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Original Research Paper

Periodicity Invariant of Finitely Generated Algebraic Structures

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Abstract. In this paper, we discuss the periodicity problems in the finitely generated algebraic structures and exhibit their natural sources in the theory of invariants of finite groups and it forms an interesting and relatively self-contained nook in the imposing edifice of group theory. One of the deepest and important results of the related theory of finite groups is a complete classification of all periodic groups, that is, the finite groups with periodic properties. For every integer $k \geq 2$ and a $k-$generated non-associative algebraic structure $S =< A >$, where $A = \{a_1, a_2, \ldots, a_k\}$, the sequence

$$x_i = \begin{cases} a_i, & 1 \leq i \leq k, \\ x_{i-k}(x_{i-k+1}(\ldots(x_{i-3}(x_{i-2}x_{i-1}))\ldots)), & i > k, \end{cases}$$

is called the k-nacci sequence of S with respect to the generating set A, denoted $k_A(S)$. If $k_A(S)$ is periodic, we call the length of the period of the sequence the periodicity length of S with respect to the generating set A, written $LEN_A(S)$ and the minimum of the positive integers of $LEN_A(S)$ will be mentioned as periodicity invariant of S, denoted by $\lambda_k(S)$. However, this invariant has been studied for groups and semigroups during the years as well as the associative property of S where

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above sequence was reduced to $x_i = x_{i-k}x_{i-k+1}\dots x_{i-3}x_{i-2}x_{i-1}$, for every $i \geq k+1$. Thus, we attempt to give explicit upper bounds for the periodicity invariant of two infinite classes of finite non-associative 3-generated algebraic structures. Moreover, two classes of non-isomorphic Moufang loops of the same periodicity length were obtained in the study.

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Keywords and Phrases: Non-associativity, Loops, Periodic sequences, Finitely generated structures.

1 Introduction

The Fibonacci sequence and its related higher-order sequences such as k-nacci are generally viewed as sequences of integers. In 1990, periodic sequences in associative algebraic structures were considered mainly in groups and semi-groups by the researchers in following of the basic k-nacci sequences and the periods of these sequences in finite groups. However, these structures are 2 or 3 generated groups, one may consult [2, 3, 6, 7, 8, 14, 15], for examples. The article Johnson [11] introduced the invariant $\lambda(G)$ for 2-generated groups and proposed a distinctive example of infinite group with an upper bound for $\lambda(G)$. Meanwhile, we are interested to use the notation for non-associative algebraic structures, the loops and specially two infinite classes of finite loops. Our notation are merely standard and following [1, 4, 5, 10, 12], and here we recall the definitions before analyzing the problem:
A quasi-group is a non-empty set with a binary operation so that is applied to all three elements x, y and z of the set, and the equation $xy = z$ has a unique solution for any two of three specified elements. A quasi-group with neutral element is called a loop. A Moufang loop is a loop that holds each of equivalent scales, $((xy)x)z = x(y(xz))$, $x(y(zy)) = ((xy)z)y)$, $(xy)(zx) = x((yz)x)$ and $(xy)(zx) = (x(yz))x$. Moufang loops are certainly well-known loops that more information one may consult [4, 6, 8, 9]. Also, Moufang loops are generally non-associative forms with more properties in the groups. In 1978, Chein [7] began to classify Moufang loops and identified all ones with n order, where $1 \leq i \leq 63$. For any more details, one may see the interesting efforts in [5] and [13]. The following table shows all of non-associative

Moufang loops with n order, obviously:

n	12	16	20	24	28	32	36	40	42	44	48	52	54	56	60	64	81	243
M(n)	1	5	1	5	1	71	4	5	1	1	51	1	2	4	5	4262	5	72

where M(n) means the number of non-isomorphic Moufang loops with n order. Also, Chein [4] introduced a class of Moufang loops $M(G,2)$ as follows:
Let G be a finite group of order n and u be a new element ($u \notin G$). The multiplication $\circ$ on $G \cup Gu$ is defined by:

$$\begin{cases} g \circ h = gh, & \text{if } g,h \in G, \\ g \circ (hu) = (hg)u, & \text{if } g \in G,\ hu \in Gu, \\ (gu) \circ h = (gh^{-1})u, & \text{if } gu \in Gu,\ h \in G, \\ (gu) \circ (hu) = h^{-1}g, & \text{if } gu, hu \in Gu. \end{cases}$$

But, the resulting loop $M(G,2) = (G \cup Gu, \circ)$ is a typical Moufang loop. It is obvious that $M(G,2)$ will be a non-associative if and only if G is non-abelian. The Moufang loops naturally considered as multiplicative loop of Octonions in algebra and as Moufang Planes in the projective geometry. Also, two Moufang loops $M(G_1,2)$ and $M(G_2,2)$ of the Chein type are said to be non-isomorphic structure if the groups G_1 and G_2 are non-isomorphic groups. By focusing on non-associative Chein type Moufang loops, for every integer $n \geq 3$, we will consider two infinite classes of finite loops, where the group G is D_{2n}, the dihedral group and is Q_{2^n}, the generalized quaternion group. Note that in the above-mentioned definition, for every 2-generated group G, the Moufang loops $M(G,2)$ of Chein type are all 3-generated.

Our next preliminary definition concerning non-associative structures is:
Definition 1.1. Let S be a finite k-generated of non-associative algebraic structure and $k \geq 2$ be a given integer. Then, the positive integer is

$$\lambda_k(S) = \min\{LEN_A(S) \mid \ S = \langle A \rangle\},$$

where $\text{LEN}_A(S)$ is the length of period of the sequence

$$x_i = \begin{cases} a_i, & 1 \leq i \leq k, \\ x_{i-k}(x_{i-k+1}(\ldots(x_{i-3}(x_{i-2}x_{i-1}))\ldots)), & i > k, \end{cases}$$

of the elements of S, and $A = \{a_1, \ldots, a_k\}$ is a generating set of S. We call $\lambda_k(S)$ as the periodicity invariant of S and $\mathrm{LEN}_A(S)$ is called the periodicity length of the given sequence with respect to the generating set A.

2 Main results

For a given integer $n \geq 3$, by concerning 2-generated non-abelian groups $D_{2n} =< a, b \mid a^n = b^2 = (ab)^2 = 1 >$, $Q_{2^n} =< a, b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ (ab)^2 = 1 >$ and the corresponding non-associative finite Moufang loops $M(D_{2n}, 2)$ and $M(Q_{2^n}, 2)$, our main results are:

Proposition 2.1. *Let $D_{2n} = \langle a, b\rangle$ be a dihedral group, where $n \geq 3$. Then, according to the generating set $A = \{a, b, u\}$, the periodicity length of the Moufang loop $M(D_{2n}, 2)$ is equal to:*

$$LEN_A(M(D_{2n}, 2)) = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ 4n, & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 2.2. *Let $Q_{2^n} = \langle a, b\rangle$ be a generalized quaternion group, where $n \geq 3$. Then, the periodicity length of the Moufang loop $M(Q_{2^n}, 2)$ is $LEN_A(M(Q_{2^n}, 2)) = 2^n$ with respect to the generating set $A = \{a, b, u\}$.*

Corollary 2.3. *For every integer $n \geq 3$, the upper bounds of the periodicity invariant of the loops $M(D_{2n}, 2)$ and $M(Q_{2^n}, 2)$ will be defined as follows:*

$$\begin{array}{ll} \lambda(M(D_{2n}, 2)) \leq 2n, & \text{if } n \text{ is even,} \\ \lambda(M(D_{2n}, 2) \leq 4n, & \text{if } n \text{ is odd,} \\ \lambda(M(Q_{2^n}, 2) \leq 2^n, & \text{for every } n. \end{array}$$

Moreover, for every integer $k \geq 3$, the Moufang loops $M(D_{2\times 2^k}, 2)$ and $M(Q_{2^{k+1}}, 2)$ are two finite non-isomorphic Moufang loops with a same periodicity length of 2^{k+1}.

3 The proofs

On the behaviour of sequences of elements of loops of Chein type, two preliminary results are necessary. In the following lemma, we checked the elements of the sequence $\{x_i\}_{i=1}^{\infty}$ of $M(G,2)$, where $G=< a,b >$ is a finite 2-generated group and the relation $aba=b$ is hold.

Lemma 3.1. *Let $G=\langle a,b\rangle$ be a 2-generated group satisfying the relation $aba=b$. Then, the elements of the sequence $\{x_i\}_{i=1}^{\infty}$ of 3-generated Moufang loop $M=M(G,2)$ meet to the generating set $A=\{a,b,u\}$ may be presented as:*

$$x_m=\begin{cases} a, & \text{if } m\equiv 1,5 \pmod 8,\\ b, & \text{if } m\equiv 2 \pmod 8,\\ a^{-(\frac{m-3}{2})}u, & \text{if } m\equiv 3 \pmod 8,\\ a^{-(\frac{m-2}{2})}bu, & \text{if } m\equiv 0,4 \pmod 8,\\ b^{-1}, & \text{if } m\equiv 6 \pmod 8,\\ a^{-(\frac{m-3}{2})}b^2u, & \text{if } m\equiv 7 \pmod 8.\end{cases}$$

Proof. By definition of periodicity length, it is obvious that $x_1=a$, $x_2=b$ and $x_3=u$. Also, to see the following relations easy in sense:

$$x_4=x_1(x_2(x_3))=a(b(u))=a(bu)=bau=a^{-1}bu,$$

$$x_5=x_2(x_3(x_4))=b(u(a^{-1}bu))=b(b^{-1}a)=bb^{-1}a=a.$$

Let $m\equiv 1 \pmod 8$ and suppose that the assertion holds for every integer less than m. Then, we get

$$\begin{cases} x_{m-3}=b^{-1}, & \text{because of } m-3\equiv -2 \pmod 8,\\ x_{m-2}=a^{-(\frac{m-5}{2})}b^2u, & \text{because of } m-2\equiv -1 \pmod 8,\\ x_{m-1}=a^{-(\frac{m-3}{2})}bu, & \text{because of } m-1\equiv 0 \pmod 8.\end{cases}$$

Therefore, by considering $x_m=x_{m-3}(x_{m-2}(x_{m-1}))$, the result is:

$$\begin{aligned} x_m &= b^{-1}(a^{-(\frac{m-5}{2})}b^2u(a^{-(\frac{m-3}{2})}bu))\\ &= b^{-1}(b^{-1}a^{(\frac{m-3}{2})}a^{-(\frac{m-5}{2})})b^2\\ &= b^{-2}a^{(\frac{m-3-m+5}{2})}b^2\\ &= a.\end{aligned}$$

Now, let $m\equiv 5 \pmod 8$ and suppose that the assertion holds for every integer less than m. Then, we get

$$\begin{cases} x_{m-3}=b, & \text{because of } m-3\equiv 2 \pmod 8,\\ x_{m-2}=a^{-(\frac{m-5}{2})}u, & \text{because of } m-2\equiv 3 \pmod 8,\\ x_{m-1}=a^{-(\frac{m-3}{2})}bu, & \text{because of } m-1\equiv 4 \pmod 8.\end{cases}$$

So, by considering $x_m = x_{m-3}(x_{m-2}(x_{m-1}))$, the result is:

$$\begin{aligned} x_m &= b(a^{-(\frac{m-5}{2})}u(a^{-(\frac{m-3}{2})}bu)) \\ &= b(b^{-1}a^{(\frac{m-3}{2})}a^{-(\frac{m-5}{2})}) \\ &= a^{(\frac{m-3-m+5}{2})} \\ &= a. \end{aligned}$$

However, these show the assertion when $m \equiv 1 \pmod 8$ and $m \equiv 5 \pmod 8$. Of course, the proofs are similar in other cases. □

As the result of the Lemma 3.1, we will have the following Lemmas:

Lemma 3.2. *Let $D_{2n} = \langle a, b\rangle$ be a dihedral group, where $n \geq 3$. The elements of the sequence $\{x_m\}_1^\infty$ is related to the Moufang loop $M(D_{2n}, 2)$ and meet to the generating set $A = \{a, b, u\}$, x_m may be obtained by*

$$x_m = \begin{cases} a, & \text{if } m \equiv 1,5 \pmod 8, \\ b, & \text{if } m \equiv 2,6 \pmod 8, \\ a^{-(\frac{m-3}{2})}u, & \text{if } m \equiv 3,7 \pmod 8, \\ a^{-(\frac{m-2}{2})}bu, & \text{if } m \equiv 0,4 \pmod 8. \end{cases}$$

Proof. By using Lemma 3.1 the assertion holds when $m \equiv 0, 1, 2, 3, 4, 5 \pmod 8$. In the remainder cases when $m \equiv 6, 7 \pmod 8$, the relation $b^2 = 1$ is considered for the group D_{2n}, which yields $x_m = b = b^{-1}$ and so the proof is complete . □

Lemma 3.3. *Let $Q_{2^n} = \langle a, b\rangle$ be a generalized quaternion group, where $n \geq 3$. The elements of the sequence $\{x_m\}_1^\infty$ in the Moufang loop $M(Q_{2^n}, 2)$ and meet to the generating set $A = \{a, b, u\}$, x_m may be obtained by*

$$x_m = \begin{cases} a, & \text{if } m \equiv 1,5 \pmod 8, \\ b, & \text{if } m \equiv 2 \pmod 8, \\ a^{-(\frac{m-3}{2})}u, & \text{if } m \equiv 3 \pmod 8, \\ a^{(-\frac{m-2}{2})}bu, & \text{if } m \equiv 0,4 \pmod 8, \\ b^3, & \text{if } m \equiv 6 \pmod 8, \\ a^{-(\frac{m-3}{2})}b^2u, & \text{if } m \equiv 3 \pmod 8. \end{cases}$$

Proof. Let $m \equiv 6 \pmod 8$. By considering the presentation of Q_{2^n} and the binary operation in $M(Q_{2^n}, 2)$, we get $x_m = b^{-1} = b^3$. The proof in other cases are similar to that of Lemma 3.1. □

Now, by using Lemmas 3.2 and 3.3, we can prove main Propositions here.

Proof.[Proof of Proposition 2.1.] By Lemma 3.2, the least integer, $l = LEN_A(M(D_{2n}, 2))$ satisfies all of the conditions $x_{l+1} = a$, $x_{l+2} = b$, $x_{l+3} = u$. If $l \equiv 1 \pmod 8$, then

$$x_{l+1} = b, \quad x_{l+2} = a^{-(\frac{l-1}{2})}u, \quad x_{l+3} = a^{-(\frac{l}{2})}bu.$$

which yields to the contradiction $a = b$. Let $l \equiv -1 \pmod 8$. Then, we have

$$x_{l+1} = a^{-(\frac{l-1}{2})}bu, \quad x_{l+2} = a, \quad x_{l+3} = b.$$

But we get a same contradiction again in $a = b$. For the case $l \equiv 2 \pmod 8$,

$$x_{l+1} = a^{-(\frac{l-2}{2})}u, \quad x_{l+2} = a^{-(\frac{l}{2})}bu, \quad x_{l+3} = a.$$

So, we get a contradiction $a = u$, that is an ideal case. For simplicity, we omit the proofs in other cases. Finally, let $l \equiv 0 \pmod 8$. Then,

$$x_{l+1} = a, \quad x_{l+2} = b, \quad x_{l+3} = a^{-(\frac{l}{2})}u.$$

Therefore, the following relation will be given:

$$a^{-(\frac{l}{2})}u = u.$$

Subsequently, this yields $a^{-(\frac{l}{2})} = 1$. So, n divides $-(\frac{l}{2})$. Since $l \equiv 0 \pmod 8$ so by letting $l = 8s$, we have:

$$s = \begin{cases} \frac{n}{4}, & \text{if } n \text{ is even,} \\ \frac{n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus,

$$l = 8s = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ 4n, & \text{if } n \text{ is odd.} \end{cases}$$

For even and odd values of n, we consider $\lambda(M(D_{2n}, 2)) \leq 2n$ and $\lambda(M(D_{2n}, 2)) \leq 4n$, respectively. □

Proof.[Proof of Proposition 2.2.] By Lemma 3.3, the least integer

$$l = LEN_A(M(Q_{2^n}, 2))$$

satisfies all of the conditions $x_{l+1} = a$, $x_{l+2} = b$, $x_{l+3} = u$. If $l \equiv 1 \pmod 8$. Then,

$$x_{l+1} = b, \quad x_{l+2} = a^{-(\frac{l-1}{2})}u, \quad x_{l+3} = a^{(\frac{l+1}{2})}bu.$$

Which it is in a contradiction by $a = b$. Let $l \equiv 2 \pmod 8$. Then,

$$x_{l+1} = a^{-(\frac{l-2}{2})}u, \quad x_{l+2} = a^{(\frac{l+1}{2})}bu, \, x_{l+3} = a.$$

We again get the contradiction as $a = au$. Now let $l \equiv 3 \pmod 8$. Then,

$$x_{l+1} = a^{(\frac{l-1}{2})}bu, \quad x_{l+2} = a, \, x_{l+3} = b^3.$$

Which yields a contradiction in $a = b$. However, the proofs when $m \equiv -3, -2, -1 \pmod 8$ are similar. Finally, let $l \equiv 0 \pmod 8$. Then,

$$x_{l+1} = a, \quad x_{l+2} = b, \quad x_{l+3} = a^{-(\frac{l}{2})}u.$$

Consequently, $x_{l+3} = a^{-(\frac{l}{2})}u = u$ which yields that,

$$a^{-(\frac{l}{2})} = 1.$$

So, 2^{n-1} divides $-(\frac{l}{2})$ and

$$(\frac{l}{2}) = 2^{n-1}k,$$

where k is a positive integer. Therefore, $l = 2 \times 2^{n-1}k = 2^n k$. So, by definition of l, we have, $l = 2^n$ and this yields, $\lambda(M(Q_{2^n}, 2)) \leq 2^n$. □

Proof.[Proof of Corollary 2.3.] The upper bounds for the periodicity invariants of $M(D_{2n}, 2)$ and $M(Q_{2^n}, 2)$ are in straightforward results of Propositions 3.1 and 3.2. In other hands, since

$$LEN_A(M(D_{2\times 2^k}, 2)) = LEN_A(M(Q_{2^{k+1}}, 2)) = 2^{k+1},$$

the result of these propositions satisfies for every $k \geq 3$. So, the Moufang loops are in the same orders but in non-isomorphic structures, since there are $2^{k+1}+2^k+2$ elements by order 2 in $M(D_{2\times 2^k}, 2)$ just against $2^{k+1}+2$ elements by order 2 in $M(Q_{2^{k+1}}, 2)$. □

At the end of this study, verifying the equation $|M| = \lambda(M)$ for possible Moufang loops M was considered in the periodicity invariant of finitely generated algebraic structures.
According to [4, 5, 10, 13] and all of the loops with an order less than 65 and loops with orders 81 and 243 as well as only known finite non-associative Moufang loops, we could get into the following table for seven 3-generated loops. Indeed, by performing a procedure in Gap [9], we examined all the generating sets for each loop. In this table $M(k, i)$ denotes the Moufang loop in order k and i is the version of the loop in order k, as same as findings in [4, 5].

The Moufang loop M	$\lambda(M)$
$M(24, 4)$	24
$M(40, 2)$	40
$M(40, 3)$	40
$M(40, 4)$	40
$M(44, 1)$	44
$M(48, 17)$	48
$M(56, 3)$	56

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