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On the ∩-Structure Spaces

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Abstract. The family $\mathcal{M}_X \subseteq \mathcal{P}(X)$ is called an ∩-structure, when it is closed under the arbitrary intersection. This concept has been studied and considered in algebra, specially in lattices. Using this concept, we define a quasi topological structure which is called ∩-structure space. By studying this space, we attempt to explain some algebraic concepts through this structure.

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1. Introduction

A lattice L is called a complete lattice if $\vee A$ exists for every $A \subseteq L$; or equivalently, $\wedge A$ exists for every $A \subseteq L$, and also is called a distributive lattice if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for every $a, b, c \in L$. Supposing X is an ordered set, a function $f : X \to X$ is said to be a closure operator (interior operator) if it has the following properties:

(i) f is an increasing function; i.e., if $a \leqslant b$, then $f(a) \leqslant f(b)$ for every $a, b \in X$.

(ii) f is idempotent; i.e., $f(f(a)) = f(a)$ for every $a \in X$.

(iii) f is extensive (contractive); i.e., $a \leqslant f(a)$ ($f(a) \leqslant a$) for every $a \in X$.

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A nonempty subset S of an ordered set is said to be directed if every pair of S has an upper bound in S. A nonempty family D of subsets of a set X is said to be closed under directed unions if $\cup_{i\in I}A_i \in D$ for any directed family $\{A_i\}_{i\in I}$ in D.

For any set X, an intersection structure (briefly, $\cap$-structure) on X is a family $\mathcal{M}_X$ of subsets of X which is closed under arbitrary intersection. We say $(X, \mathcal{M}_X)$, briefly X, is an $\cap$-structure space. Clearly, if $(X, \mathcal{M}_X)$ is an $\cap$-structure space, then $\mathcal{M}_X$ is a complete lattice in which for every nonempty family $\{A_i\}_{i\in I}$, we have

$$\wedge_{i\in I}A_i = \cap_{i\in I}A_i \quad , \quad \vee_{i\in I} A_i = \cap\{B \in \mathcal{M}_X : \cup_{i\in I}A_i \subseteq B\}.$$

If $\mathcal{M}_X$ is a distributive lattice, we say $(X, \mathcal{M}_X)$ is a distributive $\cap$-structure space. Obviously, X is the top element of $\mathcal{M}_X$. The least element of this complete lattice is denoted by $\circ$ and we call it zero.

It is clear that if X is any algebraic structure (for example, module, ring, group, vector space) and $\mathcal{M}_X$ is the collection of all substructures of X (resp., submodules, ideals, subgroups, subspaces), then $(X, \mathcal{M}_X)$ is an $\cap$-structure space. Hence this concept is a suitable model for studying and generalizing algebraic structures. Throughout this article R is a commutative ring with $1 \neq \circ$. We use the notations $\mathrm{Id}(R)$, $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$ for the set of all ideals , the set of all prime and the set of all maximal ideals of the ring R, respectively. We denote the annihilator of a subset $S \subseteq R$ by Ann(S), i.e., $\mathrm{Ann(S)} = \{\mathrm{r} \in \mathrm{R} : \mathrm{rS} = \circ\}$. In Section 2, we define the closure and interior of a subset in an $\cap$-structure space and study their properties. We will see that the concept of closure and interior are the same as closure and interior in a topological space. Hence they introduced two topologies τ and τ_0, which in general we show that the topology τ is stronger than the topology τ_0. In Section 3, we define the cocepts of compactness, join-compactness and connectivity in an $\cap$-structure space and investigate their properties.

2. General Properties of ∩-Structure Spaces

Definition 2.1. *Let* $(X, \mathcal{M}_X)$ *be an* ∩*-structure space and* $A \subseteq X$*. The element* $\langle A \rangle$ *of* $\mathcal{M}_X$ *generated (or spanned) by* A *is the intersection of all elements* $u \in \mathcal{M}_X$ *that contain* A*. In the case that* A *is the finite set* $\{a_1, a_2, \ldots, a_n\}$*,* $\langle A \rangle$ *is written as* $\langle a_1, a_2, \ldots, a_n \rangle$*, and is referred as the element generated by* $a_1, a_2, \ldots, a_n$*. If for an element* x *of* X*, we have* $\langle x \rangle = X$*, then we call* x *is an invertible element of* X*. The set of all invertible elements of* X *is denoted by* $\mathrm{U}(X)$*.*

Definition 2.2. *If* $(X, \mathcal{M}_X)$ *is an* ∩*-structure space and* $E \subseteq X$*, the closure of* E*, i.e.,* $\bar{E}$ *is defined by*

$$\bar{E} = \{x \in X : x \in E \text{ or } E \cap u \nsubseteq \circ \text{ for all } u \in \mathcal{M}_x\},$$

where $\mathcal{M}_x$ *is the set of all elements* u *of* $\mathcal{M}_X$ *containing* x*. If we want to emphasize the set* X*, we use the notation* $\mathrm{cl}_X E$ *for the closure of* E*.*

Lemma 2.3. *The mapping* $E \to \bar{E}$ *in an* ∩*-structure space* $(X, \mathcal{M}_X)$ *is a closure operator on* X *and moreover has the following properties:*
(a) For any collection $\{A_i\}_{i \in I}$ *of subsets of* X*,* $\overline{\cup_{i \in I} A_i} = \cup_{i \in I} \bar{A}_i$*.*
(b) For all $u \in \mathcal{M}$*:*
i) $\overline{u \setminus \circ} = u \setminus \circ$.
ii) $\overline{X \setminus u} = X \setminus u$.
iii) $\overline{X \setminus (u \setminus \circ)} = X \setminus (u \setminus \circ)$.

Proof. The proof is straightforward. □

Example. Let R be a ring. In the ∩-structure space $(R, \mathrm{Id}(R))$, for any nonzero ideal I, we have $\bar{I} = R$ if and only if I is an essential ideal of R.

Definition 2.4. *If* $(X, \mathcal{M}_X)$ *is an* ∩*-structure space and* $A \subseteq X$*, the interior of* A*, i.e.,* A° *is the set*

$$A^\circ = \{x \in X : u \subseteq A \text{ for some } u \in \mathcal{M}_x\}.$$

It is easy to see that $A^\circ = \{x \in X : \langle x \rangle \subseteq A\}$*.*

Lemma 2.5. *The operation* $A \to A^\circ$ *in an* ∩*-structure space* $(X, \mathcal{M}_X)$ *is an interior operation and moreover has the following properties:*

(a) For any collection $\{A_i\}_{i\in I}$ of subsets of X, $(\cap_{i\in I}A_i)^\circ = \cap_{i\in I}A_i^\circ$.
(b) For all $u \in \mathcal{M}_X$, $u^\circ = u$.

Proof. The proof is straightforward. $\square$

The previous lemmas show that the closure and interior have the same properties as in the topological spaces. We denote the topologies induced by closure and interior maps, by τ and $\tau_\circ$, respectively. By the definition and previous lemma, it is clear that $\mathcal{M}_X$ is a base for the topology $\tau_\circ$.

The next lemma gives us more information regarding τ and $\tau_\circ$.

Lemma 2.6. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space and $A \subseteq X$. Then the folloeing statements hold.*
(a) $\beta_\circ = \{\langle x\rangle : \ x \in X\}$ is a base (in fact the smallest base) for the topology $\tau_\circ$.
(b) The set $\beta = \{\langle x\rangle \setminus \circ : \ x \in X \setminus \circ\} \cup \circ$ is the smallest base for the topology τ. Hence if $\circ \neq \varnothing$, then $\tau_\circ \subsetneq \tau$.
(c) $x \in int_\tau A$ if and only if $x \in A$ and $\langle x\rangle \setminus \circ \subseteq A$.
(d) If $\circ \subseteq A$, then $int_{\tau_\circ}A = int_\tau A$.
(e) If $A \cap \circ = \varnothing$, then $cl_\tau A = cl_{\tau_\circ}A$. Clearly if $\circ = \varnothing$, then these two toplogies coincide.
(f) $cl_\tau A = cl_{\tau_\circ}(A \setminus \circ) \cup A$ and $int_\tau A = int_{\tau_\circ}(A \cup \circ) \setminus (\circ \setminus A)$.

Proof. (a). Clearly $\beta_\circ$ is a base for the topology $\tau_\circ$. Now, suppose that β is a base for $\tau_\circ$ and $\langle x\rangle \in \beta_\circ$. Thus, $B \in \beta$ exists such that $x \in B \subseteq \langle x\rangle$ and consequently $\langle x\rangle \subseteq B \subseteq \langle x\rangle$. Therefore, $\langle x\rangle = B \in \beta$.
(b). It is similar to (a).
(c). It is evident by part (b) and the fact that every point of $\circ$ is isolated with respect to the topology τ.
(d). Since $\tau_\circ \subseteq \tau$, clearly $int_{\tau_\circ}A \subseteq int_\tau A$. Assume that $x \in int_\tau A$. Then by (c) we have $\langle x\rangle \setminus \circ \subseteq A$ and so $\langle x\rangle = (\langle x\rangle \setminus \circ) \cup \circ \subseteq A$. Therefore, $x \in int_{\tau_\circ}A$.
(e). Since $\tau_\circ \subseteq \tau$, clearly $cl_\tau A \subseteq cl_{\tau_\circ}A$. Assume that $x \in cl_{\tau_\circ}A$. Then, clearly $\varnothing \neq \langle x\rangle \cap A \subseteq A$ and consequently $\langle x\rangle \cap A \nsubseteq \circ$. Hence, $x \in cl_\tau A$.
(f). By (e) we can write

$$cl_\tau A = cl_\tau((A \setminus \circ) \cup (A \cap \circ) = cl_\tau(A \setminus \circ) \cup A = cl_{\tau_\circ}(A \setminus \circ) \cup A.$$

Also, by (d) and the fact that $\circ\setminus A$ is clopen with respect to the topology τ, we can write

$$int_{\tau}A = int_{\tau}((A\cup\circ)\setminus(\circ\setminus A)) = (int_{\tau}(A\cup\circ))\setminus(\circ\setminus A) = (int_{\tau_{\circ}}(A\cup\circ))\setminus(\circ\setminus A). \quad \square$$

Definition 2.7. *Let $(X, \mathcal{M}_X)$ be an ∩-structure space and $Y \subseteq X$. We define:*

$$\mathcal{M}_Y = \{u \cap Y : \ u \in \mathcal{M}_X\}.$$

Obviously, $\mathcal{M}_Y$ is closed under arbitrary intersection. So $(Y, \mathcal{M}_Y)$ is an ∩-structure space and we call it the subspace of $(X, \mathcal{M}_X)$. It is clear that if $Y \in \mathcal{M}_X$ then $\mathcal{M}_Y = \{u \in \mathcal{M}_X : \ u \subseteq Y\}$.

It goes without saying that $(Y, \mathcal{M}_Y)$ is a topological subspace of $(X, \mathcal{M}_X)$ with respect to $\tau_{\circ}$. The following proposition shows that this is true for τ, too.

proposition 2.8. *If $(X, \mathcal{M}_X)$ is an ∩-structure space and $A \subseteq Y \subseteq X$, then $cl_Y A = cl_X A \cap Y$.*

Proof. It is easy to see that $\langle y\rangle \cap A \nsubseteq \circ_Y$ if and only $\langle y\rangle \cap A \nsubseteq \circ_X$. By this fact, one can easily see that $cl_Y A = cl_X A \cap Y$. $\square$

Remark 2.9. *Let R be a ring, in the ∩-structure space $(R, \mathrm{Id}(R))$, for a nonzero subset A of R, it is clear that $A \cup \mathrm{U}(R) \subseteq \bar{A}$, where $\mathrm{U}(R)$ is the set of all invertible elements of R. The natural question is, when does the equality $A \cup \mathrm{U}(R) = \bar{A}$ hold. The next proposition is a partial answer to this question.*

proposition 2.10. *Let R be a ring. In the ∩-structure space $(R, \mathrm{Id}(R))$, for any proper nonzero ideal I of R, $I \cup \mathrm{U}(R) = \bar{I}$ if and only if one of the following conditions hold.*
(a) $\mathrm{Max}(R) = \{M_{\circ}\}$ *and* $I = M_{\circ}$.
(b) $\mathrm{Spec}(R) = \{M_{\circ}, M_1\}$, *where* $I = M_{\circ}$ *and* $M_1 = \mathrm{Ann(I)}$.
(c) $\mathrm{Max}(R) = \{M_{\circ}\}$, $I \subseteq \mathrm{Ann(I)} = \mathrm{M}_{\circ}$ *and* $I \subseteq P$ *for each* $P \in \mathrm{Spec}(R)$.

Proof. $\Rightarrow$) First we notice that $R \setminus \mathrm{U}(R) = I \cup \mathrm{Ann(I)}$. Let $I \subseteq M_{\circ} \in \mathrm{Max}(R)$, then $M_{\circ} \subseteq I \cup \mathrm{Ann(I)}$, and so $M_{\circ} \subseteq \mathrm{Ann(I)}$ or $M_{\circ} \subseteq I$. Hence,

$M_\circ = I$ or otherwise $I \subsetneq M_\circ = \mathrm{Ann(I)}$. Now assume that $I = M_\circ$. In this case we show that if (a) does not hold, then (b) hold. Suppose that M_1 is an arbitrary maximal ideal different from $M_\circ$. Clearly, $M_1 \subseteq R \setminus \mathrm{U}(R) = I \cup \mathrm{Ann(I)}$ and so $M_1 = \mathrm{Ann(I)}$. Now, we show that $\mathrm{Spec}(R) = \{M_\circ, M_1\}$. To see this, suppose that $P \in \mathrm{Spec}(R)$. Then $M_\circ M_1 = \circ \subseteq P$ and so $M_\circ = P$ or $M_1 = P$.
$\Leftarrow$) If (a) or (c) hold, then it is clear that $\overline{I} = I \cup \mathrm{U}(R)$. Otherwise, suppose that $\mathrm{Spec}(R) = \{M_\circ, M_1\}$, $I = M_\circ$ and $M_1 = \mathrm{Ann(I)}$. Now, assume that $x \notin I \cup \mathrm{U}(R)$, then it is sufficient to show that $\langle x \rangle \cap I = \circ$. Let $y = rx \in I = M_\circ$. Clearly $x \in M_1$ and so $xM_\circ = \circ$. Therefore, $y^2 = r^2x^2 = yrx \in M_\circ M_1 = \circ \subseteq M_\circ$ and so $r \in M_\circ$. Consequently, $y = rx \in M_\circ M_1 = \circ$. $\square$

The following corollary is immediate.

Corollary 2.11. *Let R be a ring, in the $\cap$-structure space $(R, \mathrm{Id}(R))$, let I be a proper ideal of R such that $I \subsetneq \mathrm{Ann(I)}$ (for example in the reduced ring this condition holds). Then $I \cup \mathrm{U}(R) = \bar{I}$ if and only if one of the following conditions hold.*
(a) $\mathrm{Max}(R) = \{M_1\}$ *and* $I = M_\circ$.
(b) $\mathrm{Spec}(R) = \{M_\circ, M_1\}$, *where* $I = M_\circ$ *and* $M_1 = \mathrm{Ann(I)}$.

proposition 2.12. *Let X be a vector space over a field F, $\mathcal{M}_X$ be the set of all subspaces of X and $\circ \in A \subseteq X$. Then the following statements are equivalent:*
(a) $A = \bar{A}$.
(b) A is closed under scaler multiplication.
(c) A is the union of a family of subspaces of X.

Proof. (a) $\Rightarrow$ (b). Let $a \in A$ and $x \in F$. If $a = 0$, then we have nothing to do. Otherwise $a = \frac{1}{x}xa$. Hence, $a \in \langle xa \rangle \cap A$ and so $xa \in \bar{A} = A$.
(b) $\Rightarrow$ (c). It is clear, since in this case $\langle a \rangle = \{xa : \ x \in F\} \subseteq A$ and so $A = \cup_{a \in A} \langle a \rangle$.
(c) $\Rightarrow$ (a). Suppose that $b \in \bar{A}$, then $\langle b \rangle \cap (\cup_{a \in A} \langle a \rangle) = \langle b \rangle \cap A \nsubseteq \circ$ and so there exists $a \in A$ such that $\langle b \rangle \cap \langle a \rangle \nsubseteq \circ$. Therefore, $b \in \langle a \rangle \subseteq A$. $\square$

The following corollary is immediate.

Corollary 2.13. *Let X be a vector space over a field F, $\mathcal{M}_X$ be the set* *of all subspaces of X and $\circ \in A \subseteq X$. If τ is the topology on X induced by the closure or interior map, then A is closed if and only if $A \in \tau$.*

3. Compactness and Connectedness

In this section we define essential concepts like compactness and connectedness in $(X, \mathcal{M}_X)$, and establish their elementary properties.

Definition 3.1. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space, $A \subseteq X$ and $\mathcal{K} \subseteq \mathcal{M}_X$. A is said to be $\mathcal{K}$-compact ($\mathcal{K}$-join compact) if each cover (join cover) of elements of $\mathcal{K}$ for A, has a finite subcover (finite join subcover). For simplicity, the $\mathcal{M}_X$-compact ($\mathcal{M}_X$-join compact) subset of X is called compact (join compact).*

The following examples show that these concepts need not imply each other.

Example. (1). Let $m_1 \subsetneq m_2 \subsetneq \cdots$ be a strictly ascending chain of sets. Assume that $A = \cup_{i=1}^{\infty} m_i$ and $X = m_\circ = A \cup \{x\}$, for some $x \notin A$. If we consider $\mathcal{M}_X = \{m_i : i = 0, 1, 2, \cdots\}$, then $(X, \mathcal{M}_X)$ is an $\cap$-structure space. In this $\cap$-structure space, according to the equality $m_\circ = \vee_{i=1}^{\infty} m_i$, it is easy to see that $m_\circ$ is compact while it is not join compact.
(2). In the $\cap$-structure space $(\mathbb{Z}, \mathrm{Id}(\mathbb{Z}))$, let A be the set of all prime numbers. Then clearly, A is join compact but it is not compact.

proposition 3.2. *Suppose $(X, \mathcal{M}_X)$ is an $\cap$-structure space in which $\mathcal{M}_X$ is closed under directed unions. If $A \subseteq X$ is compact, then it is join compact.*

Proof. Suppose $\mathcal{U} \subseteq \mathcal{M}_X$ is a join cover of A. Set

$$\mathcal{V} = \{\vee_{m \in F} m : \ F \text{ is a finite set of } \mathcal{U}\}.$$

By assumption $\cup_{n \in \mathcal{V}} n \in \mathcal{M}_X$ and also we have $\cup_{n \in \mathcal{V}} n = \vee_{n \in \mathcal{V}} n = \vee_{m \in \mathcal{U}} m$. Since A is compact, there exists a finite set $F_\circ \subseteq \mathcal{V}$ such that $A \subseteq \cup_{n \in F_\circ} n$. Clearly, there exists a finite subset $F \subseteq \mathcal{U}$ such that $A \subseteq \cup_{n \in F_\circ} n \subseteq \vee_{n \in F_\circ} n = \vee_{m \in F} m$. $\square$

The previous proposition immediately shows that if X is an R-module and $\mathcal{M}_X$ is the set of all submodules of X, then every compact subset of X is join compact.

proposition 3.3. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space in which $\mathcal{M}$ is closed under directed unions, and $A \subseteq X$. The following statements are equivalent:*
(a) A is join compact.
(b) $A \subseteq \vee_{i=1}^{n} \langle a_i \rangle$, where $\{a_1, \ldots, a_n\}$ is a finite subset of A.
Furthermore, if $A \in \mathcal{M}$ then
(c) A is finitely generated.

Proof. The proof is straightforward. □

Corollary 3.4. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space. The following statements are equivalent:*
(a) X is Noetherian (i.e., satisfies the ascending chain condition on elements of $\mathcal{M}_X$).
(b) Each $u \in \mathcal{M}_X$ is join compact.
(c) Each $u \in \mathcal{M}_X$ is finitely generated.

Proof. It is enough to prove (a) ⇔ (c). The proof of this equivalence is similar to what we have seen in algebra. □

Definition 3.5. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space and $u, v \in \mathcal{M}_X$. We say that $\{u, v\}$ is a separation of X if u is a complement of v. A separation $\{u, v\}$ of X is called trivial if one of them is zero. We say that X is $\mathcal{M}_X$-connected (briefly, connected) if each separation of X is trivial.*

For a subspace of an $\cap$-structure space, we can define different types of connectivity as below.

Definition 3.6. *Suppose $(X, \mathcal{M}_X)$ is an $\cap$-structure space and $Y \subseteq X$. We say that Y is*
i) *$\mathcal{M}_X$-connected if whenever $u, v \in \mathcal{M}_X$ and $u \wedge v = \circ$ such that $Y \subseteq u \vee v$, then $Y \subseteq u$ or $Y \subseteq v$;*
ii) *weakly $\mathcal{M}_X$-connected if for $u, v \in \mathcal{M}_X$ with $u \wedge v = \circ$ such that $Y \subseteq u \vee v$, we have $Y \cap u \subseteq \circ$ or $Y \cap v \subseteq \circ$.*

If X is a ring, $\mathcal{M}_X$ is the set of all ideals and I is an ideal of X, then

the previous definition is rewritten as follows:
i) I is $\mathcal{M}_X$-connected , if $I \subseteq J \oplus K$, where J and K are two ideals of X, then $I \subseteq J$ or $I \subseteq K$.
ii) I is Weakly $\mathcal{M}_X$-connected, if J and K are two ideals of X and $I \subseteq J \oplus K$, then $I \cap J = \circ$ or $I \cap K = \circ$.

The next proposition shows that the $\mathcal{M}_X$-connectedness implies the weakly $\mathcal{M}_X$-connectedness.

proposition 3.7. *Let $(X, \mathcal{M}_X)$ be an ∩-structure space and $Y \subseteq X$. If Y is $\mathcal{M}_X$-connected, then it is weakly $\mathcal{M}_X$-connected.*

Proof. Assume that $u, v \in \mathcal{M}_X$, $u \wedge v = \circ$ and $Y \subseteq u \vee v$. By assumption and without loss of generality, we may assume that $Y \subseteq u$. We show that $Y \cap v \subseteq \circ$. Let $y \in Y \cap v$, then $y \in u \cap v = \circ$. Thus $Y \cap v \subseteq \circ$ and the proof is complete. □

In the next examples, first we show that the converse of the previous proposition is not true in general. This examples also show that connected and weakly $\mathcal{M}_X$-connected are independent from each other, and in addition, weakly $\mathcal{M}_X$-connected does not implies $\mathcal{M}_X$-connected.

Example. Let X be the vector space $\mathbb{R}^2$ over $\mathbb{R}$ and $\mathcal{M}_X$ be the set of all subspaces of X. Then we can easily see that every one dimensional subspace is weakly $\mathcal{M}_X$-connected but it is not $\mathcal{M}_X$-connected.

Example. (1). Let $X = \mathbb{Z}$, $\mathcal{M}_X = \mathrm{Id}(\mathbb{Z})$ and let I and J be two incomparable ideals of $\mathbb{Z}$. If $Y = I \Delta J$, then we have $(Y \cap I) \cap (Y \cap J) = Y \cap I \cap J = \varnothing$ and $(Y \cap I) \cap (Y \cap J) = I \Delta J = Y$. Hence, Y is not connected whereas it is $\mathcal{M}_X$-connected , for, $\mathbb{Z}$ is a uniform ring.
(2). In the ∩-structure space $(\mathbb{Z}_{30}, \mathrm{Id}(\mathbb{Z}_{30}))$, the ideal $\langle 4 \rangle \in \mathcal{M}$ is not indecomposable; i.e., $\langle 4 \rangle$ is connected whereas it is not even weakly $\mathcal{M}_X$-connected.

Example. Let X be any set and Y, A, and B are subsets of X, such that $A \cap B = \varnothing$. In addition, suppose that $A_1 = Y \cap A$ and $B_1 = Y \cap B$ are nonempty sets for which $A_1 \cup B_1$ is a proper nonempty set. Set $\mathcal{M}_X = \{\varnothing, A_1, B_1, A_1 \cup B_1, A, B, X\}$ and $\mathcal{M}_Y = \{\varnothing, A_1, B_1, A_1 \cup B_1, Y\}$. Now, we show that Y is $\mathcal{M}_Y$-connected whereas is not weakly $\mathcal{M}_X$-

connected. We have $Y \subseteq A \vee B$, $A \cap B = \varnothing$, $Y \cap A = A_1 \neq \varnothing$ and $Y \cap B = B_1 \neq \varnothing$, then Y is not weakly $\mathcal{M}_X$-connected. Now, let $Y = D \vee C$ such that $C \cap D = \varnothing$. By the assumption and the definition of $\mathcal{M}_Y$, we see that D or C must be Y. Thus Y is $\mathcal{M}_Y$-connected.

In the following two propositions we give some conditions such that the weakly $\mathcal{M}_X$-connectedness implies the $\mathcal{M}_X$-connectedness.

proposition 3.8. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space and $Y \subseteq X$. The following statements are hold.*
(a) If Y is weakly $\mathcal{M}_X$-connected and $(X \setminus Y) \cup \circ$ includes no any nonzero element of $\mathcal{M}_X$, then Y is $\mathcal{M}_X$-connected.
(b) If $Y \in \mathcal{M}_X$ is weakly $\mathcal{M}_X$-connected, then it is $\mathcal{M}_X$-connected.
(c) If $Y \in \mathcal{M}_X$ is $\mathcal{M}_X$-connected, then Y is connected.

Proof. (a). Assume that $u, v \in \mathcal{M}_X$, $(u \cap Y) \cap (v \cap Y) = \circ_Y$ and $Y = (u \cap Y) \vee (v \cap Y)$. In this case it is clear that $Y \subseteq u \vee v$ and $u \cap v \subseteq (X \setminus Y) \cup \circ$. Hence $Y \subseteq u \vee v$ and $u \cap v \subseteq \circ$. Therefore, by assumption $Y \cap u \subseteq \circ$ or $Y \cap v \subseteq \circ$.
(b). Assume that $u, v \subseteq Y$, $u \wedge v = \circ$ and $Y = u \vee v$. Then by assumption $Y \cap u = u \subseteq \circ$ or $Y \cap v = v \subseteq \circ$.
(c). Let $Y = u \vee v$, where $u, v \in \mathcal{M}_X$ are two subsets of Y and $u \wedge v = \circ$. Since Y is $\mathcal{M}_X$-connected, we have $Y \subseteq u$ or $Y \subseteq v$. Therefore, $Y = u$ or $Y = v$ and consequently Y is connected. □

proposition 3.9. *Let $(X, \mathcal{M}_X)$ be a distributive $\cap$-structure space and let $Y \in \mathcal{M}_X$. If Y is a connected, then Y is $\mathcal{M}_X$-connected subspace.*

Proof. Suppose that $Y \subseteq m \vee n$ and $m \wedge n = m \cap n = \circ$. In this case by the distributive assumption we have: $Y \cap (m \vee n) = (Y \cap m) \vee (Y \cap n) = Y$. Since Y is connected, we must have $Y \cap m = Y$ or $Y \cap n = n$. Hence $Y \subseteq m$ or $Y \subseteq n$. Thus, Y is a $\mathcal{M}_X$-connected subspace. □

proposition 3.10. *Let $(X, \mathcal{M}_X)$ be an $\cap$-structure space and $Y \subseteq X$. Then the following statements are hold.*
(a) If Y is weakly $\mathcal{M}_X$-connected and $Y \subseteq B \subseteq \bar{Y}$, then B is weakly $\mathcal{M}_X$-connected.
(b) If for each $\lambda \in \Lambda$, $Y_\lambda \subseteq X$ is $\mathcal{M}_X$-connected and $\cap_{\lambda \in \Lambda} Y_\lambda \nsubseteq \circ$, then

$\cup_{\lambda\in\Lambda}Y_\lambda$ *is* $\mathcal{M}_X$*-connected.*
(c) If $Y\subseteq X$ *is* $\mathcal{M}_X$*-connected, then* $\langle Y\rangle$ *is* $\mathcal{M}_X$*-connected.*
(d) If for each $\lambda\in\Lambda$, $Y_\lambda\subseteq X$ *is* $\mathcal{M}_X$*-connected and* $\cap_{\lambda\in\Lambda}Y_\lambda\nsubseteq\circ$*, then* $\vee_{\lambda\in\Lambda}Y_\lambda$ *is* $\mathcal{M}_X$*-connected.*

Proof. (a). Suppose that $B\subseteq m\vee n$, where $m,n\in\mathcal{M}_X$ and $m\vee n=\circ$. By assumption, $Y\cap m\subseteq\circ$ or $Y\cap n\subseteq\circ$. This is equivalent to the fact that $B\cap m\subseteq\circ$ or $B\cap n\subseteq\circ$.
(b). Let $\cup_{\lambda\in\Lambda}Y_\lambda\subseteq m\vee n$ such that $m,n\in\mathcal{M}_X$ and $m\vee n=\circ$. By hypothesis, for every $\lambda\in\Lambda$ we have $Y_\lambda\subseteq m$ or $Y_\lambda\subseteq n$. If, on the contrary, there exist $\lambda_1,\lambda_2\in\Lambda$ such that $Y_{\lambda_1}\subseteq m$ and $Y_{\lambda_2}\subseteq n$, then $\circ\neq Y_{\lambda_1}\cap Y_{\lambda_2}\subseteq m\cap n=\circ$ and this is a contradiction. Therefore, $Y_\lambda\subseteq m$ for every $\lambda\in\Lambda$ or $Y_\lambda\subseteq n$ for every $\lambda\in\Lambda$ and consequently $\cup_{\lambda\in\Lambda}Y_\lambda\subseteq m$ or $\cup_{\lambda\in\Lambda}Y_\lambda\subseteq n$.
(c). Suppose that $\langle Y\rangle\subseteq m\vee n$ such that $m,n\in\mathcal{M}_X$ and $m\vee n=\circ$. Hence, by hypothesis, $Y\subseteq m$ or $Y\subseteq n$ and consequently $\langle Y\rangle\subseteq m$ or $\langle Y\rangle\subseteq n$. □

proposition 3.11. *In the* ∩*-structure space* $(R,\mathrm{Id}(R))$*, let* I *be an ideal such that any element* $a\in I$ *has a root; i.e., there exists a natural number* $n>1$ *and* $b\in I$ *such that* $b^n=a$*. Then the three kinds of connectedness for* I *are equivalent.*

Proof. By part (b), (c) of Proposition 3.8 and by proposition 3.12, it is enough to show that if I is $\mathcal{M}_X$-connected, then I is connected. To see this, suppose that J,H are two ideals where $J\cap H=\circ$ and $I\subseteq J\vee H=J+H$. It suffices to show that $I=I\cap J+I\cap H$. Clearly, we have $I\cap J+I\cap H\subseteq I$. Now, assume that $a\in I$. Then, there exist $b\in I$ and $n>1$ such that $b^n=a$. By hypothesis, there exist $c\in J$ and $d\in H$ for which $b=c+d$. Hence, $a=bb^{n-1}=cb^{n-1}+db^{n-1}\in I\cap J+I\cap H$. □

proposition 3.12. *Let* X *be a ring,* $\mathcal{M}_X$ *be the set of all semiprime ideals of* X *and* $I\in\mathcal{M}_X$*. Then the three kinds of connectedness for* I *are equivalent.*

Proof. Similar to previus proposition, it is enough to show that for every $J,H\in\mathcal{M}_X$ with $J\cap H=\circ$, if $I\subseteq J+H$, then $I\subseteq I\cap J+I\cap H$. Assuming $a\in I$, there exist $b\in J$ and $c\in H$ for which $a=b+c$. It suffices to show that $b \in I \cap J$ and $c \in I \cap H$. Clearly, $ab = b^2 + bc = b^2$ and so $b^2 \in I \cap J$, hence $b \in I \cap J$. The proof of $c \in I \cap H$ is similar. Therefore $a \in I \cap J + I \cap H$, and the proof is complete. $\square$

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