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# Gauss Summation Formula for Limit Summand Functions and Related Results

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**Abstract.** Regarding the Gauss multiplication formula for Γ-type functions, we introduce its dual formula for limit summand functions, namely Gauss summation formula. Also, we show that not only the Gauss multiplication for Γ-type functions is a simple result of this formula, but also provide an its improvement with several consequences and applications. Finally, as a note, we mention that a condition in two Webster's theorems is extra.

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**Keywords and Phrases:** Γ-type function, gauss summation formula, limit summability, monotone function, summand function

## 1. Introduction

One of the most famous and useful mathematical constants is Euler-Mascheroni constant which is limit of the sequence $\sum_{k=1}^{n} \frac{1}{k} - \log(n)$, denoted by $\gamma = 0.57722156649...$ (see [1]). There are many Euler-type constants and thier generalization some of them were studied in [3, 8]. On the other hand, regarding to uniqueness of the gamma function, Bohr and Mollerup proved that the only log-convex solution $f$ of the functional equation $f(x+1) = xf(x)$, for $x > 0$, satisfying $f(1) = 1$ is the gamma function Γ. As a generalization of the theorem, Webster studied Γ-type functions satisfying the functional equation $f(x+1) = g(x)f(x)$, in 1997

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[9]. In the paper, he coressponded to every function $g : \mathbb{R}^+ \to \mathbb{R}^+$, the following limit function $g^*$ defined by

$$g^*(x) := \lim_{n\to\infty} g_n^*(x) = \lim_{n\to\infty} \frac{g(n)\cdots g(1)g(n)^x}{g(n+x)\cdots g(x)}; \quad x > 0,$$

which we call it gamma type function of $g$. In 2001, M.H. Hooshmand introduced a new concept entitled limit summability of functions [4] and he proved that Γ-type functions can be considered as an its sub-topic. Also, in 2016 and 2017 he introduced two others type of such summabilities entitled analytic and trigonometric summability of functions [6, 7].

## 2. Limit Summability and Γ-Type Functions

Let $f$ be a real or complex function with $\mathbb{N}^* \subseteq D_f$, where $\mathbb{N}^* = \{1, 2, 3, ...\}$ and $\mathbb{N} = \{0, 1, 2, ...\}$. The summand set of $D_f$ is defined by

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\} = \{x | \{x + 1, x + 2, x + 3, \cdots\} \subseteq D_f\}.$$

Hence $x \in \Sigma_f$ if and only if $\{x + 1, x + 2, \cdots, x + n, \cdots\} \subseteq D_f$. Also, for any positive integer $n$ and $x \in \Sigma_f$ set

$$R_n(f, x) := R_n(x) = f(n) - f(x + n),$$

$$f_{\sigma_n}(x) = xf(n) + \sum_{k=1}^{n} R_k(x).$$

When $x \in D_f$, we may use the notation $\sigma_n(f(x))$ instead of $f_{\sigma_n}(x)$. The function $f$ is called limit summable at $x_0 \in \Sigma_f$ (resp. on $S \subseteq \Sigma_f$) if the sequence $\{f_{\sigma_n}(x)\}$ is convergent at $x_0$ (resp. on S). The limit function of $f_{\sigma_n}(x)$ (resp. $R_n(f, x)$) is denoted by $f_\sigma(x)$ (resp. $R(f, x)$ or $R(x)$) and it is called the limit summand function of $f$. Note that the domain of $f_\sigma$ is

$$D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is limit summable at } x\},$$

and if $x \in D_{f_\sigma}$ and $R_n(f, 1)$ is convergent, then $R(f, x) = R(x) = xR(1)$. It is important to know that $D_f \cap \Sigma_f = \Sigma_f + 1$, $f_\sigma(0) = 0$ so $0 \in D_f$. Also, if $0 \in D_f$, then $-1 \in D_{f_\sigma}$, and we have $f_\sigma(-1) = -f(0)$. If

$R_n(f,1)$ is convergent, then $D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1$. A necessary condition for limit summability of $f$ at $x$ is

$$\lim_{n\to\infty}(R_n(x) - xR_{n-1}(1)) = 0.$$

If $R(1) = 0$, then

$$f_\sigma(m) = \sum_{j=1}^{m} f(j), \tag{1}$$

for all $m \in \mathbb{N}^*$ and

$$f_\sigma(m) = -\sum_{j=0}^{-m-1} f(-j), \tag{2}$$

if $m \in \mathbb{Z}^- \cap \Sigma_f$. It is proved that the following conditions are equivalent and every function satisfying one of them is called limit summable:

**a)** $f$ is limit summable;

**b)** $f_\sigma(x) = f(x) + f_\sigma(x-1)$ for all $x \in D_f$;

**c)** $D_{f_\sigma} = \Sigma_f$, $R(1) = 0$.

The most important criteria for limit summability which were introduced in [5] stated that convexity or concavity of $f$ together with boundedness of $R_n(f,1)$, or monotonity with boundedness of $f_n$ imply limit summability of real functions $f$. In [4], some connections between limit summand and gamma type functions (if exist) are stated. Also, it is shown that gamma type functions can be considered as a subtopic of limit summability. It is proved that Theorem 1.3 of [9] is a result of Corollary 3.3 of [4] for the special case $M = 0$. Moreover, a main relation between gamma type function of $g : \mathbb{R}^+ \to \mathbb{R}^+$ and limit summand function of $\ln g$ is proved as follow

$$g^*(x) = e^{(\ln g)_\sigma(x-1)}; \quad x > 0.$$

Also, one of the results for the generalized $\Gamma$-type functions is Gauss multiplication formula which was introduced in Theorem 5.2 of [9].

## 3. Limit Summand of Shifted Functions and Gauss Summation Formula

In order to find a sum relationship corresponding to Gauss multiplication formula (in the topic of $\Gamma$-type functions) for limit summand functions, we are led to a fact which not only fulfills that objective but also it provides a more developed and indicates that the related summation formula is a consequence of the limit summand of the integer-shifted $f(x-m)$ or $f(x+m)$. We mention that if $m$ is a positive integer number, then the functions $f(x-m)$ and $f(x+m)$ are called backward and forward shift function of $f$, respectivly. The following theorem in fact presents a relationship for summand function of backward and forward shift and ential several important results.

**Theorem 3.1.** *Let $f$ be a real or complex function with domain $D_f$ such that $R(f,1)=0$ and $m$ a fixed integer number. If $N^* \subseteq D_f+m$, then*

$$\sigma(f(x-m)) = f_\sigma(x-m) - f_\sigma(-m), \tag{3}$$

*for $x \in D_{f_\sigma}+m+1$. Moreover, if $m>0$ then*

$$\sigma(f(x+m)) = f_\sigma(x+m) - f_\sigma(m) = f_\sigma(x) - f_{\sigma_m}(x) + xf(m), \tag{4}$$

*for $x \in D_{f_\sigma}-m+1$.*

**Proof.** Let $m \geqslant 0$. If $x \in D_{f_\sigma}+m+1$, then $x-m \in D_{f_\sigma}+1 = D_{f_\sigma} \cap D_f$. Thus $x-m \in D_f$ and $x-m \in D_{f_\sigma}$. Hence the both $f(x-m)$ and $f_\sigma(x-m)$ are defined. Now, put $g(x) := f(x-m)$. Then, due to $R(f,1)=0$ and relation (2), we can write

$$\begin{aligned} g_{\sigma_n}(x) &= xf(n-m) + \sum_{k=1}^{n} f(k-m) - f(k+x-m) \\ &= xf(n-m) + \sum_{k=0}^{m-1} f(-k) - f(-k+x) + \sum_{k=1}^{n-m} f(k) - f(k+x) \end{aligned}$$

$$
\begin{aligned}
g_{\sigma_n}(x) &= xf(n-m) + \sum_{k=1}^{n} f(k-m) - f(k+x-m) \\
&= xf(n-m) + \sum_{k=0}^{m-1} f(-k) - f(-k+x) + \sum_{k=1}^{n-m} f(k) - f(k+x) \\
&= x(f(n-m) - f(n)) + f_{\sigma_n}(x) + \sum_{k=0}^{m-1} f(-k) - f(-k+x) \\
&- \sum_{k=0}^{m-1} R_{n-k}(x).
\end{aligned}
$$

Therefore,

$$
g_{\sigma_n}(x) = f_{\sigma_n}(x) - \sum_{k=0}^{m-1} f(-k+x) - f_\sigma(-m) + xR_{n-m}(m) - \sum_{k=0}^{m-1} R_{n-k}(x).
$$

Now, If $x \in D_{f_\sigma} + m + 1$, then $x \in D_{f_\sigma} + m \subseteq D_{f_\sigma}$ and considering $R(f,1) = 0$, we conclude that $R(f,x) = \lim_{x\to\infty} R_n(x) = 0$. Thus, $g$ is limit summable at $x$ as $n \to \infty$, and

$$
\begin{aligned}
g_\sigma(x) &= f_\sigma(x) - \sum_{k=0}^{m-1} f(-k+x) - f_\sigma(-m) \\
&= f_\sigma(x-m) - f_\sigma(-m),
\end{aligned}
$$

by Corollary 2.13 of [4]. Therefore, $f$ satisfies (3).

Also, if $x \in D_{f_\sigma} - m + 1$, then $x + m \in D_{f_\sigma} + 1 = D_f \cap D_{f_\sigma}$. Thus $x + m \in D_f$ and $x + m \in D_{f_\sigma}$. Now, putting $h(x) := f(x+m)$, and similar to the above part, we obtain

$$\begin{aligned}
h_{\sigma_n}(x) &= xf(n+m) + \sum_{k=1}^{n} f(k+m) - f(k+x+m) \\
&= xf(n+m) + \sum_{k=1}^{n} f(k) - f(k+x) + \sum_{k=n+1}^{n+m} f(k) - f(k+x) \\
&- \sum_{k=1}^{m} f(k) - f(k+x) \\
&= xf(n+m) + f_{\sigma_n}(x) - xf(n) - \sum_{k=1}^{m} f(k) - f(k+x) \\
&+ \sum_{k=1}^{m} R_{n+k}(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
h_{\sigma_n}(x) &= f_{\sigma_n}(x) + \sum_{k=1}^{m} f(k+x) - f_\sigma(m) - xR_n(m) + \sum_{k=1}^{m} R_{n+k}(x) \\
&= f_{\sigma_n}(x) - f_{\sigma_m}(x) + xf(m) - xR_n(m) + \sum_{k=1}^{m} R_{n+k}(x).
\end{aligned}$$

On the other hand, if $x \in D_{f_\sigma} - m + 1$, then $x + m \in D_{f_\sigma} + 1 \subseteq D_{f_\sigma}$. Since $x + m \in D_{f_\sigma}$ and considering $R_n(x) = R_n(-m) + R_{n-m}(x+m)$, so $x \in D_{f_\sigma}$ and due to $R(f,1) = 0$, we conclude that $R(f,x) = 0$. Thus, $h$ is limit summable at $x$ as $n \to \infty$, and

$$\begin{aligned}
h_\sigma(x) &= f_\sigma(x) + \sum_{k=1}^{m} f(k+x) - f_\sigma(m) \\
&= f_\sigma(x) - f_{\sigma_m}(x) + xf(m).
\end{aligned}$$

Hence, we arrive at (4) similar to the first case. Therefore, the proof is complete.

**Corollary 3.2.** *If $f$ be a real or complex limit summable function with $D_f = [1, +\infty)$, then*

$$\sigma(f(x-m)) = f_\sigma(x-m) - f_\sigma(-m); \qquad x \geqslant m+1, \tag{5}$$

**Example 3.4.** For any fixed real numbers $\alpha > 0$ and $\beta < 0$, put $f(x) = x^\alpha e^{\beta x}$. Then, $D_f = [0, +\infty)$, $R(f,1) = 0$ and for every integer $m \leqslant 1$ we have

$$\begin{aligned} f_\sigma(x-m) &= \lim_{x\to\infty}((x-m)n^\alpha e^{\beta n} + \sum_{k=1}^{n} k^\alpha e^{\beta k} - (k+x-m)^\alpha e^{\beta(k+x-m)}) \\ &= \sum_{k=1}^{\infty} \frac{e^{\beta k}}{k^{-\alpha}} + \frac{e^{\beta(x-m)}}{(x-m)^{-\alpha}} - \frac{e^{\beta(x-m)}}{(x-m)^{-\alpha}} - \sum_{k=1}^{\infty} \frac{e^{\beta(k+x-m)}}{(k+x-m)^{-\alpha}} \\ &= Li_{-\alpha}(e^\beta) + \frac{e^{\beta(x-m)}}{(x-m)^{-\alpha}} - e^{\beta(x-m)} \sum_{k=0}^{\infty} \frac{e^{\beta k}}{(k+x-m)^{-\alpha}} \\ &= Li_{-\alpha}(e^\beta) - e^{\beta(x-m)} L(\frac{\beta}{2\pi i}, x-m, -\alpha) + \frac{e^{\beta(x-m)}}{(x-m)^{-\alpha}}, \end{aligned}$$

for $x > m$, where $L(\lambda, x, t) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+x)^t}$ is the Lerch zeta function and $Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ is the polylogarithm function. Also, we have

$$f_\sigma(-m) = Li_{-\alpha}(e^\beta) - e^{-\beta m} L(\frac{\beta}{2\pi i}, -m, -\alpha) + \frac{e^{-\beta m}}{(-m)^{-\alpha}}$$

thus,

$$\begin{aligned} \sigma((x-m)^\alpha e^{\beta(x-m)}) = \frac{e^{\beta(x-m)}}{(x-m)^{-\alpha}} - e^{\beta(x-m)} L(\frac{\beta}{2\pi i}, x-m, -\alpha) + \\ e^{-\beta m} L(\frac{\beta}{2\pi i}, -m, -\alpha) - \frac{e^{-\beta m}}{(-m)^{-\alpha}}. \end{aligned}$$

Therefore, putting $m = 0$, we obtain

$$\sigma(x^\alpha e^{\beta x}) = L(\frac{\beta}{2\pi i}, 0, -\alpha) - e^{\beta x} L(\frac{\beta}{2\pi i}, x, -\alpha) + x^\alpha e^{\beta x}; \qquad x > 0,$$

that is a closed form for limit summand function of $x^\alpha e^{\beta x}$.

In order to obtain a summation dual of Gauss multiplication formula, it is sufficient, we collect the relationships sides obtained from (5) for values $1, 2, \cdots, m-1$, up to we reach to the Gauss summation formula (for limit summand functions) as follows.

**Corollary 3.5.** *(Gauss Summation Formula). Let $f : [1, +\infty) \to \mathbb{R}$ be a concave function such that $R(f, 1) = 0$ and $m$ a fixed positive integer number. Then,*

$$\sigma(\sum_{j=0}^{m-1} f(x-j)) = \sum_{j=0}^{m-1} f_\sigma(x-j) - f_\sigma(-j).$$

**Proof.** This is a direct consequence of Corollary 2. and Theorem 3.3 in [5]. □

**Example 3.6.** If $f(x) = \log_b(x)$, then $f$ is concave and $R(f, 1) = 0$. Thus, by using Example 2.5 of [4], we get

$$\sum_{j=0}^{m-1} f_\sigma(x-j) = \frac{1}{\ln b} \sum_{j=0}^{m-1} \Gamma(x+1-j),$$

$$\sum_{j=0}^{m-1} f_\sigma(-j) = \frac{1}{\ln b} \sum_{j=0}^{m-1} \Gamma(1-j),$$

for $x \geqslant 1$. Therefore

$$\sigma(\sum_{j=0}^{m-1} \log_b (x-j)) = \frac{1}{\ln b}(\sum_{j=0}^{m-1} \Gamma(x+1-j) - \Gamma(1-j)),$$

for $x \geqslant 1$.

Due to the basic relationship between the limit summand function and the $\Gamma$-type function, the following result can also be obtained.

**Corollary 3.7.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and $m$ a fixed positive integer number. If $\frac{f(n)}{f(n+1)} \to 1$ as $n \to \infty$, then,*

$$\sigma(logf(x-m)) = \log f^*(x-m+1) - logf^*(1-m),$$

*for $x \in D_{(\log f)_\sigma} + m + 1$. Notice that, since $\log f^*(x) = (\log f)_\sigma(x-1)$, then $D_{\log f^*} = D_{(\log f)_\sigma}$.*

**Corollary 3.8.** *Suppose that $f : [1, +\infty) \to \mathbb{R}$ is a convex or concave (resp. monotone) function such that $R(f,1) = 0$ (resp. $f(n)$ is convergent) and $m$ a fixed positive integer number. Then, $f$ satisfies (3) for $x \geqslant m+1$.*

Now, by making use of a technique which is used in [4], we generalize the Theorem 3. in the case of $R_n(f,1)$ being convergent (not necessary to zero).

**Theorem 3.9.** *Let $f$ be a real or complex function with domain $D_f$ such that $R_n(f,1)$ is convergent and $m$ a fixed integer number. If $N^* \subseteq D_f + m$, then*

$$\sigma(f(x-m)) = f_\sigma(x-m) - f_\sigma(-m) + mR(f,1)x; \;\; x \in D_{f_\sigma} + m + 1. \tag{6}$$

**Proof.** Put $h(x) := f(x) + R(f,1)x$. Then $\Sigma_h = \Sigma_f$, $h_{\sigma_n}(x) = f_{\sigma_n}(x)$ and $R_n(h,x) = R_n(f,x) - R(f,1)x$. Since $x \in D_{f_\sigma}$ and $R_n(f,1) \to R(f,1)$ as $n \to \infty$, then $R_n(f,x) \to xR(f,1)$, as $n \to \infty$, thus $R(h,x) = \lim_{x\to\infty} R_n(h,x) = 0$ and $h$ is limit summable at $x$ and

$$\sigma(h(x-m)) = h_\sigma(x-m) - h_\sigma(-m),$$

so,

$$\sigma(f(x-m) + R(f,1)(x-m)) = f_\sigma(x-m) - f_\sigma(-m),$$

thus,

$$\begin{aligned}\sigma(f(x-m)) &= f_\sigma(x-m) - f_\sigma(-m) - R(f,1)\sigma(x-m) \\ &= f_\sigma(x-m) - f_\sigma(-m) + mR(f,1)x,\end{aligned}$$

therefore, the relation (6) is proved. □

**Note**. In the studies conducted in this resarch, we realized that in the Theorem 5.1 (ii) in [9] the condition $\frac{g_1}{g_2} \in G$ is extra, for if $g_1, g_2 \in G$, then there exists $g_1^*$ and $g_2^*$, and so $\frac{g_1^*}{g_2^*} = (\frac{g_1}{g_2})^*$ holds. Also, the condition $h_m \in G$ can be removed from Theorem 5.2 in the paper (because $g \in G$ implies that $g^*$ exists and $g_m \in G$, thus $g_m^*$ and also $\frac{g^*}{g_m^*} = h_m^*$ exist).

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