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Original Research Paper

Harmonic Univalent Functions Defined by q–Derivative and Hypergeometric Function

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Abstract. We study a family of harmonic univalent functions using an operator involving q-derivative and hypergeometric functions. We then obtain necessary and sufficient condition bounds for functions in this family. Extreme points and convex set for such functions are also introduced.

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1 Introduction

Let $\mathcal{S}_{\mathcal{H}}$ denote the class of functions which are harmonic, univalent, complex valued and sense preserving in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized

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by $f(0)=f_z(0)-1=0$. Each $f\in\mathcal{S}_{\mathcal{H}}$ can be expressed by $f=h+\overline{g}$ where h and g are analytic in $\mathbb{U}$. We call h and g analytic part and co-analytic part of f respectively. Also f is locally univalent and sense preserving in $\mathbb{U}$ if and only if $|h'(z)|>|g'(z)|$ in $\mathbb{U}$, see [2]. Thus, for $f=h+\overline{g}\in\mathcal{S}_{\mathcal{H}}$, we may consider

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k,\qquad g(z)=\sum_{k=1}^{\infty}b_kz^k,\qquad (0\leqslant b_1<1).\tag{1}$$

The q–shifted factorial for $|q|<1$ is defined by

$$(\alpha,q)_k=\begin{cases}1 & ,\quad k=0,\\ (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{k-1}) & ,\quad k\in\mathbb{N},\end{cases}\tag{2}$$

where $\mathbb{N}$ denotes the set of positive integers and α is a complex number.

For complex parameters α_i, β_j and q where $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, $\beta_j\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$ and $|q|<1$, we consider the basic hypergeometric function ${}_m\Phi_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z)$ defined by

$${}_m\Phi_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1,q)_k\cdots(\alpha_m,q)_k}{(q,q)_k(\beta_1,q)_k\cdots(\beta_n,q)_k}z^k,\tag{3}$$

where $m=n+1$, $m,n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $z\in\mathbb{U}$ and the q–shifted factorial $(\alpha,q)_k$ is given by (2).

We note that

$$\begin{aligned}&\lim_{q\to1^-}\Big({}_m\Phi_n\big(q^{\alpha_1},\ldots,q^{\alpha_m};q^{\beta_1},\ldots,q^{\beta_n},q,(q-1)^{1+n-m}z\big)\Big)\\&={}_mF_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z),\end{aligned}\tag{4}$$

where ${}_mF_n(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_n,q,z)$ is the well-known hypergeometric function. For more details, one may refer to [3, 5] and [6].

The q–derivative of a function G is defined by

$$\partial_q\big(G(z)\big)=\frac{G(qz)-G(z)}{(q-1)z},\qquad (q\neq1,\quad z\neq0).\tag{5}$$

We can easily observe that

$$\partial_q\big(z^k\big) = \frac{1-q^k}{1-q}z^{k-1} = [k]_q z^{k-1}, \tag{6}$$

where $[k]_q = \frac{1-q^k}{1-q}$ is the q–integer number, see [7] and [10].

We conclude that

$$\lim_{q\to 1}\partial_q\big(G(z)\big) = G'(z).$$

For more properties of q–derivative, see [4] and [7]. Now, we consider the linear operator

$$\begin{aligned}
&\mathcal{H}_n^m(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_n;q)f(z)\\
&= \big(z\ {}_m\Phi_n(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_n;q,z)\big) * f(z)\\
&= z + \sum_{k=2}^{\infty}\Gamma(\alpha_i,\beta_j,q,k)a_k z^k,
\end{aligned} \tag{7}$$

where "$*$" stands for the well-known convolution (or Hadamard product) and

$$\Gamma(\alpha_i,\beta_j,q,k) = \frac{(\alpha_1,q)_{k-1}\cdots(\alpha_m,q)_{k-1}}{(q,q)_{k-1}(\beta_1,q)_{k-1}\cdots(\beta_n,q)_{k-1}}. \tag{8}$$

It is convenient to write

$$\mathcal{H}_n^m(\alpha_1,\dots,\alpha_m;\beta_1,\dots,\beta_n;q)f(z) = \mathcal{H}_n^m(\alpha,\beta,q)f(z). \tag{9}$$

Aldweby and Darus [1] defined the operator (7) on harmonic function $f = h + \overline{g}$ given by (1) as follows

$$\mathcal{H}_n^m(\alpha,\beta,q)f(z) = \mathcal{H}_n^m(\alpha,\beta,q)h(z) + \overline{\mathcal{H}_n^m(\alpha,\beta,q)g(z)}. \tag{10}$$

For more properties of operators given in (7) and (10), see [3].

We denote by $\mathcal{S}_{\overline{\mathcal{H}}}$ the class of functions $f = h + \overline{g}$, where

$$h(z) = z - \sum_{k=2}^{\infty}|a_k|z^k, \qquad g(z) = \sum_{k=1}^{\infty}|b_k|z^k, \qquad (|b_1| < 1). \tag{11}$$

For $\gamma \geqslant 0$, $0 \leqslant \delta, \eta \leqslant 1$, $0 \leqslant \sigma < 1$ and $t \in \mathbb{R}$ let $\mathcal{S}_{\mathcal{H}}^{t}(\gamma,\delta,\eta,\sigma)$ denote the class of functions in $\mathcal{S}_{\mathcal{H}}$ of the type (1) such that

$$\operatorname{Re}\left\{ \left(\eta e^{it} - \gamma\delta\right) - \eta e^{it}\frac{\left(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\right)''}{z''} + (\gamma+\delta)\frac{\left(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\right)'}{z'} + (1-\gamma)(1-\delta)\frac{\mathcal{H}_n^m(\alpha,\beta,q)f(z)}{z} \right\} \geqslant \sigma, \tag{12}$$

where

$$z' = \frac{\partial}{\partial\theta}(z) = iz, \qquad z'' = \frac{\partial^2}{\partial\theta^2}(z) = -z, \tag{13}$$

and

$$\begin{aligned}\left(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\right)' &= \frac{\partial}{\partial\theta}\left(\mathcal{H}_n^m(\alpha,\beta,q)f(re^{i\theta})\right)\\ &= iz\left(\mathcal{H}_n^m(\alpha,\beta,q)h\right)' - iz\overline{\left(\mathcal{H}_n^m(\alpha,\beta,q)g\right)'},\end{aligned} \tag{14}$$

$$\begin{aligned}\left(\mathcal{H}_n^m(\alpha,\beta,q)f(z)\right)'' &= \frac{\partial^2}{\partial\theta^2}\left(\mathcal{H}_n^m(\alpha,\beta,q)f(re^{i\theta})\right)\\ &= -z\left(\mathcal{H}_n^m(\alpha,\beta,q)h\right)' - z^2\left(\mathcal{H}_n^m(\alpha,\beta,q)h\right)''\\ &\quad - z\overline{\left(\mathcal{H}_n^m(\alpha,\beta,q)g\right)'} - z^2\overline{\left(\mathcal{H}_n^m(\alpha,\beta,q)g\right)''}.\end{aligned} \tag{15}$$

Also we denote by $\mathcal{S}_{\overline{\mathcal{H}}}^{t}(\gamma,\delta,\eta,\sigma)$ the subclass of $\mathcal{S}_{\mathcal{H}}^{t}(\gamma,\delta,\eta,\sigma)$ consisting of functions $f \in \mathcal{S}_{\overline{\mathcal{H}}}$ of the type (11) which satisfy the condition (12).

2 Main Results

In this section, we first give the sufficient coefficient bounds for $f(z) \in \mathcal{S}_{\mathcal{H}}^{t}(\gamma,\delta,\eta,\sigma)$ and then we show these sufficient conditions are also necessary for $f(z) \in \mathcal{S}_{\overline{\mathcal{H}}}^{t}(\gamma,\delta,\eta,\sigma)$. By using the results of Theorem 2.1 in [9], we obtain the following Theorem.

Theorem 2.1. *Suppose $f = h + \overline{g}$, h and g be given by (1) and*

$$\begin{aligned}&\sum_{k=2}^{\infty}\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|+\\&\sum_{k=1}^{\infty}\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\leqslant 1-\sigma.\end{aligned} \tag{16}$$

Then $f(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma, \delta, \eta, \sigma)$.

Proof. By using the fact that

$$\mathrm{Re}\{W\} \geqslant \sigma \Longleftrightarrow |W + 1 - \sigma| \geqslant |W - 1 - \sigma|\,,$$

and letting

$$\begin{aligned}W = \eta e^{it} - \gamma\delta) &- \eta e^{it}\frac{\left(\mathcal{H}^m_n(\alpha,\beta,q)f(z)\right)''}{z''}\\&+(\gamma+\delta)\frac{\left(\mathcal{H}^m_n(\alpha,\beta,q)f(z)\right)'}{z'}\\&+(1-\gamma)(1-\delta)\frac{\mathcal{H}^m_n(\alpha,\beta,q)f(z)}{z}.\end{aligned}$$

It is enough to show that

$$|W + 1 - \sigma| - |W - 1 - \sigma| \geqslant 0.$$

But by using (13), (14) and (15) we have

$$\begin{aligned}|W+1-\sigma| = \Bigg|&\eta e^{it} - \gamma\delta - \eta e^{it}\Bigg(1 + \sum_{k=2}^{\infty} k\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} +\\&+\sum_{k=2}^{\infty}k(k-1)\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1} + \sum_{k=1}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\\&+\sum_{k=1}^{\infty}k(k-1)\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Bigg)\\&+(\gamma+\delta)\Bigg(1+\sum_{k=2}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)a_k z^k - \sum_{k=1}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Bigg)\end{aligned}$$

$$+ (1-\gamma)(1-\delta)\Big(1+\sum_{k=2}^{\infty}\Gamma(\alpha_i,\beta_j,q,k)a_k z^{k-1}$$
$$+\sum_{k=1}^{\infty}\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Big)\Big|$$
$$\geqslant 2-\sigma-\sum_{k=2}^{\infty}\left|1+(\gamma+\delta)(k-1)+\gamma\delta-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|\left|\frac{z^k}{z}\right|$$
$$-\sum_{k=1}^{\infty}\left|1-(\gamma+\delta)(k-1)+\gamma\delta-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\left|\frac{z^k}{z}\right|,$$

and

$$|W-1-\sigma|\leqslant\sigma$$
$$+\sum_{k=2}^{\infty}\left|1+(\gamma+\delta)(k-1)+\gamma\delta-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|\left|\frac{z^k}{z}\right|$$
$$+\sum_{k=1}^{\infty}\left|1-(\gamma+\delta)(k-1)+\gamma\delta-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\left|\frac{z^k}{z}\right|,$$

where $\Gamma(\alpha_i,\beta_j,q,k)$ is defined by (8).

So by using (16), we have

$$|W+1-\sigma|-|W-1-\sigma|\geqslant$$
$$2\Big[1-\sigma-\sum_{k=2}^{\infty}\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|a_k|$$
$$-\sum_{k=1}^{\infty}\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|\Big]\geqslant 0.$$

□

Remark 2.2. All the techniques are similar to the proofs of theorems in [8] and in special case on parameters we get the same results.

Remark 2.3. The coefficient bound (16) is sharp for the function

$$H(z)=z+\sum_{k=2}^{\infty}\frac{x_k}{|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2|\,\Gamma(\alpha_i,\beta_j,q,k)}z^k$$

$$+\sum_{k=1}^{\infty}\frac{\overline{y_k}}{|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2|\,\Gamma(\alpha_i,\beta_j,q,k)}(\overline{z})^k,$$

where

$$\frac{1}{1-\sigma}\Big(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\Big)=1.$$

Theorem 2.4. *$f=h+\overline{g}\in\mathcal{S}^t_{\overline{\mathcal{H}}}(\gamma,\delta,\eta,\sigma)$ if and only if*

$$\begin{aligned}&\sum_{k=2}^{\infty}\bigg(\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\,|a_k|\\&+\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\,|b_k|\bigg)\Gamma(\alpha_i,\beta_j,q,k)\\&\leqslant 1-\sigma-\big(2(\gamma+\delta)-(1+\gamma\delta+\eta)\big)|b_1|.\end{aligned}\tag{17}$$

Proof. From Theorem 2.1 $\mathcal{S}^t_{\overline{\mathcal{H}}}(\gamma,\delta,\eta,\sigma)\subset\mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$, and since (16) is equivalent to (17) we conclude the "if part". For the "only if part", suppose that $f(z)\in\mathcal{S}^t_{\overline{\mathcal{H}}}(\gamma,\delta,\eta,\sigma)$. Then for $z=re^{i\theta}\in\mathbb{U}$, we have

$$\begin{aligned}&\operatorname{Re}\bigg\{\big(\eta e^{it}-\gamma\delta\big)-\eta e^{it}\frac{\big(\mathcal{H}^m_n(\alpha,\beta,q)f(z)\big)''}{z''}+(\gamma+\delta)\frac{\big(\mathcal{H}^m_n(\alpha,\beta,q)f(z)\big)'}{z'}\\&+(1-\gamma)(1-\delta)\frac{\mathcal{H}^m_n(\alpha,\beta,q)f(z)}{z}\bigg\}\geqslant\sigma,\\&=\operatorname{Re}\bigg\{\eta e^{it}-\gamma\delta\\&-\eta e^{it}\bigg(1+\sum_{k=2}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)a_kz^{k-1}+\sum_{k=2}^{\infty}k(k-1)\Gamma(\alpha_i,\beta_j,q,k)a_kz^{k-1}\\&+\sum_{k=1}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}+\sum_{k=1}^{\infty}k(k-1)\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\bigg)\\&+(\gamma+\delta)\bigg(1+\sum_{k=2}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)a_kz^{k-1}-\sum_{k=1}^{\infty}k\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\bigg)\\&+(1-\gamma)(1-\delta)\bigg(1+\sum_{k=2}^{\infty}\Gamma(\alpha_i,\beta_j,q,k)a_kz^{k-1}\end{aligned}$$

$$+\sum_{k=1}^{\infty}\Gamma(\alpha_i,\beta_j,q,k)b_k(\overline{z})^{k-1}\Big)\Big\}$$

$$\geqslant 1-\sum_{k=2}^{\infty}\left|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\,|a_k|\Gamma(\alpha_i,\beta_j,q,k)$$

$$+\big(2(\gamma+\delta)-(1+\gamma\delta+\eta)\big)\,|b_1|$$

$$+\sum_{k=2}^{\infty}\left|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2\right|\Gamma(\alpha_i,\beta_j,q,k)|b_k|r^{k-1}\geqslant\sigma.$$

The above inequality holds for all $z\in\mathbb{U}$. So if $z=r\to 1$. We obtain the required result (17). Now the proof of theorem is complete. □

3 Geometric Properties

In this section we introduce extreme points of $\mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$ and show that this class is a convex set.

Theorem 3.1. *$f=h+\overline{g}\in\mathcal{S}^t_{\mathcal{H}}(\gamma,\delta,\eta,\sigma)$ if and only if it can be expressed by*

$$f(z)=X_1z+\sum_{k=2}^{\infty}X_kh_k(z)+\sum_{k=1}^{\infty}Y_kg_k(z),\qquad(z\in\mathbb{U}),\tag{18}$$

where

$$h_k(z)=z-\frac{1-\sigma}{|(\gamma+\delta)k+(1-\gamma-\delta+\gamma\delta)-\eta k^2|\,\Gamma(\alpha_i,\beta_j,q,k)}z^k,$$

and

$$g_k(z)=\frac{1-\sigma}{|(\gamma+\delta)k-(1-\gamma-\delta+\gamma\delta)-\eta k^2|\,\Gamma(\alpha_i,\beta_j,q,k)}(\overline{z})^k.$$

Furthermore $\sum_{k=1}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1, X_k\geqslant 0, Y_k\geqslant 0$ *for* $k=1,2,\ldots,$ *and* $\Gamma(\alpha_i,\beta_j,q,k)$ *is given by* (8).

Proof. If f be given by (18), then

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\sigma}{|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2| \, \Gamma(\alpha_i, \beta_j, q, k)} X_k z^k$$
$$+ \sum_{k=1}^{\infty} \frac{1-\sigma}{|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2| \, \Gamma(\alpha_i, \beta_j, q, k)} Y_k (\overline{z})^k.$$

Since by (17), or equivalently by (16), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k) \times \\
&\times \frac{(1-\sigma)|X_k|}{|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2| \, \Gamma(\alpha_i, \beta_j, q, k)} \\
&+ \sum_{k=1}^{\infty} \left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k) \times \\
&\times \frac{(1-\sigma)|Y_k|}{|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2| \, \Gamma(\alpha_i, \beta_j, q, k)} \\
&= (1-\sigma)\Big(\sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k|\Big) \\
&= (1-\sigma)(1-X_1) \leqslant 1-\sigma.
\end{aligned}$$

So $f(z) \in \mathcal{S}_{\mathcal{H}}^t(\gamma, \delta, \eta, \sigma)$.

Conversely, suppose $f(z) \in \mathcal{S}_{\mathcal{H}}^t(\gamma, \delta, \eta, \sigma)$. By letting

$$X_1 = 1 - \Big(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\Big),$$

where

$$X_k = \frac{\left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k)}{1-\sigma} |a_k|,$$
$$Y_k = \frac{\left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k)}{1-\sigma} |b_k|,$$

we conclude the required representation and so the proof is complete.
□

Theorem 3.2. *If* $f_n(z)$, $n = 1, 2, \ldots$, *belongs to* $\mathcal{S}^t_{\mathcal{H}}(\gamma, \delta, \eta, \sigma)$, *then the function* $F(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$ *is also in* $\mathcal{S}^t_{\mathcal{H}}(\gamma, \delta, \eta, \sigma)$, *where* $f_n(z)$ *is defined by*

$$f_n(z) = z - \sum_{k=2}^{\infty} a_{k,n} z^k + \sum_{k=1}^{\infty} b_{k,n} (\overline{z})^k, \quad \big(n = 1, 2, \ldots, \quad \textstyle\sum_{n=1}^{\infty} \lambda_n = 1\big). \tag{19}$$

Proof. Since $f(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma, \delta, \eta, \sigma)$, by (17) or equivalently (16), for $n = 1, 2, \ldots$ we have

$$\sum_{k=2}^{\infty} \left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k)|a_{k,n}|$$
$$+ \sum_{k=1}^{\infty} \left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k)|b_{k,n}| \leqslant 1 - \sigma.$$

Also

$$F(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{k=2}^{\infty} \Big(\sum_{n=1}^{\infty} \lambda_n a_{k,n}\Big) z^k + \sum_{k=1}^{\infty} \Big(\sum_{n=1}^{\infty} \lambda_n b_{k,n}\Big) (\overline{z})^k,$$

Now according to (17) or equivalently (16), we have

$$\sum_{k=2}^{\infty} \left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \left|\sum_{n=1}^{\infty} \lambda_n a_{k,n}\right| \Gamma(\alpha_i, \beta_j, q, k)$$
$$+ \sum_{k=1}^{\infty} \left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \left|\sum_{n=1}^{\infty} \lambda_n b_{k,n}\right| \Gamma(\alpha_i, \beta_j, q, k)$$
$$= \sum_{n=1}^{\infty} \Big\{ \sum_{k=2}^{\infty} \left|(\gamma+\delta)k + (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k)|a_{k,n}|$$
$$+ \sum_{k=1}^{\infty} \left|(\gamma+\delta)k - (1-\gamma-\delta+\gamma\delta) - \eta k^2\right| \Gamma(\alpha_i, \beta_j, q, k)|b_{k,n}| \Big\} \lambda_n$$
$$\geqslant (1-\sigma) \sum_{n=1}^{\infty} \lambda_n = 1 - \sigma.$$

Thus $F(z) \in \mathcal{S}^t_{\mathcal{H}}(\gamma, \delta, \eta, \sigma)$. $\square$

Remark 3.3. We note that $\mathcal{S}^t_{\mathcal{H}}(\gamma, \delta, \eta, \sigma)$ is a convex set.

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