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Dominions and Zigzag Theorem for Γ-Semigroups

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Abstract. Dominions have been studied from different perspectives, however their major application lies to study the closure property for monoids. The most useful characterization of semigroup dominions was provided by the famous Isbell's Zigzag Theorem. In this paper, we introduce the dominion of a Γ-semigroup and give the analogue of Isbell's zigzag theorem for Γ-semigroups.

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1 Introduction

Sen [6] defined the concept of a Γ-semigroup as a generalization of a semigroup. The investigation on Γ-semigroups was done by several math-

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ematicians which are parallel to the results in semigroup theory; for example one may see [5, 8, 9]. A generalization of S-act to the Γ-act over a Γ-semigroup can be found in [7]. Moreover, some basic properties of Γ-acts can be found in [2]. Dominions and zigzags were first studied in 1965 by Isbell [4] in connection with epimorphisms. It was again revived in 1980's and was studied extensively by Hall, Higgins, Khan and others resulting in the appearance of various interesting research articles (see for example [3] for all these references).

In this paper, we define dominion of a Γ-semigroup and generalize the concepts of bi-systems and tensor products of semigroups to Γ-semigroups and, later on, give the characterization of dominion of a Γ-semigroup.

Let S and Γ be any non-empty sets. Then S is called a Γ-semigroup if there exists a mapping from $S \times \Gamma \times S$ to S which maps (a, α, b) to $a\alpha b$ satisfying the condition $(a\alpha b)\gamma c = a\alpha(b\gamma c)$ for all $a, b, c \in S$ and for all $\alpha, \gamma \in \Gamma$. Let S_1 and S_2 be a Γ_1-semigroup and a Γ_2-semigroup respectively. A pair of mappings $f_1 : S_1 \to S_2$ and $f_2 : \Gamma_1 \to \Gamma_2$ is said to be a homomorphism from (S_1, Γ_1) to (S_2, Γ_2) if $(s_1\gamma s_2)f_1 = ((s_1)f_1)((\gamma)f_2)((s_2)f_1)$ for all $s_1, s_2 \in S_1$ and $\gamma \in \Gamma_1$. A non-empty subset U of a Γ-semigroup S is called a Γ-subsemigroup of S if $U\Gamma U \subseteq U$. A Γ-semigroup S is called a Γ-monoid if there exists $1\in S$ such that $1\gamma s = s = s\gamma 1$ for all $s \in S$ and $\gamma \in \Gamma$. Similarly one may define a Γ-submonoid. Let U be a Γ-subsemigroup of a Γ-semigroup S. Then we say that U dominates an element d of S if for every Γ-semigroup T and for all homomorphisms $\alpha, \beta : S \to T$ and $\alpha', \beta' : \Gamma \to \Gamma'$ such that $u\alpha = u\beta$ and $\gamma\alpha' = \gamma\beta'$ for all $u \in U$, implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S and is denoted by Γ-$Dom(U, S)$. It may easily be seen that Γ-$Dom(U, S)$ is a Γ-subsemigroup of S containing U.
The semigroup theoretic notations and conventions of Clifford and Preston [1] and Howie [3] will be used throughout without explicit mention.

2 Bi-systems and tensor products on Γ-semigroups

In this section, we generalize the concepts of bi-systems and tensor product of semigroups (see [3]) to Γ-semigroups.

Definition 2.1. Let S be a Γ-monoid and X be any non-empty set. Then X is said to be a right S-system, denoted by Γ_S-system, if there exists a map $X \times \Gamma \times S \to X$ defined by $(x, \gamma, s) \longmapsto x\gamma s$ such that

(i) $x\gamma_1(s\gamma_2 t) = (x\gamma_1 s)\gamma_2 t \ \ (\forall x \in X, \forall \gamma_1, \gamma_2 \in \Gamma, \forall s, t \in S)$; and

(ii) $x\gamma_1 1 = x$.

Dually one may define a left Γ_S-system.

Definition 2.2. Let S be a Γ'-semigroup, T a Γ''-semigroup and X a non-empty set. Then X is said to be a (Γ'_S, Γ''_T)-bisystem if X is both a left Γ'_S-system as well as a right Γ''_T-system and

$$s\gamma'(x\gamma''t) = (s\gamma'x)\gamma''t \ (\forall x \in X, \gamma' \in \Gamma', \gamma'' \in \Gamma'', \forall s \in S, t \in T).$$

Note 1. If X is a (Γ'_S, Γ''_T)-bisystem, then we denote it as $X \in \Gamma'_S$-ENS-Γ''_T.

Definition 2.3. Let X and Y be left Γ_S-systems. Then a map $\phi : X \longrightarrow Y$ satisfying $(s\gamma x)\phi = s\gamma(x\phi)$ $(\forall x \in X, s \in S, \gamma \in \Gamma)$ is called a Γ_S-map from X to Y.

Definition 2.4. A relation ρ on a left Γ_S-system X is called a congruence if ρ is an equivalence relation on X such that $x\rho y \Rightarrow (s\gamma x)\rho(s\gamma y)$ for all $x, y \in X, \gamma \in \Gamma$ and $s \in S$.

Let $X/\rho = \{x\rho \mid x \in X\}$. Then it can be easily verified that X/ρ is a left Γ_S-system with the action defined by $(s, \gamma, (x\rho)) \to (s\gamma x)\rho$.

For any left Γ'_S-system X and right Γ''_T-system Y, it may be easily checked that $Z = X \times Y$ is a (Γ'_S, Γ''_T)-bisystem with respect to actions defined by $(s, \gamma', (x, y)) \to (s\gamma' x, y)$ and $((x, y), \gamma'', t) \to (x, y\gamma'' t)$ $(\forall\ (x, y) \in Z,\ \gamma' \in \Gamma',\ \gamma'' \in \Gamma'',\ s \in S,\ t \in T)$.

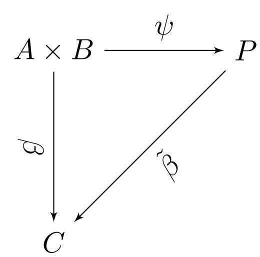
Definition 2.5. Let $A \in \Gamma'_T$-ENS-Γ''_S, $B \in \Gamma''_S$-ENS-Γ'''_U, and $C \in \Gamma'_T$-ENS-Γ'''_U. A map $\beta : A \times B \longrightarrow C$ is said to be a (Γ'_T, Γ'''_U) map if for all $a \in A, b \in B, \gamma' \in \Gamma', \gamma''' \in \Gamma''', t \in T$ and $u \in U$, we have

$$(t\gamma'(a, b))\beta = t\gamma'((a, b))\beta \text{ and } ((a, b)\gamma'''u)\beta = (a, b)\beta\gamma'''u.$$

Definition 2.6. A (Γ'_T, Γ'''_U) map $\beta : A \times B \longrightarrow C$ is called a bimap if

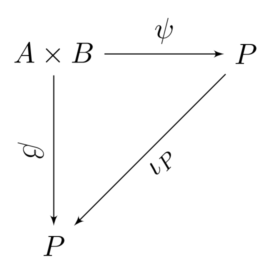
$$(a\gamma'' s, b)\beta = (a, s\gamma''_1 b)\beta \ (\forall a \in A, b \in B, \gamma'', \gamma''_1 \in \Gamma'').$$

Definition 2.7. A pair (ρ, ψ) consisting of a (Γ'_T, Γ'''_U)-bisystem P and a bimap $\psi : A \times B \longrightarrow P$ will be called a tensor product of A and B if for every (Γ'_T, Γ'''_U)-bisystem C and every bimap $\beta : A \times B \longrightarrow C$, there exists a unique (Γ'_T, Γ'''_U) map $\tilde{\beta} : P \longrightarrow C$ such that the diagram



commutes.

Moreover, when $C = P$ and $\beta = \psi$, the unique $\tilde{\beta}$ in the above diagram is ι_P (the identity map on P)



Lemma 2.8. *If there exists a tensor product of A and B over S, then it is unique up to isomophism.*

Proof. The proof is similar to the proof of 8.17 [3]. □

Define $A \otimes B = A \times B/\tau$, where τ is the equivance relation on $A \times B$ generated by the relation

$$T = \{(a\gamma s, b), (a, s\gamma b) \mid a \in A, b \in B, s \in S, \gamma \in \Gamma\}.$$

We denote the τ-class $(a,b)\tau$ of (a,b) by $a \otimes_{\Gamma_S} b$.

Note 2. $a\gamma s \otimes_{\Gamma_S} b = a \otimes_{\Gamma_S} s\gamma b$ for all $a \in A,\ b \in B,\ s \in S$ and $\gamma \in \Gamma$.

Let $\mathcal{C}$ be the class of all γ-semigroups and let $\mathcal{D} = \{S \in \mathcal{C} : a\gamma_1 b = c\gamma_2 d \implies \gamma_1 = \gamma_2$ for all $a,b,c,d \in S$ and $\forall \gamma_1, \gamma_2 \in \Gamma\}$.
Throughout rest of the paper, we prove results for the class $\mathcal{D}$ of γ-semigroups without further mention.

Proposition 2.9. *Let $a \otimes_{\Gamma_S} b, c \otimes_{\Gamma_S} d \in A \otimes_{\Gamma_S} B$. Then $a \otimes b = c \otimes d$ iff there exist $a_1, a_2, \cdots a_{n-1} \in A$, $b_1, b_2, \cdots b_{n-1}, b_n \in B$, $s_1, s_2, \cdots s_{n-1}, s_n, t_1, t_2, \cdots t_{n-1} \in S$ such that:*

$$\begin{aligned} a &= a_1\gamma s_1, & s_1\gamma b &= t_1\gamma b_1, \\ a_1\gamma t_1 &= a_2\gamma s_2, & s_2\gamma b_1 &= t_2\gamma b_2, \\ a_2\gamma t_2 &= a_3\gamma s_3, & s_3\gamma b_2 &= t_2\gamma b_3, \\ &\vdots \\ a_{n-1}\gamma t_{n-1} &= c\gamma s_n, & s_n\gamma b_{n-1} &= d. \end{aligned}$$

Proof.

$$\begin{aligned} a \otimes_{\Gamma_S} b &= a_1\gamma s_1 \otimes b \\ &= a_1 \otimes s_1\gamma b \\ &= a_1 \otimes t_1\gamma b_1 \\ &= a_1\gamma t_1 \otimes b_1 \\ &= a_2\gamma s_2 \otimes b_1 \\ &= a_2 \otimes s_2\gamma b_1 \\ &= a_2 \otimes t_2\gamma b_2 \\ &\ \vdots \\ &= a_{n-1} \otimes t_{n-1}\gamma b_{n-1} \\ &= a_{n-1}\gamma t_{n-1} \otimes b_{n-1} \\ &= c\gamma s_n \otimes b_{n-1} \\ &= c \otimes s_n\gamma b_{n-1} \\ &= c \otimes_{\Gamma_S} d. \end{aligned}$$

Conversely, suppose that $a \otimes b = c \otimes d$. Then by Theorem 1.4.10 [3],

$$(a,b) = (p_1,q_1) \to (p_2,q_2) \to \cdots \to (p_{n-1},q_{n-1}) \to (p_n,q_n) = (c,d),$$

where $((p_{i-1},q_{i-1}),(p_{i+1},q_{i+1})) \in T \cup T^{-1}$. We can assume that the sequence begins and ends with right move $(a,b) \to (a\gamma b)$.

$$\begin{aligned}
(a,b) = (p_1,q_1) &= (a_1\gamma s_1, b) \to (a_1, s_1\gamma b),\\
&= (a_1, t_1\gamma b_1) \to (a_1\gamma t_1, b_1),\\
&= (a_2\gamma s_2, b_1) \to (a_2, s_2\gamma b_1),\\
&= (a_2, t_2\gamma b_2) \to (a_2\gamma t_2, b_2),\\
&= (a_3\gamma s_3, b_2) \to (a_2, s_3\gamma b_2),\\
&= (a_3, t_3\gamma b_3) \to (a_3\gamma t_3, b_3),\\
&\vdots\\
&= (a_{n-1}, t_{n-1}\gamma b_{n-1}) \to (a_{n-1}\gamma t_{n-1}, b_{n-1}),\\
&= (c\gamma s_n, b_{n-1}) \to (c, s_n\gamma b_{n-1}) = (c,d).
\end{aligned}$$

This gives

$$\begin{aligned}
a &= a_1\gamma s_1, & s_1\gamma b &= t_1\gamma b_1,\\
a_1\gamma t_1 &= a_2\gamma s_2, & s_2\gamma b_1 &= t_2\gamma b_2,\\
a_2\gamma t_2 &= a_3\gamma s_3, & s_3\gamma b_2 &= t_2\gamma b_3,\\
&\vdots\\
a_{n-1}\gamma t_{n-1} &= c\gamma s_n, & s_n\gamma b_{n-1} &= d,
\end{aligned}$$

as required. □

Proposition 2.10. *The equivalence relation* τ *defined on* $A \times B$ *is a* (Γ_S, Γ_S)*-congruence and* $t\gamma(a \otimes b) = (t\gamma a) \otimes b$, $(a \otimes b)\gamma a = a \otimes (b\gamma a)$.

Proof. Suppose that $(a,b)\tau(c,d)$. So $(a,b)\tau = (c,d)\tau$. It follows that $a \otimes b = c \otimes d$. Then, by Proposition 2.9, we have

$$t\gamma(a \otimes b) = (t\gamma a) \otimes b \text{ and } (a \otimes b)\gamma a = a \otimes (b\gamma a),$$

for all $s, t \in S, a, c \in A$ and $b, d \in B$. This implies that

$$\begin{aligned} &(t\gamma a) \otimes b = (t\gamma c) \otimes d \\ \Rightarrow &((t\gamma a), b)\tau = ((t\gamma c), d)\tau \\ \Rightarrow &(t\gamma(a, b))\tau = (t\gamma(c, d))\tau. \end{aligned}$$

Similarly

$$((a, b)\gamma s)\tau = ((c, d)\gamma s)\tau.$$

So τ is a congruence. □

Proposition 2.11. *Let $A, B \in \Gamma_S$-ENS-Γ_S. Then $(A \otimes_{\Gamma_S} B, \tau^{\#})$ is a tensor product of A and B over S.*

Proof. We have $\tau^{\#} : A \times B \to (A \times B)/\tau = A \otimes_{\Gamma_S} B$ defined by $(a, b)\tau^{\#} = (a, b)\tau = a \otimes_{\Gamma_S} b$. Now

$$\begin{aligned} (a, b)\tau^{\#} &= (s\gamma a, b)\tau^{\#} \\ &= (s\gamma a) \otimes b \\ &= s\gamma(a \otimes b) \\ &= s\gamma(a, b)\tau^{\#} \end{aligned}$$

and

$$\begin{aligned} (a, b)\gamma s\tau^{\#} &= (a, b\gamma s)\tau^{\#} \\ &= a \otimes (b\gamma s) \\ &= (a \otimes b)\gamma s \\ &= (a, b)\tau^{\#}\gamma s. \end{aligned}$$

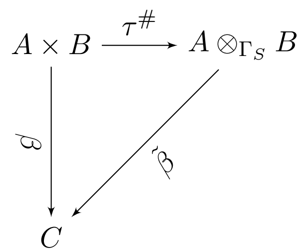
Therefore $\tau^{\#}$ is a (Γ_S, Γ_S)-map. Again

$$\begin{aligned} (a\gamma s, b)\tau^{\#} &= a\gamma s \otimes b \\ &= a \otimes s\gamma b \\ &= (a, s\gamma b)\tau^{\#}. \end{aligned}$$

Therefore $\tau^{\#}$ is a bimap. Now, let $C \in \Gamma_S$-ENS-Γ_S and let $\beta : A \times B \longrightarrow C$ be a bimap. Define $\tilde{\beta} : A \otimes_{\Gamma_S} B \longrightarrow C$ by

$$(a \otimes b)\tilde{\beta} = (a, b)\beta. \tag{1}$$

By using Proposition 2.9, we can easily verify that $\tilde{\beta}$ is well defined and the diagram



commutes.
Since $((a,b)\tau^{\#}) = a \otimes b$, from (1), we have

$$((a,b)\tau^{\#})\tilde{\beta} = (a,b)\beta \Rightarrow \tilde{\beta} = \beta.$$

Thus $(A \otimes_{\Gamma_S} B, \tau^{\#})$ is a tensor product. $\square$

3 Isbell *zigzag* theorem for Γ-semigroups

In [4], Isbell gave the characterization of semigroup dominion. In the next theorem, we generalize the Isbell *zigzag* theorem for Γ-semigroups and give the charaterization of Γ-semigroup dominion. Infact we prove the following theorem:

Theorem 3.1. *Let U be a Γ-submonoid of a Γ-monoid S. Then $d \in \Gamma$-Dom(U,S) iff $d \in U$ or there exists a series of factorization for d as follows:*

$$d = a_0\gamma t_1 = y_1\gamma a_1\gamma t_1 = y_1\gamma a_2\gamma t_2 = y_2\gamma a_3\gamma t_2$$
$$= \cdots = y_m\gamma a_{2m-1}\gamma t_m = y_m\gamma a_{2m}.$$

Where $m \geq 1, a_i \in U$ $(i = 0,1,2,...,2m)$, $y_i, t_i \in S$ $(i = 1,2,...,m)$, $\gamma \in \Gamma$ and

$$a_0 = y_1\gamma a_1, \qquad a_{2m-1}\gamma t_m = a_{2m},$$
$$a_{2i-1}\gamma t_i = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1}\ (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, a_2, ..., a_{2m}$.

To prove the above theorem, we first prove the following lemma.

Lemma 3.2. *Let U be a Γ-submonoid of a Γ-monoid S. Then $d \in \Gamma$-$Dom(U, S)$ iff $d \otimes_\Gamma 1 = 1 \otimes_\Gamma d$ in $S \otimes_{\Gamma_U} S$.*

Proof. Suppose $d \otimes_\Gamma 1 = 1 \otimes_\Gamma d$. Let T be a Γ'-monoid and $\alpha, \beta : S \longrightarrow T$ and $\alpha', \beta' : \Gamma \longrightarrow \Gamma'$ such that $u\alpha = u\beta$ $\forall u \in U$ and $(\gamma)\alpha' = (\gamma)\beta'$ $(\forall \gamma \in \Gamma)$. We show that $d\alpha = d\beta$. First, we show that T is (Γ_U, Γ_U)-bisystem. Define:

$$u\gamma' t = (u\alpha)\gamma' t \text{ and } t\gamma' u = t\gamma'(u\beta)$$

and $\psi : S \times S \longrightarrow T$ by $(s, s')\psi = (s\alpha)\gamma'(s'\beta)$. Then ψ is a (Γ_U, Γ_U) map and is also a bimap. Now

$$\begin{aligned}(u\gamma(s, s'))\psi &= (u\gamma s, s')\psi \\ &= (u\gamma s)\alpha\gamma'(s')\beta \\ &= (u)\alpha(\gamma)\alpha'(s)\alpha\gamma'(s')\beta \\ &= u\gamma'_1(s\alpha)\gamma'(s')\beta \\ &= u\gamma'_1(s, s')\psi.\end{aligned}$$

Similarly,

$$((s, s')\gamma a)\psi = (s, s')\psi\gamma' u \text{ where } \gamma'\alpha' = \gamma'_1.$$

$$\begin{aligned}(s\gamma u, s')\psi &= (s\gamma u)\alpha\gamma' s'\beta \\ &= (s\alpha)\gamma'_1(u\alpha)\gamma' s'\beta \\ &= (s\alpha)\gamma'_1(u\beta)\gamma' s'\beta \\ &= (s\alpha)\gamma'_1(u\gamma s')\beta \\ &= (s, u\gamma s')\psi.\end{aligned}$$

Therefore (T, ψ) is a tensor product. But $(S \otimes_\Gamma S, \tau^\psi)$ is also a tensor product. Therefore, by Proposition 2.11, there exists a map $\tilde{\psi} : S \otimes_\Gamma S \longrightarrow T$ such that

$$(s \otimes s')\tilde{\psi} = (s, s')\psi = (s\alpha)\gamma'(s'\beta) \ (\forall s \otimes s' \in S \otimes_\Gamma S). \tag{2}$$

Now

$$\begin{aligned} d\alpha &= (d\gamma 1)\alpha \\ &= (d\alpha)(\gamma\alpha')(1\alpha) \\ &= (d\alpha)(\gamma\alpha')(1\beta) \\ &= (d\alpha)\gamma'(1\alpha) \\ &= (d\otimes 1)\tilde{\psi} \quad \text{(by 3)} \\ &= (1\otimes d)\tilde{\psi} \\ &= (1\alpha)\gamma' d\beta \\ &= d\beta. \end{aligned}$$

Therefore $d\alpha = d\beta \Rightarrow d \in \Gamma\text{-}Dom(U,S)$.
Conversely, suppose that $d \in \Gamma\text{-}Dom(U,S)$. Let $A = S \otimes_{\Gamma_U} S$. Then A is a (Γ_S, Γ_S)-bisystem as

$$s\gamma(x\otimes y) = (s\gamma x)\otimes y \text{ and } (x\otimes y)\gamma s = x\otimes (y\gamma s).$$

Let $(Z(A),+)$ be a free abelian group on A *i.e.*

$$Z(A) = \{\Sigma z_i a_i \ : \ z_i \in Z, a_i \in A\}.$$

Then $Z(A)$ is also a (Γ_S, Γ_S)-bisystem with respect to the actions defined by

$$(s,\gamma,(\Sigma z_i a_i)) \to \Sigma z_i(s\gamma a_i) \text{ and } ((\Sigma z_i a_i),\gamma,s) \to \Sigma z_i(a_i\gamma s).$$

Now we show that $S\times Z(A)$ is a Γ-semigroup. Define a map $\phi : (S \times Z(A)) \times \Gamma \times (S\times Z(A)) \longrightarrow S\times Z(A)$ by $((p,x)\gamma(q,y))\phi = (p\gamma y, x\gamma q + p\gamma y)$ $(\forall\ p,q \in S,\ x,y\in Z(A),\ \gamma\in\Gamma)$. Then $S\times Z(A)$ is a Γ-semigroup.

Define $\alpha : S \longrightarrow S\times Z(A)$ by $s\alpha = (s,0)$ and $\beta : S \longrightarrow S\times Z(A)$ by $s\beta = (s, s\otimes 1 - 1\otimes s)$. Since $u\otimes 1 = 1\otimes u$, therefore

$$u\alpha = (u,0) = (1u, u\otimes 1 - 1\otimes u) = u\beta \ (\forall u\in U).$$

This implies that $d\alpha = d\beta$ (since $d\in\Gamma\text{-}Dom(U,S)$). So

$$(d,0) = (d, d\otimes 1 - 1\otimes d) \Rightarrow d\otimes 1 = 1\otimes d.$$

To complete the proof of the theorem, take any $d \in \Gamma\text{-}Dom(U, S)$. By above lemma $d \otimes 1 = 1 \otimes d$. Now, by Proposition 2.9, the proof of the *zigzag* theorem is completed. □

Now we show that Isbell's zigzag theorem is also aplicable to Γ-semigroups as well as to Γ-monoids.

Theorem 3.3. *Let U be a Γ-subsemigroup of a Γ-semigroup S and let $d \in S$. Then $d \in \Gamma\text{-}Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0\gamma t_1 = y_1\gamma a_1\gamma t_1 = y_1\gamma a_2\gamma t_2 = y_2\gamma a_3\gamma t_2 = \cdots = y_m\gamma a_{2m-1}\gamma t_m = y_m\gamma a_{2m}.$$

Where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$ and

$$a_0 = y_1\gamma a_1, \qquad a_{2m-1}\gamma t_m = a_{2m},$$
$$a_{2i-1}\gamma t_i = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1} \quad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

Proof. We begin by adjoining an identity element 1 to the Γ-semigroup S whether or not it already has one. If we call the resultant Γ-monoid and write $U \cup 1$ as $U^\star$, then we can easily verify that an element $d \in S \setminus U$ is in $\Gamma\text{-}Dom(U, S)$ iff $d \in \Gamma\text{-}Dom(U^\star, S^\star)$ and, hence, if $d \otimes 1 = 1 \otimes d$ in the tensor product of $S^\star \otimes S^\star$. So to obtain Isbell's theorem, we simply observe that the *zigzag* in which the element 1 appears can be shorten. For example, if $a_{2i} = 1$, then the equalities

$$a_{2i-3}\gamma t_{i-1} = a_{2i-2}\gamma t_i, \qquad y_{i-1}\gamma a_{2i-2} = y_i\gamma a_{2i-1},$$
$$a_{2i-1}\gamma t_{i+1} = a_{2i}\gamma t_{i+1}, \qquad y_i\gamma a_{2i} = y_{i+1}\gamma a_{2i+1},$$
$$a_{2i+1}\gamma t_{i+1} = a_{2i+2}\gamma t_{i+2}, \qquad y_{i+1}\gamma a_{2i+2} = y_{i+2}\gamma a_{2i+3},$$

can be collapsed to

$$a_{2i-3}\gamma t_{i-1} = a_{2i-2}\gamma t_i, \qquad y_{i-1}\gamma a_{2i-2} = y_{i+1}\gamma a_{2i+1}\gamma a_{2i-1},$$
$$a_{2i+1}a_{2i-1}\gamma t_i = a_{2i+1}\gamma t_{i+1} = a_{2i+2}\gamma t_{i+2}, \quad y_{i+1}\gamma a_{2i+2} = y_{i+2}\gamma a_{2i+3}.$$

Similarly, if $a_{2i+1} = 1$, then the length of the zigzag can be reduced.

Also, $t_1 \neq 1$. Because otherwise, we have $d = a_0\gamma 1 = a_0 \in U$ which is a contradiction. Now suppose that t_k for $k > 1$ is the first variable such that $t_k = 1$, then

$$\begin{aligned} d &= a_0\gamma t_1 \\ &= y_1\gamma a_1\gamma t_1 \\ &= y_1\gamma a_2\gamma t_2 \\ &= y_2\gamma a_3\gamma t_3 \\ &\vdots \\ &= y_k\gamma a_{2k-1}\gamma t_k \\ &= y_k\gamma a_{2k-1} \end{aligned}$$

is a *zigzag* of length k with value d in which every t_i is in S. Therefore all t_i's and a_{2i}'s are members of S. Similarly, all y_i's and a_{2i-1}'s are in S. Hence the theorem follows. □

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