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Conditional R-norm Entropy and R-norm Divergence in Quantum Logics

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Abstract. This contribution deals with the mathematical modeling of R-norm entropy and R-norm divergence in quantum logics. We extend some results concerning the R-norm entropy and conditional R-norm entropy given in (Inf. Control 45, 1980), to the quantum logics. Firstly, the concepts of R-norm entropy and conditional R-norm entropy in quantum logics are introduced. We prove the concavity property for the notion of R-norm entropy in quantum logics and we show that this entropy measure does not have the property of sub-additivity in a true sense. It is proven that the monotonicity property for the suggested type of conditional version of R-norm entropy, holds. Furthermore, we introduce the concept of R-norm divergence of states in quantum logics and we derive basic properties of this quantity. In particular, a relationship between the R-norm divergence and the R-norm entropy of partitions is provided.

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1 Introduction

A quantum logic is a mathematical representation of the set of all random events of a physical experiment and is a σ-orthomodular lattice, with a state. The quantum logic approach was introduced in [1] by Birkhoff and Neumann. The entropy is a tool to measure the amount of uncertainty in random events and has been applied in physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and many other fields. We remind that if $P = \{p_1, ..., p_n\} \subset \mathbb{R}^n$ is a probability distribution, then the Shannon entropy of P [18] is defined as the number $H_S(P) = \sum_{i=1}^{n} S(p_i)$, where $S : [0, 1] \to [0, \infty)$ is the Shannon entropy function defined, for every $x \in [0, 1]$ by the formula:

$$S(x) = \begin{cases} -x \log x & \text{if } x \in (0, 1]; \\ 0 & \text{if } x = 0. \end{cases}$$

Some extensions of Shannon entropy were presented as alternatives of entropy measure. One of the entropy measures is R-norm entropy. Assume that $P = \{p_1, ..., p_n\}$ is a probability distribution, then R-norm entropy of P is introduced by Boekke and Lubbe in [2] as the number:

$$H_R(P) = \frac{R}{R-1}\left(1 - \big(\sum_{i=1}^{n} p_i^R\big)^{\frac{1}{R}}\right),$$

for $R \in (0, 1) \cup (1, +\infty)$. Some papers ([7, 8, 9, 10, 11, 12, 13, 14]) were published about this information measure.

In ([3, 4, 5], [15, 16], [19]) using the notion of state of quantum logic, the concepts of entropy and conditional entropy of partitions and dynamical systems in quantum logics, have been defined and studied. The notion of entropy in quantum logics is very useful in studying quantum information theory [20] and has been applied in the isomorphism of quantum dynamical systems. The concept of Kullback-Leibler divergence as the distance measure between two probability distributions was introduced in [12] which is a useful tool in many fields including physics, information theory, machine learning, statistics, neuroscience, computer science, etc. Recently, in [17] we studied the concept of R-norm entropy and R-norm divergence in a product MV-algebra. Note that the version of conditional R-norm entropy studied in the present article is different with the conditional version of R-norm entropy defined in [17]. The aim of this paper is to study the R-norm entropy, conditional R-norm entropy and R-norm divergence in quantum logics. We generalize some results of the R-norm entropy and conditional R-norm entropy given in [2], to the quantum logics.

The rest of the paper is structured as follows. In section 2, basic concepts and facts are provided that will be used throughout this paper. In section 3, R-norm entropy of partitions in quantum logics with respect to a state s, is defined. We prove the concavity property for the notion of R-norm entropy in quantum logics. We show that this information measure does not have the property of sub-additivity in a general. We also define the notion of conditional R-norm entropy of partitions in quantum logics and establish the monotonicity property for this conditional version of R-norm entropy. We study R-norm entropy and conditional R-norm entropy of s- independent partitions. In section 4, the concept of R-norm divergence in quantum logics is defined and and examine their properties. Especially, the convexity of R-norm divergence with respect to states, is shown. Our results are summarized in the final section.

2 Basic Definitions

In this section, some basic concepts, notations and facts are presented that will be useful in the next sections.

Definition 2.1. [16] *A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leq, \vee, \wedge, 0, 1)$ with the smallest element 0 and the greatest element 1, an operation $' : L \to L$ such that the following properties hold for all $a, b, c \in L$:*
(i)$(a^{'})^{'} = a$, $a \leq b \Rightarrow b^{'} \leq a^{'}$;
(ii) Given any finite sequence $(a_i)_{i\in\mathbb{N}}$, $a_i \leq a_j^{'}$, $i \neq j$, *the join* $\vee_{i\in\mathbb{N}} a_i$ *exists in* L;
(iii)L *is orthomodular:* $a \leq b \Rightarrow b = a \vee (b \wedge a^{'})$.

In quantum logics we have $a \wedge a^{'} = 0$, *and as a consequence of the orthomodular law, we get* $a \vee a^{'} = 1$. *(see [16]).*

Two elements $a, b \in QL$ are called orthogonal if $a \leq b'$ and denoted by $a \perp b$.

Definition 2.2. [16] *Let L be a quantum logic. A map $s : L \to [0, 1]$ is a state if:*
$(i) s(1) = 1$;
(ii) for $a, b \in L$ with $a \perp b, s(a \vee b) = s(a) + s(b)$.

It may be observed that $s(0) = 0$, s is monotone and $s(a') = 1 - s(a), a \in L$.

Definition 2.3. [16] *Let $P = \{a_1, ..., a_n\}$ be a finite set of elements of a quantum logic. P is called to be $\vee$-orthogonal iff $\vee_{i=1}^{k} a_i \perp a_{k+1}$, for $k = 1, 2, ..., n - 1$.*

Definition 2.4. [16] *Assume that L is a quantum logic. $P = \{a_1, ..., a_n\} \subset L$ is said to be a partition of L corresponding to a state s on L, (a partition P of couple (L, s)) if:*
$(i) P$ is $\vee$-orthogonal;
$(ii) s(\vee_{i=1}^{n} a_i) = 1$.

Note that from Definition 2.2, we obtain $\sum_{i=1}^{n} s(a_i) = 1$.

Definition 2.5. [4] *Let $P = \{a_1, a_2, ..., a_n\}$ and $Q = \{b_1, b_2, ..., b_m\}$ be partitions of a couple (L, s). We say Q is a s-refinement of P, denoted by $P \preceq_s Q$, if there exists a partition $I(1), ..., I(n)$ of the set $\{1, ..., m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, ..., n$.*

Two partitions P and Q of a couple (L, s), are called s-independent if $s(a \wedge b) = s(a)s(b)$ for all $a \in P$, and $b \in Q$.

Definition 2.6. [16] *Let $\{b_1, ..., b_m\}$ be any partition of a couple (L, s), and $a \in L$. The state s is said has Bayes' property if*

$$s(\vee_{j=1}^{m} (a \wedge b_j)) = s(a).$$

In this case we get

$$\sum_{j=1}^{m} s(a \wedge b_j) = s(a).$$

Suppose $P = \{a_1, ..., a_n\}$ and $Q = \{b_1, ..., b_m\}$ are two partitions of (L, s). Then the common refinement of partitions is defined as $P \vee Q = \{a_i \wedge b_j : a_i \in P, b_j \in Q\}$. If s has Bayes' property, then $P \vee Q$ is also a partition of (L, s) [15].

In this paper, we will use the following known Minkowski inequality: for $R > 1$,

$$\left(\sum_{i=1}^{n} x_i^R\right)^{\frac{1}{R}} + \left(\sum_{i=1}^{n} y_i^R\right)^{\frac{1}{R}} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^R\right)^{\frac{1}{R}},$$

and for $0 < R < 1$,

$$\left(\sum_{i=1}^{n} x_i^R\right)^{\frac{1}{R}} + \left(\sum_{i=1}^{n} y_i^R\right)^{\frac{1}{R}} \leq \left(\sum_{i=1}^{n} (x_i + y_i)^R\right)^{\frac{1}{R}},$$

where $x_1, ..., x_n, y_1, ...y_n$ are nonnegative numbers.
Furthermore, we will use the following known Jensen inequality: for a real convex function φ, real numbers $x_1, x_2, ..., x_m$ in its domain and non-negative real numbers $a_1, a_2, ..., a_m$ with $\sum_{i=1}^{n} a_i = 1$, it holds

$$\varphi\left(\sum_{j=1}^{m} a_i x_i\right) \leq \sum_{j=1}^{m} a_i \varphi(x_i), \tag{1}$$

and the inequality is reversed if φ is a real concave function.

3 R-norm Entropy of Partitions in Quantum Logics

In this section, we define the notions of R-norm entropy and conditional R-norm entropy of finite partitions on a quantum logic. We prove some ergodic properties of the suggested measures. Especially, we prove the concavity property of R-norm entropy of partitions on a quantum logic. We also show that the R-norm entropy does not satisfy the property of sub-additivity, in general. An example is presented to illustrate the obtained results.

We recall that, by a partition P of a couple (L, s) we mean that P is a partition of L corresponding to the state s. We shall now define the notion of R-norm entropy of partitions.

Definition 3.1. *Let $P = \{a_1, a_2, ..., a_n\}$ be a partition of a couple (L, s). The R-norm entropy of P with respect to the state s is defined by:*

$$H_R^s(P) = \frac{R}{R-1}\Big(1 - \big(\sum_{i=1}^{n} s(a_i)^R\big)^{\frac{1}{R}}\Big),$$

for $R \in (0, 1) \cup (1, +\infty)$.

Note that, we write $s(a_i)^R$ instesd of $(s(a_i))^R$.

Remark 3.2. *If $P = \{a_1, a_2, ..., a_n\}$ is a partition of a couple (L, s), then $H_R^s(P) \geq 0$, because:*
for the case of $0 < R < 1$, we have $s(a_i)^R \geq s(a_i), i = 1, 2, ..., n$. Therefore $\sum_{i=1}^{n} s(a_i)^R \geq \sum_{i=1}^{n} s(a_i) = 1$. It follows that $\Big(\sum_{i=1}^{n} s(a_i)^R\Big)^{\frac{1}{R}} \geq 1$. Since for $0 < R < 1$, we have $\frac{R}{R-1} < 0$, we get $H_R^s(P) = \frac{R}{R-1}\Big(1 - \big(\sum_{i=1}^{n} s(a_i)^R\big)^{\frac{1}{R}}\Big) \geq 0$. Now if $R > 1$, it holds that $s(a_i)^R \leq s(a_i), i = 1, 2, ..., n$. Thus $\sum_{i=1}^{n} s(a_i)^R \leq \sum_{i=1}^{n} s(a_i) = 1$, and we conclud that $\Big(\sum_{i=1}^{n} s(a_i)^R\Big)^{\frac{1}{R}} \leq 1$. It follows that for $R > 1, H_R^s(P) = \frac{R}{R-1}\Big(1 - \big(\sum_{i=1}^{n} s(a_i)^R\big)^{\frac{1}{R}}\Big) \geq 0$.

Example 3.3. Let L be a quantum logic and $a \in L$, and let s be a state such that $s(a) = s(a') = \frac{1}{2}$. Then $P = \{a, a'\}$ is a partition of (L, s). By simple calculations we obtain:

$$H_R^s(P) = \frac{R}{R-1}(1 - 2^{\frac{1-R}{R}}).$$

In particular, if $R = 2$, then $H_2^s(P) = 2 - \sqrt{2}$.

In the following, we will use the symbol Γ to denote the family of all states on quantum logics. It is easy to prove the following proposition.

Proposition 3.4. *If* $s, t \in \Gamma$, *then, for every real number* $\lambda \in [0, 1]$, $\lambda s + (1 - \lambda)t \in \Gamma$.

In the following theorem, it is shown that the concavity of R-norm entropy $H_R^s(P)$ as a function on Γ.

Theorem 3.5. *Assume that* $s, t \in \Gamma$, *and* P *is a partition of* $(L, s), (L, t)$. *Then for every real number* $\lambda \in [0, 1]$, *the following inequality holds:*

$$\lambda H_R^s(P) + (1 - \lambda)H_R^t(P) \le H_R^{\lambda s+(1-\lambda)t}(P).$$

Proof. Suppose $P = \{a_1, a_2, ..., a_n\}$. If put $x_i = \lambda s(a_i)$ and $y_i = (1 - \lambda)t(a_i)$, in the Minkowski inequality, then we obtain for $R > 1$,

$$\lambda\left(\sum_{i=1}^{n} s(a_i)^R\right)^{\frac{1}{R}} + (1-\lambda)\left(\sum_{i=1}^{n} t(a_i)^R\right)^{\frac{1}{R}} \ge \left(\sum_{i=1}^{n}(\lambda s(a_i)+(1-\lambda)t(a_i))^R\right)^{\frac{1}{R}},$$

and for $0 < R < 1$,

$$\lambda\left(\sum_{i=1}^{n} s(a_i)^R\right)^{\frac{1}{R}} + (1-\lambda)\left(\sum_{i=1}^{n} t(a_i)^R\right)^{\frac{1}{R}} \le \left(\sum_{i=1}^{n}(\lambda s(a_i)+(1-\lambda)t(a_i))^R\right)^{\frac{1}{R}}.$$

Since $\dfrac{R}{R-1} > 0$ for $R > 1$ and $\dfrac{R}{R-1} < 0$ for $0 < R < 1$, we get

$$\lambda H_R^s(P) + (1 - \lambda)H_R^t(P) \le H_R^{\lambda s+(1-\lambda)t}(P).$$

□

Theorem 3.6. *Let* $P = \{a_1, a_2, ..., a_n\}$ *and* $Q = \{b_1, b_2, ..., b_m\}$ *be partitions of a couple* (L, s), *and* $P \preceq_s Q$. *Then* $H_R^s(P) \le H_R^s(Q)$.

Proof. Since $P \preceq_s Q$, there exists a partition $I(1), ..., I(n)$ of the set $\{1, ..., m\}$ such that $a_i = \vee_{j \in I(i)} b_j$ for every $i = 1, ..., n$. So from Definition 2.2, we have $s(a_i) = \sum_{j \in I(i)} s(b_j)$. Then we obtain for $R > 1$ and each $i = 1, ..., n$,

$$s(a_i)^R = \left(\sum_{j \in I(i)} s(b_j)\right)^R \ge \sum_{j \in I(i)} s(b_j)^R.$$

Since $I(1), ..., I(n)$ is a partition of the set $\{1, ..., m\}$, we have

$$\bigcup_{i=1}^{n} I(i) = \bigcup_{j=1}^{m} \{j\}, \ \ \forall i, k \in \{1, ..., n\}, I(i) \cap I(k) = \emptyset.$$

Thus

$$\sum_{i=1}^{n} \sum_{j \in I(i)} s(b_j)^R = \sum_{j=1}^{m} s(b_j)^R.$$

Hence by the relation

$$s(a_i)^R = (\sum_{j \in I(i)} s(b_j)^R)^R \geq \sum_{j \in I(i)} s(b_j)^R$$

since the state s is nonnegative, we obtain

$$\sum_{i=1}^{n} s(a_i)^R \geq \sum_{i=1}^{n} \sum_{j \in I(i)} s(b_j)^R = \sum_{j=1}^{m} s(b_j)^R,$$

therefore $\sum_{i=1}^{n} s(a_i)^R \geq \sum_{j=1}^{m} s(b_j)^R$. Then

$$\left(\sum_{i=1}^{n} s(a_i)^R \right)^{\frac{1}{R}} \geq \left(\sum_{j=1}^{m} s(b_j)^R \right)^{\frac{1}{R}},$$

hence we get

$$\begin{aligned} H_R^s(P) &= \frac{R}{R-1} \left(1 - (\sum_{i=1}^{n} s(a_i)^R)^{\frac{1}{R}} \right) \\ &\leq \frac{R}{R-1} \left(1 - (\sum_{j=1}^{m} s(b_j)^R)^{\frac{1}{R}} \right) = H_R^s(Q). \end{aligned}$$

If $0 < R < 1$, then we have for every $i = 1, ..., n$,

$$s(a_i)^R = \left(\sum_{j \in I(i)} s(b_j) \right)^R \leq \sum_{j \in I(i)} s(b_j)^R,$$

therefore $\sum_{i=1}^{n} s(a_i)^R \leq \sum_{j=1}^{m} s(b_j)^R$, then

$$\left(\sum_{i=1}^{n} s(a_i)^R\right)^{\frac{1}{R}} \leq \left(\sum_{j=1}^{m} s(b_j)^R\right)^{\frac{1}{R}}.$$

Since $\dfrac{R}{R-1} < 0$, analogously as in the above we obtain the assertion.
□

Corollary 3.7. *For two partitions P, Q of a quantum logic L corresponding to a state s having Bayes' property, we have:*

$$H_R^s(P \vee Q) \geq \max\{H_R^s(P), H_R^s(Q)\}.$$

Proof. Since $P \preceq_s P \vee Q$, $Q \preceq_s P \vee Q$, by the previous theorem, it holds. □

If partitions P, Q are s-independent partitions of a quantum logic L corresponding to a state s having Bayes' property, then it is easy to see that the Shannon entropy $H^s(P)$ as $H^s(P) = -\sum_{i=1}^{n} s(a_i)\, log\, s(a_i)$ of partitions in (L, s) (defined in [16]) has additivity property, i.e.

$$H^s(P \vee Q) = H^s(P) + H^s(Q).$$

In the case of R-norm entropy, we have the following property.

Theorem 3.8. *Assume that P and Q are s-independent partitions of a quantum logic L corresponding to a state s having Bayes' property. Then*

$$H_R^s(P \vee Q) = H_R^s(P) + H_R^s(Q) - \frac{R-1}{R} H_R^s(P).H_R^s(Q).$$

Proof. Let $P = \{a_1, a_2, ..., a_n\}$ and $Q = \{b_1, b_2, ..., b_m\}$. By the assumption, we have $s(a_i \wedge b_j) = s(a_i)s(b_j)$ for $i = 1, ..., n, j = 1, ..., m$. Let us calculate:

$$H_R^s(P) + H_R^s(Q) - \frac{R-1}{R} H_R^s(P).H_R^s(Q)$$

$$= \frac{R}{R-1}\Bigg(1 - \big(\sum_{i=1}^{n} s(a_i)^R\big)^{\frac{1}{R}} + 1 - \big(\sum_{j=1}^{m} s(b_j)^R\big)^{\frac{1}{R}}\Bigg)$$

$$+\frac{R-1}{R}(\frac{R}{R-1})^2\Big(-1+(\sum_{i=1}^{n}s(a_i)^R)^{\frac{1}{R}}\Big)\Big(1-(\sum_{j=1}^{m}s(b_j)^R)^{\frac{1}{R}}\Big)$$

$$=\frac{R}{R-1}\Big(1-(\sum_{i=1}^{n}s(a_i)^R)^{\frac{1}{R}}(\sum_{j=1}^{m}s(b_j)^R)^{\frac{1}{R}}\Big)$$

$$=\frac{R}{R-1}\Big(1-(\sum_{i=1}^{n}\sum_{j=1}^{m}(s(a_i)s(b_j))^R)^{\frac{1}{R}}\Big)$$

$$=\frac{R}{R-1}\Big(1-(\sum_{i=1}^{n}\sum_{j=1}^{m}(s(a_i\wedge b_j)^R)^{\frac{1}{R}}\Big)=H_R^s(P\vee Q).$$

□

Definition 3.9. *Let $P=\{a_1,a_2,...,a_n\}$ and $Q=\{b_1,b_2,...,b_m\}$ be partitions of a couple (L,s). The conditional R-norm entropy of P given Q with respect to state s is defined by:*

$$H_R^s(P\mid Q)=\frac{R}{R-1}\Big(1-(\sum_{j=1}^{m}s(b_j)\sum_{i=1}^{n}(\frac{s(a_i\wedge b_j)}{s(b_j)})^R)^{\frac{1}{R}}\Big),$$

for $R\in(0,1)\cup(1,+\infty)$.

It is easy to see that, for two s-independent partitions P,Q of a couple (L,s), we get $H_R^s(P\mid Q)=H_R^s(P)$.

In the next theorem, we prove that the conditional R-norm entropy $H_R^s(P\mid Q)$ has the property of monotonicity.

Theorem 3.10. *Let P and Q be partitions of a quantum logic L corresponding to a state s having Bayes' property. Then*

$$H_R^s(P\mid Q)\le H_R^s(P).$$

Proof. Since s has Bayes' property, using Jensen's inequality(1) we obtain for $R>1$,

$$\sum_{j=1}^{m}s(b_j)(\frac{s(a_i\wedge b_j)}{s(b_j)})^R\ge\Big(\sum_{j=1}^{m}s(b_j)(\frac{s(a_i\wedge b_j)}{s(b_j)})\Big)^R=s(a_i)^R,$$

and for $0 < R < 1$,

$$\sum_{j=1}^{m} s(b_j)\big(\frac{s(a_i \wedge b_j)}{s(b_j)}\big)^R \leq \left(\sum_{j=1}^{m} s(b_j)\big(\frac{s(a_i \wedge b_j)}{s(b_j)}\big)\right)^R = s(a_i)^R.$$

Then for $R > 1$,

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m} s(b_j)\big(\frac{s(a_i \wedge b_j)}{s(b_j)}\big)^R\right)^{\frac{1}{R}} \geq \left(\sum_{i=1}^{n} s(a_i)^R\right)^{\frac{1}{R}},$$

and for $0 < R < 1$,

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{m} s(b_j)\big(\frac{s(a_i \wedge b_j)}{s(b_j)}\big)^R\right)^{\frac{1}{R}} \leq \left(\sum_{i=1}^{n} s(a_i)^R\right)^{\frac{1}{R}}.$$

Since for $R > 1$, we have $\frac{R}{R-1} > 0$, and for $0 < R < 1$, $\frac{R}{R-1} < 0$, by the above last two inequalities we get $H_R^s(P \mid Q) \leq H_R^s(P)$. □

The following example, shows that the R-norm entropy does not have the property of sub-additivity in general. The results of Corollary 3.7 and Theorem 3.10 are illustrated in this example.

Example 3.11. Let $([0,1], M)$ be a measurable space, where M is the σ-algebra of all Borel subsets of $[0,1]$. Suppose L is the family of all M-measurable functions $f : [0,1] \to \{0,1\}$. Then $QL = (L, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$ with an operation $' : L \to L$ is a quantum logic such that for all $f, g \in L$: $f \vee g := \min\{f + g, 1\}$, $f \wedge g := f.g$, and $f' := 1 - f$. Consider $s : L \longrightarrow [0,1]$ defined by $s(f) = \int_0^1 f(x)dx$. Evidently, the sets $P = \{f_1, f_2, f_3\} = \{\chi_{[0,\frac{1}{3}]}, \chi_{(\frac{1}{3},\frac{2}{3}]}, \chi_{(\frac{2}{3},1]}\}$ and $Q = \{g_1, g_2\} = \{\chi_{[0,\frac{1}{2}]}, \chi_{(\frac{1}{2},1]}\}$ are partitions of QL corresponding to the state s. The s-state values of the corresponding elements of P and Q, are $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$, and $\frac{1}{2}, \frac{1}{2}$ respectively. By simple calculations we obtain:

$$H_R^s(P) = \frac{R}{R-1}\left(1 - \big(\sum_{i=1}^{3} s(f_i)^R\big)^{\frac{1}{R}}\right) = \frac{R}{R-1}\big(1 - 3^{\frac{1}{R}-1}\big),$$

$$H_R^s(Q) = \frac{R}{R-1}\Big(1 - (\sum_{j=1}^{2} s(g_i)^R)^{\frac{1}{R}}\Big) = \frac{R}{R-1}(1 - 2^{\frac{1}{R}-1}),$$

$$\begin{aligned} H_R^s(P \mid Q) &= \frac{R}{R-1}\Big(1 - (\sum_{j=1}^{2} s(g_j) \sum_{i=1}^{3} (\frac{s(f_i \wedge g_j)}{s(g_j)})^R)^{\frac{1}{R}}\Big) \\ &= \frac{R}{R-1}\Big(1 - (\frac{2+2^{1-R}}{3})^{\frac{1}{R}}\Big). \end{aligned}$$

The join of P and Q is $P \vee Q = \{\chi_{[0,\frac{1}{3}]}, \chi_{(\frac{1}{3},\frac{1}{2}]}, \chi_{(\frac{1}{2},\frac{2}{3}]}, \chi_{(\frac{2}{3},1]}, \mathbf{0}\}$ with the s-state values $\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, 0$ of the corresponding elements. The R-norm entropy of $P \vee Q$ is obtained as:

$$\begin{aligned} H_R^s(P \vee Q) &= \frac{R}{R-1}\Big(1 - (\sum_{i=1}^{3} \sum_{j=1}^{2} s(f_i \wedge g_j)^R)^{\frac{1}{R}}\Big) \\ &= \frac{R}{R-1}\Big(1 - \frac{\big(2+2^{1+R}\big)^{\frac{1}{R}}}{6}\Big). \end{aligned}$$

It can be verified that:

$$H_R^s(P \mid Q) \leq H_R^s(P),$$

$$H_R^s(P \vee Q) \geq \max\{H_R^s(P), H_R^s(Q)\}.$$

Elementary calculations show that $H^s_{\frac{1}{2}}(P) \doteq 0.586$, $H^s_{\frac{1}{2}}(Q) = 2$, $H^s_3(Q) \doteq 0.555$, $H^s_3(P \mid Q) \doteq 0.137$, $H^s_{\frac{1}{2}}(P \vee Q) \doteq 2.886$, $H^s_3(P \vee Q) \doteq 0.845$. It holds that $H^s_3(P \mid Q) \neq H^s_3(P \vee Q) - H^s_3(Q)$. On the other hand it can be seen that $H^s_{\frac{1}{2}}(P \vee Q) > H^s_{\frac{1}{2}}(P) + H^s_{\frac{1}{2}}(Q)$. This means that the R-norm entropy $H^s_R(P)$ of order $R \in (0,1)$ does not satisfy the sub-additivity property in general.

4 R-norm Diveregence in Quantum Logics

In this section we introduce the concept of R-norm divergence of states in quantum logics and prove some properties of this measure. The proposed notion is an analogy of the concept R-norm divergence introduced by Hooda and Sharma in [9].

Definition 4.1. *Let $P = \{a_1, ..., a_n\}$ be a partition of two coupls (L, s) and (L, t). The R-norm divergence of s, t is defined by:*

$$D_R^P(s \parallel t) = \frac{R}{R-1}\Big(\big(\sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R} \big)^{\frac{1}{R}} - 1 \Big).$$

In the following theorem we show that the R-norm divergence of states on quantum logics is always nonnegative. Using this result, we obtain an information measure in computing of distance between two states on quantum logics.

Theorem 4.2. *Assume $P = \{a_1, ..., a_n\}$ is a partition of two coupls (L, s) and (L, t). Then $D_R^P(s \parallel t) \geq 0$, with the equality if and only if $s(a_i) = t(a_i)$, for $i = 1, ..., n$.*

Proof. By Jensen's inequality (1) for $R > 1$, we have

$$\begin{aligned} 1 = \Big(\sum_{i=1}^{n} s(a_i) \big(\frac{t(a_i)}{s(a_i)} \big) \Big)^{1-R} &\leq \sum_{i=1}^{n} s(a_i) \big(\frac{t(a_i)}{s(a_i)} \big)^{1-R} \\ &= \sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R}. \end{aligned}$$

Consequently $(\sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R})^{\frac{1}{R}} \geq 1$. Since $\dfrac{R}{R-1} > 0$, we get $D_R^P(s \parallel t) \geq 0$. For $0 < R < 1$, we have

$$\begin{aligned} 1 = \Big(\sum_{i=1}^{n} s(a_i) \big(\frac{t(a_i)}{s(a_i)} \big) \Big)^{1-R} &\geq \sum_{i=1}^{n} s(a_i) \big(\frac{t(a_i)}{s(a_i)} \big)^{1-R} \\ &= \sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R}. \end{aligned}$$

This applies that $\left(\sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R}\right)^{\frac{1}{R}} \leq 1$. Since $\dfrac{R}{R-1} < 0$, it holds that $D_R^P(s \parallel t) \geq 0$.
If $s(a_i) = t(a_i)$, for $i = 1, ..., n$, then $(\sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R})^{\frac{1}{R}} = 1$, therefore $D_R^P(s \parallel t) = 0$. If $D_R^P(s \parallel t) = 0$, then $\dfrac{t(a_i)}{s(a_i)} = c$, for $i = 1, ..., n$, where c is constant, thus $\sum_{i=1}^{n} s(a_i) = c \sum_{i=1}^{n} t(a_i)$, which implies that $c = 1$. Hence $s(a_i) = t(a_i)$, for $i = 1, ..., n$. □

In the following example it is shown that the R-norm divergence is not symmetric, i.e., the equality $D_R^P(s \parallel t) = D_R^P(t \parallel s)$ is not in a true sense. Therefore it is not a metric necessarily.

Example 4.3. Let L be a quantum logic, and $a \in L$, then $P = \{a, a'\}$ is a partition. Let s, t be two states on L such that $s(a) = \frac{1}{2}, s(a') = \frac{1}{2}$ and $t(a) = \frac{1}{3}, t(a') = \frac{2}{3}$, then

$$D_R^P(s \parallel t) = \frac{R}{R-1}\left(((\frac{1}{2})^R (\frac{1}{3})^{1-R} + (\frac{1}{2})^R (\frac{2}{3})^{1-R})^{\frac{1}{R}} - 1 \right),$$

and

$$D_R^P(t \parallel s) = \frac{R}{R-1}\left(((\frac{1}{3})^R (\frac{1}{2})^{1-R} + (\frac{2}{3})^R (\frac{1}{2})^{1-R})^{\frac{1}{R}} - 1 \right).$$

Now if $R = 2$, then we have

$$D_2^A(s \parallel t) = \frac{3}{\sqrt{2}} - 2 \neq \frac{2\sqrt{10}}{3} - 2 = D_2^A(t \parallel s).$$

If $R = \frac{1}{3}$, then

$$D_{\frac{1}{3}}^P(s \parallel t) = \frac{-(1 + \sqrt[3]{2})^3}{36} + \frac{1}{2} \neq \frac{-(\sqrt[3]{2} + 1)^3}{24} + \frac{1}{2} = D_{\frac{1}{3}}^P(t \parallel s).$$

The result means that $D_R^P(s \parallel t) \neq D_R^P(t \parallel s)$, in general.

The next theorem shows that the R-norm divergence is a convex function on the family of states in a quantum logic.

Theorem 4.4. *Let $P = \{a_1, ..., a_n\}$ be a partition of coupls (L, s_1), (L, s_2), (L, t). Then for any real number $\lambda \in [0,1]$, we have the following inequality:*

$$D_R^P(\lambda s_1 + (1-\lambda)s_2 \parallel t) \leq \lambda D_R^P(s_1 \parallel t) + (1-\lambda)D_R^P(s_2 \parallel t).$$

Proof. By the convexity of the function $f(x) = x^R$, for $R > 1$, we have for any a_i,

$$\left(\sum_{i=1}^{n} \big(\lambda s_1(a_i) + (1-\lambda)s_2(a_i)\big)^R t(a_i)^{1-R}\right)^{\frac{1}{R}}$$

$$= \left(\sum_{i=1}^{n} \big((\lambda s_1(a_i) + (1-\lambda)s_2(a_i))t(a_i)^{\frac{1-R}{R}}\big)^R\right)^{\frac{1}{R}}$$

$$\leq \left(\sum_{i=1}^{n} \big(\lambda s_1(a_i)t(a_i)^{\frac{1-R}{R}}\big)^R\right)^{\frac{1}{R}} + \left(\sum_{i=1}^{n} \big((1-\lambda)s_1(a_i)t(a_i)^{\frac{1-R}{R}}\big)^R\right)^{\frac{1}{R}}.$$

Since $\frac{R}{R-1} > 0$, we get

$$D_R^P(\lambda s_1 + (1-\lambda)s_2 \parallel t) \leq \lambda D_R^P(s_1 \parallel t) + (1-\lambda)D_R^P(s_2 \parallel t).$$

For $0 < R < 1$, by the concavity of the function $f(x) = x^R$, we obtain for each a_i,

$$\left(\sum_{i=1}^{n} \big(\lambda s_1(a_i) + (1-\lambda)s_2(a_i)\big)^R t(a_i)^{1-R}\right)^{\frac{1}{R}}$$

$$= \left(\sum_{i=1}^{n} \big((\lambda s_1(a_i) + (1-\lambda)s_2(a_i))t(a_i)^{\frac{1-R}{R}}\big)^R\right)^{\frac{1}{R}}$$

$$\geq \left(\sum_{i=1}^{n} \big(\lambda s_1(a_i)t(a_i)^{\frac{1-R}{R}}\big)^R\right)^{\frac{1}{R}} + \left(\sum_{i=1}^{n} \big((1-\lambda)s_1(a_i)t(a_i)^{\frac{1-R}{R}}\big)^R\right)^{\frac{1}{R}}.$$

and since $\frac{R}{R-1} < 0$, we obtain

$$D_R^P(\lambda s_1 + (1-\lambda)s_2 \parallel t) \leq \lambda D_R^P(s_1 \parallel t) + (1-\lambda)D_R^P(s_2 \parallel t).$$

□

Theorem 4.5. *Suppose $P = \{a_1, ..., a_n\}$ is a partition of two coupls (L, s), (L, t), such that t is uniform over P. Then, for the R-norm entropy of P with respect to s, we have:*

$$H_R^s(P) = \frac{R}{R-1}\left(1 - n^{\frac{R-1}{-R}}\right) - n^{\frac{R-1}{-R}} D_R^P(s \parallel t).$$

Proof. Let $P = \{a_1, ..., a_n\}$. Then $t(a_i) = \frac{1}{n}$, for $i = 1, ..., n$. We can write:

$$\begin{aligned}
D_R^P(s \parallel t) &= \frac{R}{R-1}\left(\Big(\sum_{i=1}^{n} s(a_i)^R t(a_i)^{1-R}\Big)^{\frac{1}{R}} - 1\right) \\
&= \left(-n^{\frac{R-1}{R}}\right)\frac{R}{R-1}\left(-\Big(\sum_{i=1}^{n} s(a_i)^R\Big)^{\frac{1}{R}} + n^{\frac{R-1}{-R}}\right) \\
&= \left(-n^{\frac{R-1}{R}}\right)\frac{R}{R-1}\left(1 - \Big(\sum_{i=1}^{n} s(a_i)^R\Big)^{\frac{1}{R}} + n^{\frac{R-1}{-R}} - 1\right) \\
&= \left(-n^{\frac{R-1}{R}}\right)\frac{R}{R-1}\left(1 - \Big(\sum_{i=1}^{n} s(a_i)^R\Big)^{\frac{1}{R}}\right) + \frac{R}{R-1}\left(n^{\frac{R-1}{R}} - 1\right) \\
&= \left(-n^{\frac{R-1}{R}}\right)H_R^s(P) + \frac{R}{R-1}\left(n^{\frac{R-1}{R}} - 1\right).
\end{aligned}$$

Thus

$$H_R^s(P) = \frac{R}{R-1}\left(1 - n^{\frac{R-1}{-R}}\right) - n^{\frac{R-1}{-R}} D_R^P(s \parallel t).$$

□

By the concept of R-norm divergence of states on quantum logics, we obtain an upper bound for the R-norm entropy in quantum logics (see the following corollary).

Corollary 4.6. *For any partition $P = \{a_1, ..., a_n\}$ of a coupl (L, s), it holds*

$$H_R^s(P) \leq \frac{R}{R-1}\left(1 - n^{\frac{R-1}{-R}}\right),$$

with the equality if and only if s is uniform over the partition P.

Proof. Consider a state t on quantum logic L uniform over P, i.e., it holds $t(a_i) = \frac{1}{n}$, for $i = 1, 2, ..., n$. Then by Theorem 4.5 we get:

$$D_R^P(s \parallel t) = -n^{\frac{R-1}{R}} H_R^s(P) + \frac{R}{R-1}\left(n^{\frac{R-1}{R}} - 1\right).$$

Since $D_R^P(s \parallel t) \geq 0$, it holds the inequality:

$$H_R^s(P) \leq \frac{R}{R-1}\left(1 - n^{\frac{R-1}{-R}}\right).$$

On the other hand, $D_R^P(s \parallel t) = 0$, if and only if $s(a_i) = t(a_i)$, for $i = 1, 2, ..., n$. This means that the equality $H_R^s(P) = \frac{R}{R-1}\left(1 - n^{\frac{R-1}{-R}}\right)$ holds if and only if $s(a_i) = \frac{1}{n}$, for $i = 1, ..., n$. $\square$

5 Conclusions

In this work we defined the notions of R-norm entropy, conditional R-norm entropy and R-norm divergence in quantum logics. Sections 3, 4 include the obtained results.

In Section 3 we started with defining the concept of R-norm entropy of order $R \in (0, 1) \cup (1, +\infty)$ of partitions in a quantum logic and proved this quantity is always nonnegative (see Remark 3.2). In Theorem 3.5, the concavity of R-norm entropy $H_R^s(P)$ as a function on the family of all states on a quantum logic was shown. After defining the conditional R-norm entropy $H_R^s(P \mid Q)$, we proved that, this measure information has the property of monotonicity (see Theorem 3.10). The results of Theorem 3.10 and Corollary 3.7 were demonstrated in this Example 3.11. In this example, we showed that the R-norm entropy $H_R^s(P)$ does not satisfy the sub-additivity property in a true sense.

In Section 4, we defined the concept of R-norm divergence of states on a quantum logic. It was shown (Theorem 4.2) that the R-norm divergence of states is nonnegative; therefore we obtained a tool for computing of

distance between two states in quantum logics. In Example 4.3, we illustrated that the R-norm divergence is not symmetrical and so it is not a metric in general. The convexity of R-norm divergence with respect to states, was proved in Theorem 4.4. We obtained a relationship between the R-norm divergence and R-norm entropy of partitions in quantum logics (Theorem 4.5). In Corollary 4.6, using the results of this section, an upper bound for the R-norm entropy of partitions was provided.
Since in the R-norm entropy formula is not used logarithms, calculations with this formula are more convenient than Shannon entropy. Therefore the suggested information measure can be used besides the Shannon entropy of partitions in quantum logics as a measure the amount of uncertainty in random events of a physical experiment.

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