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Original Research Paper

On Linear Operators that Preserve $BJ-$Orthogonality in $2-$Normed Spaces

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Abstract. Let X be a real $2-$Banach space. We follow Gunawan, Mashadi, Gemawati, Nursupiamin and Siwaningrum in saying that x is orthogonal to y if there exists a subspace V of X with $codim(V)=1$ such that $\|x+\lambda y, z\| \geqslant \|x,z\|$ for every $z \in V$ and $\lambda \in \mathbb{R}$. In this paper, we prove that every linear mapping $T : X \longrightarrow X$ which preserve orthogonality is a $2-$isometry multiplied by a constant.

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1. Introduction

In a real normed linear space $(X, \| \ . \ \|)$, there are many defnitions of orthogonality between two elements $x, y \in X$. Sikorska [18], studied the properties of some orthogonalities and compared them together. The most usefull version of orthogonality in a normed space $(X, \| \ . \ \|)$, commonly used by mathematicians, is the Birkhof-James orthogonality which says that x is orthogonal to y in X (and in this case we write

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not symmetric in general, but it happens when the norm comes from an inner product.

The concept of linear 2−normed spaces has been initially investigated by S. Gahler [7] and have been developed extensively by Y. J. Cho, C. Diminnie, R. Freese, S. Gahler, A.White and many other mathematicians, see, e.g. [4, 5, 6]. Let X be a linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(2) $\|x, y\| = \|y, x\|$;

(3) $\|x, \lambda y\| = |\lambda|\|x, y\| = \|\lambda x, y\|$;

(4) $\|x + y, z\| \leqslant \|x, z\| + \|y, z\|$.

Then $\|\cdot,\cdot\|$ is called a 2−norm on X and the pair $(X, \|\cdot,\cdot\|)$ is a 2−norm space.

A sequence (x_n) in 2−normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to x if the 2-norm $\|x_n - x, y\|_{n\in\mathbb{N}}$ tends to zero for all $y \in X$. In this case, we write $\lim_{n\to\infty} x_n = x$ and we call x the limit of (x_n). (x_n) is called a Cauchy sequence if there exist linearly independent elements y and z in X such that $(\|x_n, y\|)$ and $(\|x_n, z\|)$ are real Cauchy sequences. If every Cauchy sequence converges to some $x \in X$, then $(X, \|\cdot,\cdot\|)$ is called a 2−Banach space.

Various notions in normed spaces have been extended to 2−normed spaces by many mathematicians. For more details, see, e.g. [5, 13, 14, 15, 16, 17]. khan and Siddiqui [10], defined another expression of the Birkhoff-James orthogonality on an arbitrary 2−normed space $(X, \|.,.\|)$ asserting that x is orthogonal to y (and denoted by $x \underset{BJ}{\perp} y$) if $\|x + \lambda y, z\| \geqslant \|x, z\|$ for every $z \neq 0$ and $\lambda \in \mathbb{R}$. However, Gunawan, Mashadi, Gemawati, Nursupiamin and Siwaningrum [8] remarked that there are no such elements satisfying orthogonality in this sense and restricted the relation $\|x+\lambda y, z\| \geqslant \|x, z\|$ to a subspace V of codimention 1. Actually,

$x \underset{BJ}{\perp} y$ if and only if there exists a subspace V of X with $codim(V) = 1$ such that $\|x + \lambda y, z\| \geqslant \|x, z\| \quad$ for every $z \in V$ and $\lambda \in \mathbb{R}$.

In 1984, A. White and Y. J. Cho gave a chractrization for continuity of linear self mapping in 2−normed spaces [21]. In 1992, Koldabsky [12] proved that a linear self mapping T on a real normed space X preserves orthogonality if and only if T is an isometry multiplied by a positive constant and then in 2006 Blanco and Turnsek [3] extend this result to the case of linear operator between two different coplex normed linear spaces. In this paper, we prove the Koldabsky theorem for 2−normed spaces.

2. Main Results

Definition 2.1. *Let X and Y be two linear 2−normed spaces. Then the map $T : X \longrightarrow Y$ is said to be a 2− isometry if*

$$\|x_1, x_2\| = \|T(x_1), T(x_2)\|$$

for all $x_1, x_2 \in X$.

Throughout this paper, X denotes a 2−Banach space and X^* is its dual.

Let x_0 and $z_0 \in X$ and $x_0 \neq 0,\ z_0 \neq 0$. We define $S^{z_0}(x_0)$ as follows

$$S^{z_0}(x_0) = \{x^* \in X^*; \sup_{x \in X\ ,\ \|x, z_0\| = 1} x^*(x) = 1\ ,\ x^*(x_0) = \|x_0, z_0\|\}.$$

Definition 2.2. [22] *Let $f : X \to \mathbb{R}$ be a function, $x_0 \in X$, $x_0^* \in X^*$, then we say that x_0 is a subgradient of f at x if*

$$f(y) - f(x) \geqslant x_0^*(y - x_0) \qquad y \in Y.$$

The subdifferential of f at x_0 is the set of all subgradients of f at x_0 and is denoted by $\partial f(x_0)$.

Theorem 2.3. [2, Theorem 2.39] *Let $f : X \to \mathbb{R}$ be a proper convex function. If f is finite and continuous at x_0, then*

$$\lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda y) - f(x_0)}{\lambda} = \sup\{x^*(y) | x^* \in \partial f(x_0)\}.$$

By [22, Theorem 2.4.14, part (iii)] we have:

$$\partial f(x_0) = \{x^* \in \partial f(0) | x^*(x_0) = f(x_0)\}. \tag{1}$$

Let $f : X \to \mathbb{R}$, $f(x) := \|x, z_0\|$ and $z_0 \in X$. Then by Definition 2.2 and (1) we have

$$\begin{aligned}\partial f(x_0) &= \{x^* \in X^*; x^*(x) \leqslant \|x, z_0\| \ , \ x^*(x_0) = \|x_0, z_0\|\} \\ &= \{x^* \in X^*; \sup_{x \in X \ , \ \|x, z_0\| = 1} x^*(x) = 1 \ , \ x^*(x_0) = \|x_0, z_0\|\} \\ &= S^{z_0}(x_0).\end{aligned}$$

Thus, by Theorem 2.3, we have

$$\lim_{\lambda \to 0^+} \frac{\|x + \lambda y, z\| - \|x, z\|}{\lambda} = \sup_{x^* \in S^z(x)} x^*(x) \tag{2}$$

$$\lim_{\lambda \to 0^-} \frac{\|x + \lambda y, z\| - \|x, z\|}{\lambda} = \inf_{x^* \in S^z(x)} x^*(x). \tag{3}$$

for every $x, y \in X, \ x \neq 0$.

Let $D^z(x, y)$ be the set of all real numbers λ at which the function $\phi^z(\lambda) = \|x + \lambda y, z\|$ is differentiable.

Lemma 2.4. *The function $\phi^z(\lambda)$ is differentiable if and only if the value of $x^*(y)$ is independent of choice of $x^* \in S^z(x + \lambda y)$.*

Proof. ϕ^z is differentiable if and only if

$$\lim_{h \to 0^+} \frac{\|x + \lambda y + hy, z\| - \|x + \lambda y, z\|}{h} = \lim_{h \to 0^-} \frac{\|x + \lambda y + hy, z\| - \|x + \lambda y, z\|}{h}$$

if and only if

$$\sup_{x^* \in S^z(x + \lambda y)} x^*(y) = \inf_{x^* \in S^z(x + \lambda y)} x^*(y).$$

Now, suppose that there exists x_1^* and $x_2^* \in X^*$ such that $x_1^*(y) \neq x_2^*(y)$. Let $x_1^*(y) < x_2^*(y)$, we have

$$\inf_{x^* \in S^z(x + \lambda y)} x^*(y) \leqslant x_1^*(y) < x_2^*(y) \leqslant \sup_{x^* \in S^z(x + \lambda y)} x^*(y),$$

therefore

$$\sup_{x^*\in S^z(x+\lambda y)} x^*(y) \neq \inf_{x^*\in S^z(x+\lambda y)} x^*(y),$$

this is a contradiction. Hence

$$x_1^*(y) = x_2^*(y) \qquad \text{for all } \; x_1^*, x_2^* \in S^z(x+\lambda y). \quad \square$$

Lemma 2.5. [19] *Every convex function $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere on $\mathbb{R}$ respect to Lebesgue measure.*

Let λ_1 and $\lambda_2 \in \mathbb{R}$. Then for every $t \in [0,1]$, we have

$$\begin{aligned} \phi^z(t\lambda_1 + (1-t)\lambda_2) &= \|x + (t\lambda_1 + (1-t)\lambda_2)y, z\| \\ &= \|tx + t\lambda_1 y, z\| + \|(1-t)x + (1-t)\lambda_1 y, z\| \\ &\leqslant t\|x + \lambda_1 y, z\| + (1-t)\|x + \lambda_1 y, z\| \\ &\leqslant t\phi^z(\lambda_1) + (1-t)\phi^z(\lambda_2). \end{aligned}$$

So by Lemma 2.5, ϕ^z is differentiable almost everywhere on $\mathbb{R}$ with respect to Lebesgue measure.

Lemma 2.6. *Let $\lambda \in D^z(x,y)$ and $a, b \in \mathbb{R}$. Then*

(1) The value of $x^*(ax+by)$ is independent of the choice of $x^* \in S^z(x+\lambda y)$.

(2) $x + \lambda y \underset{BJ}{\perp} ax + by$ if and only if $S^z(ax+by) = 0$ for every $x^* \in S^z(x+\lambda y)$.

Proof.

(1) . Let $x^* \in S^z(x+\lambda y)$. By Lemma 2.4, $x^*(y)$ is independent of the choice x^*. Furthermore,

$$x^*(x) = x^*(x+\lambda y) - \lambda x^*(y) = \|x - \lambda y, z\| - \lambda x^*(y) \qquad (4)$$

for every $x^* \in S^z(x+\lambda y)$. By the independence of $x^*(y)$ and equation 4, the result holds.

(2) . If $x+\lambda y \underset{BJ}{\perp} ax+by$ then there exists subspace V with $codim(V) = 1$ such that

$$\|x+\lambda y+t(ax+by), z\| \geqslant \|x+\lambda y, z\| \qquad \text{for all } t\in\mathbb{R} \text{ and } z\in V.$$

Therefore,

$$\lim_{t\to 0^+} \frac{\|x+\lambda y+t(ax+by), z\| - \|x+\lambda y, z\|}{t} \geqslant 0;$$

$$\lim_{t\to 0^-} \frac{\|x+\lambda y+t(ax+by), z\| - \|x+\lambda y, z\|}{t} \leqslant 0.$$

Hence

$$\inf_{x^*\in S^z(x+\lambda y)} x^*(ax+by) \leqslant 0 \leqslant \sup_{x^*\in S^z(x+\lambda y)} x^*(ax+by). \qquad (5)$$

The fact that the value of $x^*(ax+by)$ is independent of the choice of x^*, makes the inequlity 5 into equlity. Therefore

$$x^*(ax+by) = 0 \qquad for\ all\ x^*\in S^z(x+\lambda y).$$

Conversely, if $x^*(ax+by) = 0$ for any $x^*\in S^z(x+\lambda y)$ then

$$x^*(x+\lambda y+\gamma(ax+by)) = x^*(x+\lambda y) = \|x+\lambda y, z\| \qquad for\ every\ \ \gamma\in\mathbb{R}.$$

Since $\underset{x\in X\ ,\ \|x,y\|=1}{x^*(x)} = 1$, we have

$$1 = \sup_{w\in\mathbf{X}} x^*\Big(\frac{w}{\|w,z\|}\Big) \geqslant x^*\Big(\frac{x+\lambda y+\gamma(ax+by)}{\|x+\lambda y+\gamma(ax+by), z\|}\Big) = \frac{\|x+\lambda y, z\|}{\|x+\lambda y+\gamma(ax+by), z\|}$$

which means

$$x+\lambda y \underset{BJ}{\perp} ax+by. \quad \square$$

Remark 2.7. *Let* $x+ay \underset{BJ}{\perp} y$. *There exists a subspace* V *of* X *with* $codim(V) = 1$ *such that*

$$\|(x+ay)+\lambda y, z\| \geqslant \|x+ay, z\| \qquad for\ every\ z\in V,\ \lambda\in\mathbb{R}.$$

We choose an element $z \in V$ and fix it. we have;
$\|(x+ay)+\lambda y, z\| \geqslant \|x+ay, z\|$ *for every* $\lambda \in \mathbb{R}$ *if and only if* $\|x+ay, z\|$ *is the smallest value of* $\|x+\lambda y, z\|$. *Since* $\|x+\lambda y, z\|$ *is continuous at* λ, *it most take its minimum. Suppose* $x+my \underset{BJ}{\perp} x$ *and* $x+My \underset{BJ}{\perp} x$. *since the function* ϕ *is convex, it follows that* $\|x+my, z\| = \|x+My, z\| = \|x+ay, z\|$ *if* a *is between* m *and* M *and that* $x+ay \underset{BJ}{\perp} y$. *So, the set of numbers* a *for which* $\|(x+ay)+\lambda y, z\| \geqslant \|x+ay, z\|$ *is closed interval* $[m, M]$ *and* $\|x+ay, z\| = \|x+my, z\| = \|x+My, z\|$ *for any* $a \in [m, M]$.

Lemma 2.8. *For each* $a \in D^z(x,y)$, *we have either* $\|(x+ay)+\lambda y, z\| \geqslant \|x+ay, z\|$ *for any* $\lambda \in \mathbb{R}$ *or there exists a unique number* $k_a \in \mathbb{R}$ *such that* $\|(x+ay)+\lambda(x-k_a y), z\| \geqslant \|x+ay, z\|$ *for every* $\lambda \in \mathbb{R}$.

Proof. Let $a \in D^z(x,y)$ and $x^* \in S^z(x+ay)$. Using Lemma 2.4 and 2.6 the value of the $x^*(x)$ and $x^*(y)$ are not depend to the choice of $x^* \in S^z(x+ay)$. If $x^*(y) = 0$, then by Lemma **??**, $\|(x+ay)+\lambda y, z\| \geqslant \|x+ay, z\|$. If $x^*(y) \neq 0$, then $\|(x+ay)+\lambda(x-by),\ z\| \geqslant \|x+ay, z\|$ if and only if $x^*(x-by) = 0$. Thus, $k_a = \dfrac{x^*(x)}{x^*(y)}$. □

Lemma 2.8, define the function f on $\mathbb{R}\backslash[m, M]$ by $f(a) = k_a$. It appeares that the $2-$norm can be expressed in terms of the function f.

Lemma 2.9. *Consider* m *and* M *as we mentioned in Remark 2.7. Let* $\lambda \in D^z(x,y)$. *If* $\lambda \in (m, M)$ *then*

$$\|x+\lambda y, z\| = \|x+My, z\|\ exp(\int_M^\lambda (t+f(t))^{-1}dt) \tag{6}$$

$$\|x+\lambda y, z\| = \|x+my, z\|\ exp(\int_\lambda^m (t+f(t))^{-1}dt) \tag{7}$$

Proof. Let $\lambda \in D^z(x,y),\ \lambda > M$. Fix $x^* \in S^z(x+\lambda y)$. By Lemma 2.8,

$$x^*(x) = f(\lambda)x^*(y). \tag{8}$$

From Equations (2),(3) and Lemma 2.4, we have

$$x^*(x) = \frac{d(\|x+\lambda y, z\|)}{d\lambda}.$$

So

$$x^*(x) = x^*(x+\lambda y) - \lambda x^*(y) = \|x+\lambda y, z\| - \lambda \frac{d(\|x+\lambda y, z\|)}{d\lambda}. \quad (9)$$

Hence, $f(\lambda)x^*(y) = \|x+\lambda y, z\| - \lambda \frac{d(\|x+\lambda y, z\|)}{d\lambda}$ which implies that

$$f(\lambda)\frac{d(\|x+\lambda y, z\|)}{d\lambda} = \|x+\lambda y, z\| - \lambda \frac{d(\|x+\lambda y, z\|)}{d\lambda}.$$

Therefore, $\frac{d(\|x+\lambda y, z\|)}{d\lambda}(f(\lambda)+\lambda) = \|x+\lambda y, z\|$ and so

$$\frac{\frac{d(\|x+\lambda y, z\|)}{d\lambda}}{\|x+\lambda y, z\|} = \frac{1}{f(\lambda)+\lambda}.$$

Since $\lambda \in D^z(x,y) \cap [M, \infty]$ and Lebesgue measure of the set $\mathbb{R} \setminus D^z(x,y)$ is zero, we get

$$\int_M^\lambda \frac{\frac{d(\|x+\omega y, z\|)}{d\omega}}{\|x+\omega y, z\|} d\omega = \int_M^\lambda \frac{1}{f(\omega)+\omega}. \quad (10)$$

The function $\phi^z(\lambda) = \|x+\lambda y, z\|$ is absolutely continuous and so the integral in the left-hand side of Equation (10) is equal to

$$ln(\frac{\|x+\lambda y, z\|}{\|x+My, z\|}).$$

This complete the proof of the Equation (6). A similar argument shows that Equation (7) is also hold. □

Theorem 2.10. *Let X be a linear $2-$normed space and $T : X \longrightarrow X$ be a linear $2-$ isometry then T is one-to-one.*

Proof. For every $a \neq 0$, since there exists a point $b \in X$ with $\|T(a), T(b)\| = \|a, b\| \neq 0$, $T(a) \neq 0$ and hence T is one-to-one. □

Let X and Y be linear $2-$normed spamathbbces. By definition of $2-$ isometry and $BJ-$orthogonality we get that, If $T : X\ longrightarrowY$ is a linear $2-$ isometry then T preserve $BJ-$orthogonality.

Now we prove the Koldobsky Theorem in 2−normed space.

Theorem 2.11. *Let X be a real 2−Banach space and $T : X \longrightarrow X$ be a linear operator that preserve BJ−orthogonality. Then there exists $k \in \mathbb{R}$ such that $T = kU$ and U is a 2−isometry.*

Proof. Let T be a linear operator preiseving BJ−orthogonality and $T \neq 0$. Fix $x, z \in X$ such that $Tx \neq 0$. Let y be an arbitrary element of X such that $x \neq \lambda y$ for every $\lambda \in \mathbb{R}$. Denote by I_1 and I_2 the intervals $[m, M]$ corresponding to the pairs of vectors (x, y) and (Tx, Ty). Since T preserves BJ−orthogonality we have $I_1 \subseteq I_2$. Now we prove that $I_1 = I_2$. Let $I = I_2 \setminus I_1 \neq \emptyset$ and let $\lambda \in I$ and $\lambda \in D^z(x, y) \cap D^z(Tx, Ty)$. Since $\lambda \in I_2$ so $Tx + \lambda Ty \underset{BJ}{\perp} Ty$. By Lemma 2.8 there exists $k_\lambda \in \mathbb{R}$ such that

$$x + \lambda y \underset{BJ}{\perp} x - k_\lambda y,$$

so

$$Tx + \lambda Ty \underset{BJ}{\perp} Tx - k_\lambda Ty.$$

By Lemma 2.6 for any $x^* \in S^{\lambda Ty}(Tx + \lambda Ty)$ we have

$$x^*(Ty) = 0 \quad , \quad x^*(Tx - k_\lambda Ty) = 0.$$

Hence

$$\|T(x + \lambda y), T(\lambda y)\| = x^*(Tx + \lambda Ty) = x^*(Tx - k_\lambda + k_\lambda + \lambda Ty)$$

$$= x^*(Tx - k_\lambda Ty) + k_\lambda x^*(Ty) + \lambda x^*(Ty) = 0$$

and so there exists $\alpha \in \mathbb{R}$ such that

$$T(x + \lambda y) = \alpha T(\lambda y)$$

or equivalently

$$T(x + \lambda y - \alpha \lambda y) = 0.$$

Concequently by Theorem 2.10 we get

$$x = \lambda(1 - \alpha)y$$

which is a contradiction. Therefore $I_1 = I_2$. Hence m, M and the function k_α are the same for both pairs (x, y) and (Tx, Ty). Therefore $\|T(x + \lambda y), Tz\|$ and $\|x + \lambda y, z\|$ are constant for every $\lambda \in [m, M]$. Since we have

$$\|x + \lambda y, z\| = \|x + My, z\| exp(\int_M^\lambda (t + f(t))^{-1} dt)$$

and

$$\|Tx + \lambda Ty, Tz\| = \|Tx + MTy, Tz\| exp(\int_M^\lambda (t + f(t))^{-1} dt).$$

cosequently, there exists $k_1, k_2 \in \mathbb{R}$ such that

$$\|x + \lambda y, z\| = k_1$$

$$\|Tx + \lambda Ty, Tz\| = k_2$$

and hence

$$\|x + \lambda y, z\| = \frac{k_1}{k_2} \|Tx + \lambda Ty, Tz\|. \quad \square$$

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