth of all of the DMUs from the second period to the third period is greater than their growth from period 3 to period 4. A similar discussion can be

Table 2: Table 2. The inputs and outputs of 15 DMUs [14]

Year 2000	Firms	y_1	y_2	y_3	y_4	x_1
1	ASE	145.86	98.37	122.87	3.5	38.26
2	SIPIN	158.16	72.21	117.09	3.56	32.84
3	OSE	146.85	41.04	100.73	2.19	31.12
4	ChipMos	128.82	55.39	118.71	4.11	33.8
5	KYEC	239.66	51.78	128.17	1.41	43.37
6	Sharp Chunh Li	284.76	55.9	121.02	3.47	50.9
7	Sharp in Taiwan	157.53	58.19	135.43	3.31	28.55
8	Greatek	154.48	45.25	114.15	2.68	44.83
9	Lingsen	153.12	43.38	110.27	2.07	26.09
10	PowerTech	344.42	42.5	118.85	1.46	56.07
11	UTC	136.54	49.02	125.65	4.49	23.01
12	KingPak	200.28	38.75	98.05	22.27	53.41
13	Hi-Sincerity	100.75	40.25	101.68	12.37	38.89
14	Formosa	143.13	41.77	110.24	2.37	58.83
15	Sigurd	135.29	41.5	114.49	1.98	32.05
Year 2001						
1	ASE	80.35	18.57	89.55	3.4	41.46
2	SIPIN	87.71	28.17	92.84	2.5	38.11
3	OSE	75.04	8.1	70.14	1.98	56.19
4	ChipMos	65.79	24.91	72.58	3.24	31.91
5	KYEC	92.71	32.08	79.57	1.44	53.48
6	Sharp Chunh Li	64.8	40.57	101.16	2.66	38.12
7	Sharp in Taiwan	78.55	37.6	94.05	2.75	25.01
8	Greatek	89.43	42.48	107.48	2.74	41.8
9	Lingsen	71.17	41.25	105.34	1.87	20.73
10	PowerTech	234.47	41.73	105.56	3.57	43.3
11	UTC	38.25	31.1	33.83	2.43	24.64
12	KingPak	33.53	39.17	96.14	7.68	48.35
13	Hi-Sincerity	70.15	40.21	102.02	11.32	37.24
14	Formosa	59.51	41.22	111.86	1.76	58.27
15	Sigurd	82.7	40.08	100.93	1.91	26.29
Year 2002						
1	ASE	125	41.29	100.5	4.2	42.5
2	SIPIN	134.9	44.25	101.91	2.79	43.28
3	OSE	119.56	7	74.16	2.65	64.18
4	ChipMos	118.57	27.92	81.49	3.21	44.48
5	KYEC	137.94	36.97	94.33	1.76	49.08
6	Sharp Chunh Li	105.22	43.66	107.09	2.29	30.66
7	Sharp in Taiwan	118.37	37.99	95.79	2.74	32.12
8	Greatek	134.67	46.34	114.19	3.36	36.48
9	Lingsen	125.4	36.33	87.51	2.13	25.67
10	PowerTech	90.74	41.87	106.63	2.8	34.86
11	UTC	159.26	36.73	84.73	3.17	22.31
12	KingPak	98.79	38.87	94.68	4.38	54.26
13	Hi-Sincerity	96.83	39.64	96.43	11.59	39.12
14	Formosa	162.59	40.92	105.5	2.51	55.16
15	Sigurd	143.22	42.38	120.12	2.31	43.77
Year 2003						
1	ASE	122.85	67.43	108.71	3.11	41.08
2	SIPIN	122.8	68.39	110.37	2.99	45.06
3	OSE	105.91	5.64	74.6	2.72	66.88
4	ChipMos	129.77	48.61	110.17	3.36	39.43
5	KYEC	126.91	47.73	111.39	2.38	33.89
6	Sharp Chunh Li	116.65	41.83	103.04	2.08	34.23
7	Sharp in Taiwan	140.26	51.91	117.79	2.68	34.58
8	Greatek	116.1	49.27	117.88	3.42	35.63
9	Lingsen	133.22	43.69	109.43	2.43	30.28
10	PowerTech	155.44	50.4	123.72	3.21	45.67
11	UTC	107.53	39.92	99.63	2.93	19.95
12	KingPak	59.82	40.94	107.4	2.87	44.82
13	Hi-Sincerity	101.98	39.35	93.68	11.05	40.29
14	Formosa	122.77	41.91	109.3	2.9	54.62
15	Sigurd	149.37	44.18	123.66	2.47	34.16

Table 3: The optimal solution of model (4) and the CEF for the different periods of time

Period	u^{t*}	v^{t*}	δ^{t*}	Cross Efficient Frontier
t=1	(0.001114, 0.001438, 0.000100, 0.005368)	0.011271	0.000021	$0.01114y_1 + 0.01438y_2 + 0.001y_3 + 0.05368y_4 = 0.11271x_1$
t=2	(0.000169, 0.000348, 0.000100, 0.002857)	0.002038	0.000004	$0.0169y_1 + 0.0348y_2 + 0.01y_3 + 0.2857y_4 = 0.2038x_1$
t=3	(0.000100, 0.001074, 0.003807, 0.011319)	0.018549	0.000038	$0.001y_1 + 0.01074y_2 + 0.03807y_3 + 0.11319y_4 = 0.18549x_1$
t=4	(0.000473, 0.000932, 0.000100, 0.001462)	0.005130	0.000011	$0.0473y_1 + 0.0932y_2 + 0.01y_3 + 0.1462y_4 = 0.513x_1$

Table 4: The cross Malmquist index and Malmquist index of DMUs

DMU	Firms	MI(1,2)	MI(2,3)	MI(3,4)	CM(1,2)	CM(2,3)	CM(3,4)
1	ASE 0.417	1.3004	1.1856	0.34	1.528	1.081	1.1856
2	SIPIN	0.469	1.17	1.057	0.4626	1.1026	1.0916
3	OSE	0.22	0.895	0.828	0.2683	1.0542	0.9089
4	ChipMos	0.573	0.892	1.451	0.5725	0.8651	1.487
5	KYEC	0.373	1.426	1.799	0.3936	1.3571	1.6318
6	Sharp Chunh Li	0.642	1.467	0.871	0.555	1.3838	0.891
7	Sharp in Taiwan	0.759	0.897	1.083	0.6813	0.8492	1.1452
8	Greatek	0.904	1.48	1.005	0.8049	1.3258	1.0151
9	Lingsen	1.035	0.756	0.983	0.8493	0.8051	0.999
10	PowerTech	0.732	0.729	0.999	1.0323	0.9817	0.9863
11	UTC	0.32	2.757	1.067	0.3835	2.3247	1.0838
12	KingPak	0.295	1.247	0.955	0.4265	0.8689	1.1333
13	Hi-Sincerity	0.982	0.992	0.923	0.9041	0.9688	0.9612
14	Formosa	0.715	1.625	1.009	0.6635	1.2704	0.9565
15	Sigurd	0.996	0.6	1.364	0.9322	0.7534	1.3294
Average		0.624	1.231	1.098	0.6231	1.1474	1.1204

made for the other columns of Table 4.

Table 5 shows the cross efficiencies using model (2) and CEF. The columns 3, 5, 7 and 9 of the Table 5 show the cross efficiency by using CEF for the time periods 1, 2, 3, and 4, respectively. The columns 4, 6, 8and 10 of Table 5 represent the cross efficiency of DMUs by using model (2).

It can be seen that, the cross scores using model (2) and the CEF are very close together. For instance, the last row of Table 5 shows the average of the cross efficiency of the DMUs using model (2) and the CEF for $t = 1$ and 3 are the same.

Table 6 illustrates the results of the weight efficiency change (WEC), cross efficiency change (CEC) and cross technical change (CTC) from period t to the period $t + 1 (t = 1, 2, 3)$, and the averages are shown in the last row of this Table. For example, the cross efficiency change for DMU_1 from period 1 to period 2 is 0.5893, i.e., there is a decrease in cross efficiency for DMU_1 from period 1 to period 2. However, there is an increase in cross efficiency for DMU_1 from period 2 to period 3.

The numerator and the denominator of WEC component represent

Table 5: The Neutral cross efficiency and its approximation by using the proposed CEF

		t=1		t=2		t=3		t=4	
DMU	Firms	CE(CEF)	CE(2)	CE(CEF)	CE(2)	CE(CEF)	CE(2)	CE(CEF)	CE(2)
1	ASE	0.777	0.7771	0.4579	0.4579	0.6177	0.6178	0.6472	0.6472
2	SIPIN	0.84	0.84	0.5283	0.5283	0.5986	0.5986	0.5938	0.5938
3	OSE	0.6971	0.6971	0.2459	0.2459	0.2787	0.2787	0.1948	0.1948
4	ChipMos	0.675	0.675	0.558	0.558	0.4708	0.4708	0.6063	0.6064
5	KYEC	0.7404	0.7405	0.3567	0.3568	0.4751	0.4751	0.6854	0.6854
6	Sharp Chunh Li	0.7469	0.7469	0.5505	0.5506	0.8634	0.8635	0.6124	0.6124
7	Sharp in Taiwan	0.9029	0.903	0.8554	0.8556	0.7525	0.7525	0.7354	0.7354
8	Greatek	0.5206	0.5206	0.5688	0.5688	0.7921	0.7921	0.6436	0.6437
9	Lingsen	0.8677	0.8678	0.9999	1	0.8586	0.8586	0.7613	0.7614
10	PowerTech	0.7353	0.7353	0.8482	0.8483	0.7604	0.7604	0.5873	0.5873
11	UTC	0.9999	1	0.5497	0.5498	1	1	0.9999	1
12	KingPak	0.6782	0.6783	0.5159	0.516	0.4587	0.4587	0.354	0.354
13	Hi-Sincerity	0.5629	0.5629	0.9008	0.9009	0.7587	0.7587	0.5344	0.5344
14	Formosa	0.367	0.367	0.3419	0.3419	0.4791	0.4792	0.4009	0.4009
15	Sigurd	0.6437	0.6438	0.811	0.8111	0.6691	0.6692	0.7295	0.7296
Average		0.717	0.717	0.6059	0.606	0.6556	0.6556	0.6057	0.6058

the weight efficiency scores of DMU_j respectively in time periods t and t+1. As stated in section 3, the weight efficiency score determines the rate of change of the efficiency of DMU_j when the proposed common cross weights are assigned to the inputs and outputs instead of the most favorable weights for this DMU. Therefore, according to Table 6 WEC(1,2)=1.0516>1 for DMU_1 indicates that, in period 2 compared to the period 1, the proposed common input/output cross weights were closer to the most favorable input/output weights of DMU_1, while WEC(3,4)=0.0818<1 for DMU_1 indicates that the difference between the common cross weights and the most favorable input/output weights of DMU_1 increases from period 3 to period 4. Also, WEC(3,4)=1 for DMU_{11} means the difference between the two sets of weights is the same in two periods 3 and 4.

Looking at the average values in Table 6, for example, we can see that the average CEC(1,2) index of all of the DMUs from the first period to the second period (0.8709) is lower than their averages CM(2,3) index from time periods 2 and 3 to time periods 3 and 4, respectively. A similar discussion can be made for the average of the other columns of Table 6.

7 Conclusions

The traditional DEA allows each DMU, to select the most advantageous weights in measuring efficiency and this leads to comparison of DMUs on uncommon base. In this paper, we proposed a method based on cross efficiency to generate a common set of weights and determine a common base so called common set of cross weights. Using the optimal solution of the presented model, we defined a hyperplane as the cross efficient frontier. Then some theorems for this concept and the continuation of discussion are presented. The cross efficient frontier was used to introduce a new Malmquist index, the cross Malmquist index and determine the productivity changes. Additionally, weight efficiency, as a new efficiency, and the decomposition of cross efficiency have been introduced. The proposed CM index decomposed into CEC and CTC, and a further decomposition for CEC and CTC have been proposed. An empirical example is used to analyze the proposed method and determine the introduced indices.

Table 6: Weight efficiency change, cross efficiency change and cross technical change of DMUs

DMU	Firms	WEC(1,2)	WEC(2,3)	WEC(3,4)	CEC(1,2)	CEC(2,3)	CEC(3,4)	CTC(1,2)	CTC(2,3)	CTC(3,4)
1	ASE	1.0516	1.1804	0.818	0.5893	1.3491	1.0477	0.7076	0.964	1.1316
2	SIPIN	1.001	1.0906	0.8121	0.6289	1.1331	0.9919	0.7356	0.9731	1.1005
3	OSE	0.8457	1.2215	0.7237	0.3527	1.1336	0.6988	0.7608	0.93	1.3007
4	ChipMos	0.9713	1.0702	1.0215	0.8266	0.8437	1.288	0.6926	1.0254	1.1545
5	KYEC	1.1056	1.046	1.0373	0.4818	1.3318	1.4426	0.8169	1.019	1.1311
6	Sharp Chunh Li	1.1834	0.9779	1.0317	0.7371	1.5683	0.7093	0.7529	0.8824	1.2563
7	Sharp in Taiwan	1.0021	1.0027	1.0198	0.9474	0.8796	0.9773	0.7191	0.9655	1.1718
8	Greatek	1.0626	0.9993	0.9692	1.0926	1.3926	0.8126	0.7367	0.952	1.2493
9	Lingsen	1.1272	0.9566	0.9751	1.1523	0.8587	0.8867	0.737	0.9377	1.1266
10	PowerTech	1.1536	1.113	0.9852	1.1536	0.8964	0.7724	0.8948	1.0951	1.2769
11	UTC	0.7758	1.289	1	0.5497	1.8192	1	0.6976	1.2779	1.0838
12	KingPak	1.2149	1.1453	0.7817	0.7607	0.889	0.7718	0.5607	0.9773	1.4684
13	Hi-Sincerity	1.4497	0.8422	0.7044	1.6003	0.8422	0.7044	0.565	1.1502	1.3647
14	Formosa	1.0022	1.0513	1.0104	0.9317	1.4015	0.8367	0.7122	0.9065	1.1432
15	Sigurd	1.0662	0.9556	0.9711	1.26	0.825	1.0902	0.7398	0.9132	1.2194
Average		1.0675	1.06287	0.9241	0.8709	1.1442	0.9354	0.7219	0.9979	1.2119

We saw in some situations the productivity change using the cross and technical efficient frontiers are not the same. The productivity change discussed using the cross efficiency for the case of the CRS. The idea is applicable to the variable returns-to-scale case. In that case, the various kinds of decomposition of the CM index can be obtained. A global CEF, global CM index and its decompositions can be defined using the cross efficiency for inputs and outputs on all of the periods of time.

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