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Operator Jensen's Type Inequalities for Convex Functions

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Abstract. This paper is mainly devoted to studying operator Jensen inequality. More precisely, a new generalization of Jensen inequality and its reverse version for convex (not necessary operator convex) functions have been proved. Several special cases are discussed as well.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*–algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As customary, we reserve m, M for scalars and $\mathbf{1}_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. A self-adjoint operator A is said to be positive (written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ holds for all $x \in \mathcal{H}$ also an operator A

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is said to be strictly positive (denoted by $A > 0$) if A is positive and invertible. If A and B are self-adjoint, we write $B \geq A$ in case $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$–isomorphism between the C^*–algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a selfadjoint operator A and the C^*–algebra generated by A and the identity operator $\mathbf{1}_{\mathcal{H}}$. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ $(t \in \sigma(A))$ implies that $f(A) \geq g(A)$.

For $A, B \in \mathcal{B}(\mathcal{H})$, $A \oplus B$ is the operator defined on $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It's said to be unital if $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. A continuous function f defined on the interval J is called an operator convex function if $f((1-v)A + vB) \leq (1-v)f(A) + vf(B)$ for every $0 \leq v \leq 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in J.

Hansen et al. [5] showed if $f : J \to \mathbb{R}$ is an operator convex function, $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators with the spectra in J, and $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ are positive linear mappings such that $\sum_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$, then

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) \leq \sum_{i=1}^{n} \Phi_i(f(A_i)). \tag{1}$$

Though in the case of convex function the inequality (1) does not hold in general, we have the following estimate [4, Lemma 2.1]:

$$f\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x \right\rangle\right) \leq \left\langle \sum_{i=1}^{n} \Phi_i(f(A_i))x, x \right\rangle \tag{2}$$

for any unit vector $x \in \mathcal{K}$. For recent results treating the Jensen operator inequality, we refer the reader to [6, 8, 9, 10].

Remark 1.1. It is shown in [12, Theorem 3] that if $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a unital positive linear map, $A \in \mathcal{B}(\mathcal{H})$ is a positive operator with the

Cartesian decomposition $A = B + iC$, then for any unit vector $x \in \mathcal{K}$,

$$f\left(\left\langle \Phi\left(A\right)x,x\right\rangle\right) \le \begin{cases} f\left(\left(\left\langle \Phi(B)^2 x,x\right\rangle + \left\langle \Phi\left(C\right)x,x\right\rangle^2\right)^{\frac{1}{2}}\right) \\ f\left(\left(\left\langle \Phi\left(B\right)x,x\right\rangle^2 + \left\langle \Phi(C)^2 x,x\right\rangle\right)^{\frac{1}{2}}\right) \end{cases}$$
$$\le \left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle,$$

where f is a non-negative function on $[0,\infty)$, such that $g\left(t\right) = f\left(\sqrt{t}\right)$ is convex.
As noticed by Dragomir [3], if A is a positive operator, then $C = 0$ and $B = A$. Thus, the above-mentioned inequality will be reduced to

$$f\left(\left\langle \Phi\left(A\right)x,x\right\rangle\right) \le \left\langle \Phi\left(f\left(A\right)\right)x,x\right\rangle.$$

Of course, this is also true if A is self-adjoint. The following result provides the analogue of [12, Theorem 3] in the case of $A \in \mathcal{B}\left(\mathcal{H}\right)$ is an arbitrary operator. Let the assumptions above hold. Then

$$\begin{aligned} g\left(\left|\left\langle \Phi\left(A\right)x,x\right\rangle\right|^2\right) &= g\left(\left\langle \Phi\left(B\right)x,x\right\rangle^2 + \left\langle \Phi\left(C\right)x,x\right\rangle^2\right) \\ &\le g\left(\left\langle \Phi(B)^2 x,x\right\rangle + \left\langle \Phi\left(C\right)x,x\right\rangle^2\right) \\ &\le g\left(\left\langle \Phi(B)^2 + \Phi(C)^2 x,x\right\rangle\right) \\ &\le g\left(\left\langle \Phi\left(B^2\right) + \Phi\left(C^2\right)x,x\right\rangle\right) \\ &= g\left(\left\langle \Phi\left(B^2 + C^2\right)x,x\right\rangle\right) \\ &= g\left(\left\langle \Phi\left(\frac{|A|^2 + |A^*|^2}{2}\right)x,x\right\rangle\right) \\ &\le \left\langle \Phi\left(g\left(\frac{|A|^2 + |A^*|^2}{2}\right)\right)x,x\right\rangle, \end{aligned}$$

where the first and the second inequality follows from the Cauchy–Schwarz inequality and the third inequality obtained from the Kadison's inequality [1, 2.3.2 Theorem]. So,

$$\begin{aligned} g\left(\left|\left\langle \Phi\left(A\right)x,x\right\rangle\right|^{2}\right) &\le g\left(\left\langle \Phi(B)^{2}x,x\right\rangle + \left\langle \Phi\left(C\right)x,x\right\rangle^{2}\right) \\ &\le \left\langle \Phi\left(g\left(\frac{|A|^{2}+|A^{*}|^{2}}{2}\right)\right)x,x\right\rangle . \end{aligned}$$

On account of the assumption on f, we get

$$\begin{aligned} f\left(\left|\left\langle \Phi\left(A\right)x,x\right\rangle\right|\right) &\le f\left(\left(\left\langle \Phi(B)^{2}x,x\right\rangle + \left\langle \Phi\left(C\right)x,x\right\rangle^{2}\right)^{\frac{1}{2}}\right) \\ &\le \left\langle \Phi\left(f\left(\left(\frac{|A|^{2}+|A^{*}|^{2}}{2}\right)^{\frac{1}{2}}\right)\right)x,x\right\rangle . \end{aligned}$$

Similarly, we have

$$\begin{aligned} f\left(\left|\left\langle \Phi\left(A\right)x,x\right\rangle\right|\right) &\le f\left(\left(\left\langle \Phi\left(B\right)x,x\right\rangle^{2} + \left\langle \Phi(C)^{2}x,x\right\rangle\right)^{\frac{1}{2}}\right) \\ &\le \left\langle \Phi\left(f\left(\left(\frac{|A|^{2}+|A^{*}|^{2}}{2}\right)^{\frac{1}{2}}\right)\right)x,x\right\rangle . \end{aligned}$$

In the current paper extensions of Jensen-type inequalities for the continuous function of self-adjoint operators on complex Hilbert spaces are given. We obtain some interesting inequalities which generalize the results in [7]. We emphasize that our method in this paper is entirely different from that appeared in [13] and [14].

2 Main Results

Let $A \in \mathcal{B}\left(\mathcal{H}\right)$ be a self-adjoint operator with $\sigma\left(A\right) \subseteq [m, M]$, and let f be a convex function on $[m, M]$, then from [11], we have for any unit vector $x \in \mathcal{H}$,

$$f\left(\left\langle Ax,x\right\rangle\right) \le \left\langle f\left(A\right)x,x\right\rangle .$$

Replace A with $\Phi\left(A\right)$, where $\Phi : \mathcal{B}\left(\mathcal{H}\right) \to \mathcal{B}\left(\mathcal{K}\right)$ is a unital positive linear map, we get

$$f\left(\left\langle \Phi\left(A\right)x,x\right\rangle\right) \le \left\langle f\left(\Phi\left(A\right)\right)x,x\right\rangle \tag{3}$$

for any unit vector $x \in \mathcal{K}$. Assume that $A_1, \ldots, A_n$ are self-adjoint operators on $\mathcal{H}$ with spectra in J and $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ are positive linear maps with $\sum_{i=1}^n \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. Now apply inequality (3) to the self-adjoint operator A on the Hilbert space $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ defined by $A = A_1 \oplus \cdots \oplus A_n$ and the positive linear map Φ defined on $\mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ by $\Phi(A) = \Phi_1(A_1) \oplus \cdots \oplus \Phi_n(A_n)$. Thus,

$$f\left(\left\langle \sum_{i=1}^n \Phi_i(A_i)x, x \right\rangle\right) \le \left\langle f\left(\sum_{i=1}^n \Phi_i(A_i)\right)x, x \right\rangle. \tag{4}$$

More generalization is discussed as follows:

Lemma 2.1. *Let $f : J \to \mathbb{R}$ be a convex and differentiable function on $\overset{o}{J}$ (the interior of J) whose derivative f' is continuous on $\overset{o}{J}$, let $A_i \in \mathcal{B}(\mathcal{H})$ self-adjoint operators with the spectra in $[m, M] \subset \overset{o}{J}$ for $(i = 1, \ldots, n)$, and let $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be positive linear mappings such that $\sum_{i=1}^n \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. Then for any unit vector $x \in \mathcal{K}$,*

$$\begin{aligned}
&\left\langle \sum_{i=1}^n \Phi_i(f(A_i))x, x \right\rangle - f\left(\left\langle \sum_{i=1}^n \Phi_i(A_i)x, x \right\rangle\right) \\
&\le \left\langle \sum_{i=1}^n \Phi_i\left(f'(A_i)A_i\right)x, x \right\rangle \\
&\quad - \left\langle \sum_{i=1}^n \Phi_i(A_i)x, x \right\rangle \left\langle \sum_{i=1}^n \Phi_i\left(f'(A_i)\right)x, x \right\rangle.
\end{aligned}$$

Proof. Since f is convex and differentiable on $\overset{o}{J}$, then we have for any $t, s \in [m, M]$,

$$f'(s)(t-s) \le f(t) - f(s) \le f'(t)(t-s).$$

It is equivalent to

$$f'(s)t - f'(s)s \le f(t) - f(s) \le f'(t)t - f'(t)s. \tag{5}$$

If we fix $s \in [m, M]$ and apply the continuous functional calculus for A_i $(i = 1, \ldots, n)$, we get

$$f'(s)A_i - f'(s)s\mathbf{1}_{\mathcal{H}} \le f(A_i) - f(s)\mathbf{1}_{\mathcal{H}} \le f'(A_i)A_i - sf'(A_i).$$

Applying the positive linear mappings Φ_i and summing on i from 1 to n, this implies

$$\begin{aligned} f'(s)\sum_{i=1}^{n}\Phi_i(A_i) - f'(s)\,s\mathbf{1}_{\mathcal{K}} &\le \sum_{i=1}^{n}\Phi_i(f(A_i)) - f(s)\,\mathbf{1}_{\mathcal{K}} \\ &\le \sum_{i=1}^{n}\Phi_i\left(f'(A_i)\,A_i\right) - s\sum_{i=1}^{n}\Phi_i\left(f'(A_i)\right). \end{aligned}$$

Therefore, for any unit vector $x \in \mathcal{K}$, we have

$$\begin{aligned} & f'(s)\left\langle \sum_{i=1}^{n}\Phi_i(A_i)x, x\right\rangle - f'(s)\,s \\ & \le \left\langle \sum_{i=1}^{n}\Phi_i(f(A_i))x, x\right\rangle - f(s) \\ & \le \left\langle \sum_{i=1}^{n}\Phi_i\left(f'(A_i)\,A_i\right)x, x\right\rangle - s\left\langle \sum_{i=1}^{n}\Phi_i\left(f'(A_i)\right)x, x\right\rangle. \end{aligned}$$

Since $\sum_{i=1}^{n}\Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$ and $\sigma(A_i) \subseteq [m, M]$, then $\sigma\left(\sum_{i=1}^{n}\Phi_i(A_i)\right) \subseteq [m, M]$. Thus, by substituting $s = \langle \sum_{i=1}^{n}\Phi_i(A_i)x, x\rangle$, we deduce the desired result. □

We now have all the tools needed to write the proof of the first theorem.

Theorem 2.2. *Let all the assumptions of Lemma 2.1 hold. Then*

$$\sum_{i=1}^{n}\Phi_i(f(A_i)) \le f\left(\sum_{i=1}^{n}\Phi_i(A_i)\right) + \delta\mathbf{1}_{\mathcal{K}}$$

where

$$\begin{aligned} \delta = \sup_{\substack{x\in\mathcal{K} \\ \|x\|=1}} & \left\{ \left\langle \sum_{i=1}^{n}\Phi_i\left(f'(A_i)\,A_i\right)x, x\right\rangle - \left\langle \sum_{i=1}^{n}\Phi_i(A_i)x, x\right\rangle \right. \\ & \left. \times \left\langle \sum_{i=1}^{n}\Phi_i\left(f'(A_i)\right)x, x\right\rangle \right\}. \end{aligned}$$

Proof. It follows from the assumptions that

$$\begin{aligned}
0 &\le \left\langle \sum_{i=1}^{n} \Phi_i \left(f\left(A_i\right)\right)x, x \right\rangle - f\left(\left\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x \right\rangle\right) \\
&\le \left\langle \sum_{i=1}^{n} \Phi_i \left(f'\left(A_i\right) A_i\right)x, x \right\rangle \\
&\quad - \left\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x \right\rangle \left\langle \sum_{i=1}^{n} \Phi_i \left(f'\left(A_i\right)\right)x, x \right\rangle \\
&\le \sup_{\substack{x\in\mathcal{K} \\ \|x\|=1}} \left\{ \left\langle \sum_{i=1}^{n} \Phi_i \left(f'\left(A_i\right) A_i\right)x, x \right\rangle - \left\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x \right\rangle \right. \\
&\quad \left. \times \left\langle \sum_{i=1}^{n} \Phi_i \left(f'\left(A_i\right)\right)x, x \right\rangle \right\} \\
&= \delta,
\end{aligned}$$

thanks to Lemma 2.1. Therefore,

$$\left\langle \sum_{i=1}^{n} \Phi_i \left(f\left(A_i\right)\right)x, x \right\rangle \le f\left(\left\langle \sum_{i=1}^{n} \Phi_i \left(A_i\right)x, x \right\rangle\right) + \delta$$

for any unit vector $x \in \mathcal{K}$.

Now we can write,

$$\begin{aligned}\left\langle \sum_{i=1}^{n} \Phi_i\left(f\left(A_i\right)\right)x, x\right\rangle &\le f\left(\left\langle \sum_{i=1}^{n} \Phi_i\left(A_i\right)x, x\right\rangle\right) + \delta \\ &\le \left\langle f\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right)x, x\right\rangle + \delta \quad (\text{by } (4)) \\ &= \left\langle f\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right)x, x\right\rangle + \delta\left\langle x, x\right\rangle \\ &\quad (\text{since } \|x\| = 1) \\ &= \left\langle f\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right)x, x\right\rangle + \left\langle \delta \mathbf{1}_{\mathcal{K}} x, x\right\rangle \\ &= \left\langle f\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right) + \delta \mathbf{1}_{\mathcal{K}} x, x\right\rangle\end{aligned}$$

for any unit vector $x \in \mathcal{K}$.
By replacing x by $\frac{y}{\|y\|}$ where y is any vector in $\mathcal{K}$, we deduce the desired inequality. □

A kind of a converse of Theorem 2.2 can be considered as follows.

Theorem 2.3. *Let all the assumptions of Lemma 2.1 hold with the additional condition that f is increasing. Then*

$$f\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right) \le \sum_{i=1}^{n} \Phi_i\left(f\left(A_i\right)\right) + \zeta \mathbf{1}_{\mathcal{K}}$$

where

$$\begin{aligned}\zeta = \sup_{\substack{x \in \mathcal{K} \\ \|x\|=1}} &\left\{\left\langle f'\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right)\sum_{i=1}^{n} \Phi_i\left(A_i\right)x, x\right\rangle - \left\langle \sum_{i=1}^{n} \Phi_i\left(A_i\right)x, x\right\rangle\right. \\ &\left.\times \left\langle f'\left(\sum_{i=1}^{n} \Phi_i\left(A_i\right)\right)x, x\right\rangle\right\}.\end{aligned}$$

Proof. Fix $t \in [m, M]$. Since $[m, M]$ contains the spectra of the A_i for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \Phi_i\left(\mathbf{1}_{\mathcal{H}}\right) = \mathbf{1}_{\mathcal{K}}$, so the spectra of $\sum_{i=1}^{n} \Phi_i\left(A_i\right)$ is also contained in $[m, M]$. Then we may replace s in the inequality (5) by $\sum_{i=1}^{n} \Phi_i(A_i)$, via a functional calculus to get

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) - f(t)\,\mathbf{1}_{\mathcal{K}} \le f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)\sum_{i=1}^{n} \Phi_i(A_i) - tf'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right).$$

This inequality implies, for any $x \in \mathcal{K}$ with $\|x\| = 1$,

$$\begin{aligned}&\left\langle f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x\right\rangle - f(t) \\ &\le \left\langle f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)\sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle - t\left\langle f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x\right\rangle.\end{aligned} \tag{6}$$

Substituting t with $\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\rangle$ in (6). Thus,

$$\begin{aligned}0 &\le \left\langle f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x\right\rangle - f\left(\left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right) \quad \text{(by (4))} \\ &\le \left\langle f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)\sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle \\ &\quad - \left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\left\langle f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x\right\rangle \\ &\le \sup_{\substack{x\in\mathcal{K} \\ \|x\|=1}} \left\{\left\langle f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)\sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\right. \\ &\quad \left. - \left\langle \sum_{i=1}^{n} \Phi_i(A_i)x, x\right\rangle\left\langle f'\left(\sum_{i=1}^{n} \Phi_i(A_i)\right)x, x\right\rangle\right\} \\ &= \zeta.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle f\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)x,x\right\rangle\\
&\le f\left(\left\langle\sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right)+\zeta\\
&\le\left\langle\sum_{i=1}^{n}\Phi_i\left(f\left(A_i\right)\right)x,x\right\rangle+\zeta\quad(\text{by }(2))
\end{aligned}$$

for any unit vector $x\in\mathcal{K}$, and the proof is complete. □

3 Some Applications

In this section, we collect some consequences of Theorems 2.2 and 2.3.
(I) By setting $f(t)=t^p$ $(p\ge 1)$ in Theorems 2.2 and 2.3 we find that:

$$\sum_{i=1}^{n}\Phi_i\left(A_i^p\right)\le\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)^p+p\delta\mathbf{1}_{\mathcal{K}}\tag{7}$$

where

$$\begin{aligned}
\delta=\sup_{\substack{x\in\mathcal{K}\\ \|x\|=1}}&\left\{\left\langle\sum_{i=1}^{n}\Phi_i\left(A_i^p\right)x,x\right\rangle-\left\langle\sum_{i=1}^{n}\Phi_i\left(A_i\right)x,x\right\rangle\right.\\
&\left.\times\left\langle\sum_{i=1}^{n}\Phi_i\left(A_i^{p-1}\right)x,x\right\rangle\right\},
\end{aligned}$$

and

$$\left(\sum_{i=1}^{n}\Phi_i\left(A_i\right)\right)^p\le\sum_{i=1}^{n}\Phi_i\left(A_i^p\right)+p\zeta\mathbf{1}_{\mathcal{K}}\tag{8}$$

where

$$\zeta = \sup_{\substack{x\in\mathcal{K}\\ \|x\|=1}} \left\{ \left\langle \left(\sum_{i=1}^{n} \Phi_i\left(A_i\right) \right)^{p} x, x \right\rangle - \left\langle \sum_{i=1}^{n} \Phi_i\left(A_i\right) x, x \right\rangle \times \left\langle \left(\sum_{i=1}^{n} \Phi_i\left(A_i\right) \right)^{p-1} x, x \right\rangle \right\}.$$

whenever $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ are positive operators and $\Phi_1, \ldots, \Phi_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ positive linear mappings such that $\sum_{i=1}^{n} \Phi_i(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$. If the operators $A_1, \ldots, A_n$ are strictly positive, then (7) and (8) are also true for $p < 0$.

(II) Assume that $w_1, \ldots, w_n$ are positive scalars such that $\sum_{i=1}^{n} w_i = 1$. If we apply Theorems 2.2 and 2.3 for positive linear mappings $\Phi_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ determined by $\Phi_i : T \mapsto w_i T$ $(i = 1, \ldots, n)$, we get

$$\sum_{i=1}^{n} w_i f\left(A_i\right) \le f\left(\sum_{i=1}^{n} w_i A_i \right) + \delta \mathbf{1}_{\mathcal{H}}$$

where

$$\delta = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \left\{ \left\langle \sum_{i=1}^{n} w_i f'\left(A_i\right) A_i x, x \right\rangle - \left\langle \sum_{i=1}^{n} w_i A_i x, x \right\rangle \times \left\langle \sum_{i=1}^{n} w_i f'\left(A_i\right) x, x \right\rangle \right\},$$

and

$$f\left(\sum_{i=1}^{n} w_i A_i \right) \le \sum_{i=1}^{n} w_i f\left(A_i\right) + \zeta \mathbf{1}_{\mathcal{H}}$$

where

$$\zeta = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \left\{ \left\langle f'\left(\sum_{i=1}^{n} w_i A_i \right) \sum_{i=1}^{n} w_i A_i x, x \right\rangle - \left\langle \sum_{i=1}^{n} w_i A_i x, x \right\rangle \times \left\langle f'\left(\sum_{i=1}^{n} w_i A_i \right) x, x \right\rangle \right\}.$$

Choi's inequality [2, Proposition 4.3] says that

$$\Phi\left(B\right)\Phi(A)^{-1}\Phi\left(B\right)\le\Phi\left(BA^{-1}B\right) \tag{9}$$

whenever B is self-adjoint and A is positive invertible. We shall show the following complementary inequality of (9):

Proposition 3.1. *Let $A,B\in\mathcal{B}\left(\mathcal{H}\right)$ such that B is self-adjoint and A is positive invertible, and let $\Phi:\mathcal{B}\left(\mathcal{H}\right)\to\mathcal{B}\left(\mathcal{K}\right)$ be a unital positive linear mapping. Then*

$$\Phi\left(BA^{-1}B\right)\le\Phi\left(B\right)\Phi(A)^{-1}\Phi\left(B\right)+2\delta\Phi\left(A\right) \tag{10}$$

where

$$\delta=\sup_{\substack{x\in\mathcal{K}\\\|x\|=1}}\left\{\left\langle\Phi(A)^{-\frac{1}{2}}\Phi\left(BA^{-1}B\right)\Phi(A)^{-\frac{1}{2}}x,x\right\rangle\right.$$
$$\left.-\left\langle\Phi(A)^{-\frac{1}{2}}\Phi\left(B\right)\Phi(A)^{-\frac{1}{2}}x,x\right\rangle^{2}\right\}.$$

Proof. It follows from Theorem 2.2 that

$$\Psi\left(T^{2}\right)\le\Psi(T)^{2}+2\delta\mathbf{1}_{\mathcal{K}} \tag{11}$$

where

$$\delta=\sup\left\{\left\langle\Psi\left(T^{2}\right)x,x\right\rangle-\left\langle\Psi\left(T\right)x,x\right\rangle^{2}:\ x\in\mathcal{K};\ \ \|x\|=1\right\}.$$

To a fixed positive $A\in\mathcal{B}\left(\mathcal{H}\right)$ we set

$$\Psi\left(X\right)=\Phi(A)^{-\frac{1}{2}}\Phi\left(A^{\frac{1}{2}}XA^{\frac{1}{2}}\right)\Phi(A)^{-\frac{1}{2}}$$

and notice that $\Psi:\mathcal{B}\left(\mathcal{H}\right)\to\mathcal{B}\left(\mathcal{K}\right)$ is a unital linear map. Now, if $T=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we infer from (11) that

$$\Phi(A)^{-\frac{1}{2}}\Phi\left(BA^{-1}B\right)\Phi(A)^{-\frac{1}{2}}\le\Phi(A)^{-\frac{1}{2}}\Phi\left(B\right)\Phi(A)^{-1}\Phi\left(B\right)\Phi(A)^{-\frac{1}{2}}$$
$$+2\delta\mathbf{1}_{\mathcal{K}}$$

where

$$\delta = \sup_{\substack{x\in\mathcal{K}\\ \|x\|=1}} \left\{ \left\langle \Phi(A)^{-\frac{1}{2}}\Phi\left(BA^{-1}B\right)\Phi(A)^{-\frac{1}{2}}x, x\right\rangle \right.$$
$$\left. -\left\langle \Phi(A)^{-\frac{1}{2}}\Phi\left(B\right)\Phi(A)^{-\frac{1}{2}}x, x\right\rangle^2 \right\}.$$

By multiplying from the left and from the right with $\Phi(A)^{\frac{1}{2}}$ we obtain (10). □

The parallel sum of two positive operators A, B is defined as the operator

$$A : B = \left(A^{-1} + B^{-1}\right)^{-1}.$$

A simple calculation shows that (see, e.g., [1, (4.6) and (4.7)])

$$A : B = A - A(A+B)^{-1}A = B - B(A+B)^{-1}B. \tag{12}$$

If Φ is any positive linear map, then (see [1, Theorem 4.1.5])

$$\Phi\left(A : B\right) \le \Phi\left(A\right) : \Phi\left(B\right). \tag{13}$$

The following result gives a reverse of inequality (13).

Proposition 3.2. *Let $A, B \in \mathcal{B}\left(\mathcal{H}\right)$ positive invertible operators and let $\Phi : \mathcal{B}\left(\mathcal{H}\right) \to \mathcal{B}\left(\mathcal{K}\right)$ be unital positive linear mapping. Then*

$$\Phi\left(A\right) : \Phi\left(B\right) \le \Phi\left(A : B\right) + 2\delta\Phi\left(A+B\right)$$

where

$$\delta = \sup\left\{ \left\langle \Phi(A+B)^{-\frac{1}{2}}\Phi\left(A(A+B)^{-1}A\right)\Phi(A+B)^{-\frac{1}{2}}x, x\right\rangle \right.$$
$$\left. -\left\langle \Phi(A+B)^{-\frac{1}{2}}\Phi\left(A\right)\Phi(A+B)^{-\frac{1}{2}}x, x\right\rangle^2 :\ x \in \mathcal{K};\ \ \|x\| = 1 \right\}.$$

Proof. Proposition 3.1 easily implies

$$\Phi\left(A(A+B)^{-1}A\right) \le \Phi\left(A\right)\Phi(A+B)^{-1}\Phi\left(A\right) + 2\delta\Phi\left(A+B\right) \tag{14}$$

where

$$\delta = \sup\left\{ \left\langle \Phi(A+B)^{-\frac{1}{2}}\Phi\left(A(A+B)^{-1}A\right)\Phi(A+B)^{-\frac{1}{2}}x, x\right\rangle \right.$$
$$\left. -\left\langle \Phi(A+B)^{-\frac{1}{2}}\Phi\left(A\right)\Phi(A+B)^{-\frac{1}{2}}x, x\right\rangle^2 :\ x \in \mathcal{K};\ \ \|x\| = 1 \right\}.$$

Then we have

$$\begin{aligned}\Phi\left(A\right):\Phi\left(B\right) &= \Phi\left(A\right)-\Phi\left(A\right)\left(\Phi\left(A\right)+\Phi\left(B\right)\right)^{-1}\Phi\left(A\right) \quad \text{(by (12))}\\ &= \Phi\left(A\right)-\Phi\left(A\right)\Phi(A+B)^{-1}\Phi\left(A\right)\\ &\quad \text{(by the linearity of } \Phi\text{)}\\ &\le \Phi\left(A\right)-\Phi\left(A(A+B)^{-1}A\right)+2\delta\Phi\left(A+B\right)\\ &\quad \text{(by (14))}\\ &= \Phi\left(A-A(A+B)^{-1}A\right)+2\delta\Phi\left(A+B\right)\\ &\quad \text{(by the linearity of } \Phi\text{)}\\ &= \Phi\left(A:B\right)+2\delta\Phi\left(A+B\right).\end{aligned}$$

Hence the conclusions follow. □

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