Journal of Mathematical Extension
Vol. 15, No. 1, (2021), 1-14
ISSN: 1735-8299
URL: https://doi.org/10.30495/JME.2021.1244
Original Research Paper

ON Semi-n-Absorbing Submodules

Sh. Khoshdel*
Payame Noor University

M. Maani-Shirazi
Payame Noor University

Abstract. Let n be a positive integer greater than 1. In this paper, we introduce the concept of semi-n-absorbing submodules. several results concerning this class of submodules and examples of them are given.

AMS Subject Classification: 13C13; 13C99
Keywords and Phrases: Flat module, faithfully flat module, multiplication module, semi-n-absorbing ideal, semi-n-absorbing submodule

1. Introduction

During this paper, R is a non-zero commutative ring with identity and all modules are unital. For a submodule N of an R-module M, we denote the ideal $\{r \in R \mid rM \subseteq N\}$ by $(N :_R M)$. The annihilator of M, denoted $ann(M)$, is $(0 :_R M)$. Also, for an element $a \in R$ and an ideal I of R, $(N :_M a)$ is the set of all elements $x \in M$ such that $ax \in N$ and $(N :_M I)$ is the submodule $\{x \in M \mid Ix \subseteq N\}$.

The concept of semi-2-absorbing submodule was introduced in [6]. A proper submodule N of an R-module M is called semi-2-absorbing submodule of M, if whenever $a \in R$, $x \in M$ and $a^2x \in N$ then $ax \in N$ or

Received: May 2019; Accepted: November 2019
*Corresponding author

$a^2 \in (N :_R M)$. We extend this concept to semi-n-absorbing submodule, where n is a positive integer greater than 1. A proper submodule N of an R-module M is called a semi-n-absorbing submodule of M if whenever $a \in R$, $x \in M$ and $a^n x \in N$ then $ax \in N$ or $a^n \in (N :_R M)$. As a new class of submodules, we would like to compare it with the class of prime submodules. The main purpose of this paper is to get new results about semi-n-absorbing submodules. Some examples are given as well.

The organization of this paper is as follows. In section 2, we introduce the notion of semi-n-absorbing submodules and investigate some results of them. It is proved that a proper submodule N in an R-module M is semi-n-absorbing if and only if for every $a \in R$ we have $(N :_M a^n) = (N :_M a)$ or $(N :_M a^n) = M$. Also, we show that in a multiplication R-module M, a submodule N is semi-n-absorbing in M if and only if $(N :_R M)$ is so in R. In proposition 2.23, it is seen that the inverse image of every semi-n-absorbing submodule is semi-n-absorbing. However, proposition 2.20 shows that this conclusion is not always true for the images of semi-n-absorbing submodules. In example 2.5, we characterize all semi-n-absorbing submodules of $\mathbb{Z}$. Also, in example 2.4, we prove that $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module does not have any semi-n-absorbing submodule, for every prime number p. In section 3, we focus on direct sums and tensor products of modules in order to find semi-n-absorbing submodules of them. It is shown that if N is a semi-n-absorbing submodule of an R-module M and F is a flat R-module such that $F \otimes N$ is proper in $F \otimes M$ then $F \otimes N$ is semi-n-absorbing in $F \otimes M$. We prove that the converse holds when F is faithfully flat.

2. Basic Properties of Semi-n-Absorbing Submodules

In this section, we first define the concept of semi-n-absorbing submodules. Then some examples and properties of these submodules are given.

Definition 2.1. *Let M be an R-module and n a positive integer greater than 1. A proper submodule N of M is called a semi-n-absorbing submodule of M if for each $a \in R$ and $x \in M$, $a^n x \in N$ implies that $ax \in N$*

or $a^n \in (N :_R M)$.

Particularly, a proper ideal I of a ring R is called semi-n-absorbing in R if, for each $a, b \in R$, $a^n b \in I$ implies that $ab \in I$ or $a^n \in I$.

Recall that a proper submodule N of an R-module M is called prime if for every $a \in R$ and every $x \in M$, $ax \in N$ implies that $x \in N$ or $a \in (N :_R M)$. Obviously, every prime submodule is semi-n-absorbing, for each positive integer n greater than 1.

Most of the results below are the same as ones in [6] when $n = 2$.

Clearly, every proper subspace in a vector space is semi-n-absorbing, where n is a fixed positive integer graeter than 1.

Lemma 2.2. *Let N be a proper submodule of an R-module M and n a positive integer greater than 1. The following statements are equivalent.*
(i) N is semi-n-absorbing in M;
(ii) For every $a \in R$ and every submodule L of M, $a^n L \subseteq N$ implies that $aL \subseteq N$ or $a^n \in (N :_R M)$.

Proof. It is clear. □

Example 2.3 Let p be a prime number and n a positive integer greater than 1. The zero submodule of $\mathbb{Z}_{p^n}$ as a $\mathbb{Z}$-module is a semi-n-absorbing submodule. For it, let $a, x \in \mathbb{Z}$ be such that $a^n \bar{x} = \bar{0} \in \mathbb{Z}_{p^n}$. Then $p^n \mid a^n x$ and so $p \mid a$ or $p^n \mid x$. If $p \mid a$ then $a^n \in (0 :_{\mathbb{Z}} \mathbb{Z}_{p^n})$. Otherwise, $p^n \mid x$ which implies that $a\bar{x} = \bar{0} \in \mathbb{Z}_{p^n}$.

Example 2.4. Let p be a fixed prime number and n a positive integer greater than 1. Each proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is of the form $G_k = (\frac{1}{p^k} + \mathbb{Z})$, for a non-negative integer k. Furthermore, for each $k \geqslant 0$, $(G_k :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$. Note that $p^n(\frac{1}{p^{n+k}} + \mathbb{Z}) = \frac{1}{p^k} + \mathbb{Z} \in G_k$, but neither $p^n \in (G_k :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$ nor $p(\frac{1}{p^{n+k}} + \mathbb{Z}) \in G_k$. Hence $\mathbb{Z}_{p^\infty}$ does not have any semi-n-absorbing submodule.

In the following example we characterize all semi-n-absorbing ideals of $\mathbb{Z}$.

Example 2.5. For a positive integer n greater than 1, the only semi-n-absorbing ideals in $\mathbb{Z}$ are 0, $p\mathbb{Z}$, $p^2\mathbb{Z}$,..., $p^n\mathbb{Z}$, where p is an arbitrary prime number. Clearly, 0 is semi-n-absorbing. Now, we show that $p^\alpha\mathbb{Z}$ is semi-n-absorbing for $1 \leqslant \alpha \leqslant n$. To see this, let $a, x \in \mathbb{Z}$ and $a^n x \in p^\alpha\mathbb{Z}$. If $p \mid a$ then $a^n \in p^\alpha\mathbb{Z}$. Otherwise, $p^\alpha \mid x$ which implies that $ax \in p^\alpha\mathbb{Z}$. Let $\alpha > n$. Then $p^\alpha = p^n p^{\alpha-n} \in p^\alpha\mathbb{Z}$. But $pp^{\alpha-n} \notin p^\alpha\mathbb{Z}$ and $p^n \notin p^\alpha\mathbb{Z}$. In this case, $p^\alpha\mathbb{Z}$ is not semi-n-absorbing. Finally, let k be a positive integer such that $k = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $k > 1$, $r > 1$ and p_i's are distinct prime numbers. Suppose that $\alpha_i \geqslant n$, for some i. Then without loss of generality, we can assume that $i = 1$. Note that $k = p_1^n p_1^{\alpha_1-n}p_2^{\alpha_2}\cdots p_r^{\alpha_r} \in k\mathbb{Z}$. But $p_1 p_1^{\alpha_1-n}p_2^{\alpha_2}\cdots p_r^{\alpha_r} \notin k\mathbb{Z}$ and $p_1^n \notin k\mathbb{Z}$. If $\alpha_i < n$, for each i, then $p_1^n(p_2^{\alpha_2}\cdots p_r^{\alpha_r}) \in k\mathbb{Z}$ but $p_1p_2^{\alpha_2}\cdots_r^{\alpha_r} \notin k\mathbb{Z}$ and $p_1^n \notin k\mathbb{Z}$. Therefore we proved our claim.

This example shows that, for each positive integer greater than 1, there are semi-n-absorbing submodules which are not prime.

Let M be an R-module. Recall that the idealization $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(a, x) + (b, y) = (a + b, x + y)$ and multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$.

Lemma 2.6. *Let n be a positive integer greater than 1. Suppose that I is a proper ideal of the ring R and M is an R-module. Then I is semi-n-absorbing in R if and only if $I(+)M$ is semi-n-absorbing in $R(+)M$.*

Proof. Suppose that $I(+)M$ is semi-n-absorbing in $R(+)M$. Let $a, b \in R$ be such that $a^n b \in I$. Then we have $(a, 0)^n(b, 0) \in I(+)M$. Since $I(+)M$ is semi-n-absorbing in $R(+)M$ so $(a, 0)(b, 0) \in I(+)M$ or $(a, 0)^n \in I(+)M$. This yields that $ab \in I$ or $a^n \in I$. Thus I is semi-n-absorbing. Similarly, one can prove the other direction. □

Lemma 2.7. *Let n be a positive integer greater than 1. If I is a semi-n-absorbing ideal of a ring R, then $\sqrt{I}$ is also semi-n-absorbing.*

Proof. Let $a, b \in R$, $a^n b \in \sqrt{I}$ and $ab \notin \sqrt{I}$. There exists a positive integer k such that $(a^n b)^k \in I$. Since I is semi-n-absorbing and $(a^k)^n b^k \in I$ so $a^k b^k \in I$ or $(a^k)^n \in I$. But $ab \notin \sqrt{I}$. Thus $a^k b^k \notin I$ and hence $(a^k)^n \in I$ which shows that $a^n \in \sqrt{I}$. □

Example 2.8. Consider the idealization $\mathbb{Z}(+)\mathbb{Z} = \mathbb{Z}\times\mathbb{Z}$ Lemma 2.6. Let n be a positive integer greater than 1, p a prime number and take

$I = p^{n+1}\mathbb{Z}$. It is seen that $\sqrt{0(+)I} = 0(+)\mathbb{Z}$ is semi-n-absorbing in $\mathbb{Z}(+)\mathbb{Z}$. But, note that $(p,0)^n(0,p) \in 0(+)I$ while $(p,0)(0,p) = (0,p^2) \notin 0(+)I$ and $(p,0)^n = (p^n,0) \notin 0(+)I$. Therefore $0(+)I$ is not semi-n-absorbing.

This example shows that the converse of Lemma 2.7 is not true.

Lemma 2.9. *Let n be a positive integer greater than 1 and M an R-module. Let N and K be two submodules of M such that $N \subseteq K$. If N is a semi-n-absorbing submodule of M then N is semi-n-absorbing in K.*

Proof. It is clear. □

Next, we give an example of a module which shows that the converse of the above lemma is not correct.

Example 2.10. Take $N = 0$ and $K = (\frac{1}{2} + \mathbb{Z})$ in the $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$. Then it is easily checked that N is semi-n-absorbing in K while N is not so in M (Example 2.4).

Lemma 2.11. *Let n be a positive integer greater than 1 and N a semi-n-absorbing submodule of an R-module M. Then, for each submodule L of M, either $L \subseteq N$ or $L \bigcap N$ is semi-n-absorbing in L.*

Proof. Suppose that $L \nsubseteq N$. Then $L \bigcap N$ is proper in L. By the definition, we can easily show that $L \bigcap N$ is semi-n-absorbing in L. □

Lemma 2.12. *Let n be a positive integer greater than 1, M an R-module and $\{N_\lambda\}_{\lambda\in\Lambda}$ be a chain of semi-n-absorbing submodules of M. If $N = \bigcup_{\lambda\in\Lambda} N_\lambda$ is proper, then N is semi-n-absorbing.*

Proof. It follows immediately from the definition of semi-n-absorbing submodules. □

The above result is to somehow similar to one in [2].

Example 2.8 shows that the converse of Lemma 2.12 is not true in general.

Let N and K be two isomorphic submodules of an R-module M and n a positive integer greater than 1. If N is semi-n-absorbing, it is not necessary that K is so. As an example, consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Note that $2^n\mathbb{Z}$ and $6^n\mathbb{Z}$ are isomorphic submodules in $\mathbb{Z}$. But $2^n\mathbb{Z}$ is a semi-n-absorbing submodule while $6^n\mathbb{Z}$ is not.

In general, the intersection of two semi-n-absorbing submodules is not essential to be semi-n-absorbing. See the following example.

Example 2.13. Let $M = \mathbb{Z}$ as a $\mathbb{Z}$-module, $N = 2^n\mathbb{Z}$ and $K = 3^n\mathbb{Z}$. We know that N and K are semi-n-absorbing submodules of M but $N \bigcap K = 6^n\mathbb{Z}$ is not semi-n-absorbing.

The following lemma will give us a characterization of semi-n-absorbing submodules.

Lemma 2.14. *Let n be a positive integer greater than 1 and N a proper submodule of an R-module M. Then N is semi-n-absorbing if and only if for every $a \in R$ we have $(N :_M a^n) = (N :_M a)$ or $(N :_M a^n) = M$.*

Proof. Suppose that N is a semi-n-absorbing submodule of M and $a \in R$. If $(N :_M a^n) = M$ then we are done. Otherwise, let $x \in M$ and $a^n x \in N$. Since N is semi-n-absorbing and $(N :_M a^n) \neq M$ so $ax \in N$ which shows that $x \in (N :_M a)$. Therefore $(N :_M a^n) \subseteq (N :_M a)$. Clearly $(N :_M a) \subseteq (N :_M a^n)$. Consequently $(N :_M a) = (N :_M a^n)$.

Conversely, let $a \in R$, $x \in M$ and $a^n x \in N$. If $(N :_M a^n) = M$, then $a^n \in (N :_R M)$. Otherwise $(N :_M a) = (N :_M a^n)$. Since $x \in (N :_M a^n)$ so $x \in (N :_M a)$ i.e., $ax \in N$. Thus N is semi-n-absorbing submodule in M. $\square$

Lemma 2.15. *Let N be a proper submodule of an R-module M and n a positive integer greater than 1. The following are equivalent.*
(i) N is semi-n-absorbing;
(ii) $(N :_M I)$ is semi-n-absorbing, for each ideal I of R with $IM \nsubseteq N$;
(iii) $(N :_M (r))$ is semi-n-absorbing, for each $r \in R$ with $rM \nsubseteq N$.

Proof. (i) $\Longrightarrow$ (ii) Assume that I is an ideal of R with $IM \nsubseteq N$. Let $a \in R$, L be a submodule of M and $a^n L \subseteq (N :_M I)$. Then $a^n IL \subseteq N$. By Lemma 2.2, $aIL \subseteq N$ or $a^n \in (N :_R M)$. If $aIL \subseteq N$ then $aL \subseteq (N :_M I)$. Consequently, $aL \subseteq (N :_M I)$ or $a^n \in (N :_R M)$. Therefore $(N :_M I)$ is semi-n-absorbing.

(ii)$\Longrightarrow$ (iii) It is clear.

(iii) $\Longrightarrow$ (i) Take $r = 1$. Then $(N :_M (1)) = N$. So N is semi-n-absorbing. $\square$

Lemma 2.16. *Let N be a proper submodule of an R-module M and n a positive integer greater than 1. If N is semi-n-absorbing in M then $(N :_R Rx)$ is a semi-n-absorbing ideal of R, for all $x \in M \setminus N$.*

Proof. Note that $(N :_R Rx) \neq R$, for each $x \in M \setminus N$. By the definition of semi-n-absorbing ideals, we can get the result. $\square$

Lemma 2.17. *Let N be a proper submodule of an R-module M and n a positive integer greater than 1. If N is a semi-n-absorbing submodule of M then $(N :_R M)$ is semi-n-absorbing as an ideal.*

Proof. Let $a, b \in R$ and $a^n b \in (N :_R M)$. Then $a^n(bM) \subseteq N$. Since N is semi-n-absorbing, so by Lemma 2.2, $abM \subseteq N$ or $a^n \in (N :_R M)$. The rest of the proof is clear. $\square$

The following proposition shows that for multiplication modules the converse of the above lemma is true.

Proposition 2.18. *Let N be a proper submodule of a multiplication R-module M and n a positive integer greater than 1. If $(N :_R M)$ is a semi-n-absorbing ideal of R then N is semi-n-absorbing as a submodule of M.*

Proof. Let $a^n x \in N$, for some $a \in R$ and $x \in M$. Then $a^n Rx \subseteq N$. Since M is multiplication so $Rx = IM$, for some ideal I of R. Hence we get $a^n IM \subseteq N$ i.e, $a^n I \subseteq (N :_R M)$. By Lemma 2.2, $aI \subseteq (N :_R M)$ or $a^n \in (N :_R M)$. The relation $aI \subseteq (N :_R M)$ implies that $aIM \subseteq N$ and so $aRx \subseteq N$. Therefore $ax \in N$ or $a^n \in (N :_R M)$. As a result, N is semi-n-absorbing. $\square$

Corollary 2.19. *Let M be a finitely generated faithful multiplication R-module, N a proper submodule of M and n be a positive integer greater than 1. Then N is a semi-n-absorbing submodule of M if and only if $N = IM$, for a semi-n-absorbing ideal I of R.*

Proof. Let N be a semi-n-absorbing submodule of M. Since M is multiplication so $N = IM$ for $I = (N :_R M)$, by [5]. Lemma 2.17 shows that I is semi-n-absorbing as an ideal.

Conversely, let $N = IM$, for a semi-n-absorbing ideal I of R. Suppose that $a \in R$, $x \in M$ and $a^n x \in N$. Then $a^n Rx \subseteq N$. Since M is multiplication so $Rx = JM$, for an ideal J of R. In this case, we have $a^n JM \subseteq N = IM$. But M is finitely generated, faithful and multiplication. Thus $a^n J \subseteq I$, (see [5]). Since I is semi-n-absorbing, by lemma (2.), $aJ \subseteq I$ or $a^n \in I$. Thus $aJM \subseteq IM = N$ or $a^n \in I$. If $aJM \subseteq N$ then $aRx \subseteq N$ and so $ax \in N$. Otherwise $a^n \in I$ which implies that $a^n M \subseteq IM = N$. $\square$

Proposition 2.20. *Let n be a positive integer greater than 1, $f : M \longrightarrow M'$ an R-module epimorphism and N be a semi-n-absorbing submodule of M such that $Kerf \subseteq N$. Then $f(N)$ is a semi-n-absorbing submodule of M'.*

Proof. It is clear. $\square$

Corollary 2.21. *Let n be a positive integer greater than 1, N a semi-n-absorbing submodule of an R-module M and K be a submodule of M such that $K \subseteq N$. Then $\frac{N}{K}$ is a semi-n-absorbing submodule of $\frac{M}{K}$.*

Proof. Let $f : M \to \frac{M}{K}$ be the canonical epimorphism. Then $K = Kerf \subseteq N$. By the above proposition, the result follows. $\square$

Note that the condition of being epimorphism in Proposition 2.20 is essential. See the following example.

Example 2.22. Let n be a positive integer greater than 1. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(k) = 3^n k$, for all $k \in \mathbb{Z}$. Then f is a $\mathbb{Z}$-module homomorphism which is not onto. Take $N = 2^n\mathbb{Z}$. We have N is semi-n-absorbing but $f(N) = 6^n\mathbb{Z}$ is not.

Proposition 2.23. *Let n be a positive integer greater than 1, $f : M \to M'$ be an R-module homomorphism and N' a semi-n-absorbing submodule of M'. Then $f^{-1}(N')$, the inverse image of N, is semi-n-absorbing in M.*

Proof. It is clear. $\square$

Corollary 2.24. *Let n be a positive integer greater than 1 and N and*

K be two submodules of an R-module M such that $K \subseteq N$. If $\frac{N}{K}$ is a semi-n-absorbing submodule of $\frac{M}{K}$, then N is semi-n-absorbing as a submodule of M.

Proof. By considering $f : M \to \frac{M}{K}$ to be the canonical epimorphism, Proposition 2.23 shows the result. □

Recall that a submodule N of an R-module M is called relatively divisible, denoted RD, if $rN = N \bigcap rM$, for all $r \in R$.

Proposition 2.25. *Let n be a positive integer greater than 1, N a semi-n-absorbing submodule of an R-module M and K be a proper RD-submodule of M such that $N \subseteq K$. Then $\frac{K}{N}$ and K are semi-n-absorbing submodules of $\frac{M}{N}$ and M, respectively.*

Proof. First we show that $\frac{K}{N}$ is a semi-n-absorbing submodule of $\frac{M}{N}$. Let $a \in R$, $x+N \in \frac{M}{N}$ and $a^n(x+N) \in \frac{K}{N}$. Then $a^n x+N = k+N$, for some $k \in K$, and so $a^n x - k \in N \subseteq K$. Therefore $a^n x \in K$. As $a^n x \in a^n M$ and K is RD, $a^n x \in a^n M \bigcap K = a^n K$ which implies that $a^n x = a^n y$, for some $y \in K$. Hence $a^n(x-y) = 0 \in N$. But N is semi-n-absorbing. Thus $a(x-y) \in N$ or $a^n \in (N :_R M)$.

If $a(x-y) \in N$, then $ax + N = ay + N$ and $a(x + N) = a(y + N) \in \frac{K}{N}$. Otherwise $a^n \in (N :_R M) \subseteq (\frac{K}{N} :_R \frac{M}{N})$. Therefore $\frac{K}{N}$ is semi-n-absorbing.

Then, we show that K is semi-n-absorbing in M. Let $a \in R$, $x \in M$ and $a^n x \in K$. Thus $a^n(x + N) \in \frac{K}{N}$. Since $\frac{K}{N}$ is semi-n-absorbing so $a(x + N) \in \frac{K}{N}$ or $a^n \in (\frac{K}{N} :_R \frac{M}{N})$. Hence $ax \in K$ or $a^n \in (\frac{K}{N} :_R \frac{M}{N}) = (K :_R M)$ and we get the result. □

Remark 2.26. *Being RD in the above proposition is necessary.*

Example 2.27. Let n be a positive integer greater than 1. Take $M = \mathbb{Z}$ as a $\mathbb{Z}$-module, $N = 0$ which is semi-n-absorbing in M and $K = 6^n\mathbb{Z}$. We have $K \bigcap 3\mathbb{Z} = 6^n\mathbb{Z} \bigcap 3\mathbb{Z} = 6^n\mathbb{Z}$. So $3K \neq K \bigcap 3\mathbb{Z}$ which shows that K is not an RD submodule. Also, $\frac{K}{N}$ and K are not semi-n-absorbing in $\frac{M}{N}$ and M, respectively.

Proposition 2.28. *Let n be a positive integer greater than 1. Assume* *that N is a semi-n-absorbing submodule of an R-module M and S a multiplicatively closed subset of R such that $S\bigcap(N:_R M)=\emptyset$. Then $S^{-1}N$ is semi-n-absorbing in the $S^{-1}R$-module $S^{-1}M$.*

Proof. Since $S\bigcap(N:_R M)=\emptyset$ so $S^{-1}N$ is proper in $S^{-1}M$. Now we show that $S^{-1}N$ is semi-n-absorbing in $S^{-1}M$. For this, let $\frac{a}{t}\in S^{-1}R$, $\frac{x}{s}\in S^{-1}M$ and $\frac{a^n}{t^n}\frac{x}{s}\in S^{-1}N$. There exists an element $u\in S$ such that $ua^nx\in N$. As N is semi-n-absorbing, $aux\in N$ or $a^n\in(N:_R M)$. If $aux\in N$, then $\frac{a}{t}\frac{x}{s}=\frac{aux}{tus}\in S^{-1}N$. In other case, $a^n\in(N:_R M)$ and so $\frac{a^n}{t^n}\in S^{-1}(N:_R M)\subseteq(S^{-1}N:_{S^{-1}R}S^{-1}M)$. Consequently, $S^{-1}N$ is semi-n-absorbing in $S^{-1}M$. $\square$

Example 2.29. Consider the $\mathbb{Z}$-module $M=\mathbb{Q}\times\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take $N=\mathbb{Z}\times 0$ and $S=\mathbb{Z}-\{0\}$. Then S is a multiplicatively closed subset of $\mathbb{Z}$ and $S^{-1}\mathbb{Z}=\mathbb{Q}$ is a field. So $S^{-1}(\mathbb{Q}\times\mathbb{Q})$ is a vector space over $S^{-1}\mathbb{Z}=\mathbb{Q}$ and the proper submodule $S^{-1}N$ is a semi-n-absorbing submodule of $S^{-1}(\mathbb{Q}\times\mathbb{Q})$, where n is a positive integer greater than 1. But N is not semi-n-absorbing in the $\mathbb{Z}$-module M. To see this, Note that $2^n(\frac{1}{2^n},0)=(1,0)\in N$ but neither $2(\frac{1}{2^n},0)=(\frac{1}{2^{n-1}},0)\in N$ nor $2^n\in(N:_{\mathbb{Z}}\mathbb{Q}\times\mathbb{Q})=0$.

3. Direct Sum, Tensor Product and Semi-n-Absorbing Submodules

In this section, we first characterize semi-n-absorbing submodules in the R-module $M=M_1\oplus M_2$. Then we find a condition under which $F\otimes N$ is semi-n-absorbing in $F\otimes M$ if and only if N is semi-n-absorbing in M.

Proposition 3.1. *Let n be a positive integer greater than 1, M_1 and M_2 be R-modules and $M=M_1\oplus M_2$. Moreover, let N_1 and N_2 be proper submodules of M_1 and M_2, respectively. Then*
(i) N_1 is a semi-n-absorbing submodule of M_1 if and only if $N_1\oplus M_2$ is semi-n-absorbing in $M=M_1\oplus M_2$.
(ii) N_2 is semi-n-absorbing in M_2 if and only if $M_1\oplus N_2$ is semi-n-absorbingin M.

Proof. (i) Let N_1 be semi-n-absorbing in M_1 and $a\in R$, $(x_1,x_2)\in M$

be such that $a^n(x_1, x_2) \in N_1 \oplus M_2$. Thus $a^n x_1 \in N_1$. By hypothesis, $ax_1 \in N_1$ or $a^n \in (N_1 :_R M_1)$. If $ax_1 \in N_1$, then $a(x_1, x_2) \in N_1 \oplus M_2$. In other case, $a^n \in (N_1 :_R M_1)$ and so $a^n \in (N_1 \oplus N_2 :_R M)$ which shows that $N_1 \oplus M_2$ is semi-n-absorbing in M.

Conversely, assume that $N_1 \oplus M_2$ is semi-n-absorbing in M. Let $a \in R$, $x_1 \in M_1$ and $a^n x_1 \in N_1$. Then $a^n(x_1, 0) \in N_1 \oplus M_2$. But $N_1 \oplus M_2$ is semi-n-absorbing. So $a(x_1, 0) \in N_1 \oplus M_2$ or $a^n \in (N_1 \oplus M_2 :_R M)$. If $a(x_1, 0) \in N_1 \oplus M_2$, then $ax_1 \in N_1$. Otherwise $a^n \in (N_1 \oplus M_2 :_R M)$ which shows that $a^n M_1 \subseteq N_1$. Therefore N_1 is semi-n-absorbing in M_1.
(ii) It is similar to part (i). □

Proposition 3.2. *Let n be a positive integer greater than 1, M_1 and M_2 two R-modules such that $annM_1 + annM_2 = R$ and N be a semi-n-absorbing submodule of the R-module $M = M_1 \oplus M_2$. Then one of the following holds.*
(i) $N = N_1 \oplus M_2$ and N_1 is semi-n-absorbing in M_1.
(ii) $N = M_1 \oplus N_2$ and N_2 is semi-n-absorbing in M_2.
(iii) $N = N_1 \oplus N_2$, where N_1 and N_2 are semi-n-absorbing in M_1 and M_2, respectively.

Proof. By the proof of Theorem 2.4 in [1], $N = N_1 \oplus N_2$, for some submodules N_1 of M_1 and N_2 of M_2. Now if $N = N_1 \oplus M_2$ or $N = M_1 \oplus N_2$, by Proposition 3.1, we are done. Otherwise, $N = N_1 \oplus N_2$, where N_1 and N_2 are proper in M_1 and M_2, respectively. Now let $a \in R$, $x_1 \in M_1$ and $a^n x_1 \in N_1$. Then $a^n(x_1, 0) \in N = N_1 \oplus N_2$. By hypothesis, $a(x_1, 0) \in N = N_1 \oplus N_2$ or $a^n \in (N_1 \oplus N_2 :_R M)$. In the first case, we get $ax_1 \in N_1$ and in the second $a^n \in (N_1 :_R M_1)$. Therefore N_1 is semi-n-absorbing in M_1. Similarly, we can show that N_2 is semi-n-absorbing in M_2. □

Lemma 3.3. *Let N be a submodule of an R-module M and $r \in R$. Then for every flat R-module F, we have $F \otimes (N :_M r) = (F \otimes N :_{F \otimes M} r)$.*

Proof. See [3]. □

Theorem 3.4. *Let n be a positive integer greater than 1, N a semi-n-absorbing submodule of an R-module M and F be a flat R-module. If*

$F \otimes N$ is a proper submodule of $F \otimes M$ then $F \otimes N$ is a semi-n-absorbing submodule of $F \otimes M$.

Proof. As N is a semi-n-absorbing submodule of M, by Lemma 2.14, we have either $(N :_M a^n) = (N :_M a)$ or $(N :_M a^n) = M$, for all $a \in R$. Assume $(N :_M a^n) = (N :_M a)$. By the above lemma, we have $(F \otimes N :_{F\otimes M} a^n) = F \otimes (N :_M a^n) = F \otimes (N :_M a) = (F \otimes N :_{F\otimes M} a)$. If $(N :_M a^n) = M$ then $(F \otimes N :_{F\otimes M} a^n) = F \otimes (N :_M a^n) = F \otimes M$. Hence $F \otimes N$ is a semi-n-absorbing submodule of $F \otimes M$, by Lemma 2.14. $\square$

Here, we give an example satisfying Theorem 3.4.

Example 3.5. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module ([4]). By Example 2.5, 0 is semi-n-absorbing in $\mathbb{Z}$, where n is a positive integer greater than 1. Also, we have $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} 0 = 0$. Therefore $\mathbb{Q} \otimes_{\mathbb{Z}} 0$ is proper in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$ and so is semi-n-absorbing, by Theorem 3.4.

Theorem 3.6. *Let n be a positive integer greater than 1 and F be a faithfully flat R-module. Then N is a semi-n-absorbing submodule of M if and only if $F \otimes N$ is semi-n-absorbing in $F \otimes M$.*

Proof. Let N be a semi-n-absorbing submodule of M. If $F \otimes N = F \otimes M$ then $0 \to F \otimes N \to F \otimes M \to 0$ is an exact sequence. Since F is faithfully flat so $0 \to N \to M \to 0$ is exact which shows that $N = M$, a contradiction. So $F \otimes N \neq F \otimes M$. By the above theorem, we have $F \otimes N$ is a semi-n-absorbing submodule of $F \otimes M$.

Conversely, suppose that $F \otimes N$ is semi-n-absorbing in $F \otimes M$. Thus $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a \in R$. By Lemma 2.14, $(F \otimes N :_{F\otimes M} a^n) = (F \otimes N :_{F\otimes M} a)$ or $(F \otimes N :_{F\otimes M} a^n) = F \otimes M$. Suppose that $(F \otimes N :_{F\otimes M} a^n) = (F \otimes N :_{F\otimes M} a)$. Then, by Lemma 3.3, we get $F \otimes (N :_M a^n) = F \otimes (N :_M a)$ and so $0 \to F \otimes (N :_M a^n) \to F \otimes (N :_M a) \to 0$ is an exact sequence. But F is faithfully flat. Therefore $0 \to (N :_M a^n) \to (N :_M a) \to 0$ is exact. Thus $(N :_M a^n) = (N :_M a)$. Now, suppose that $(F \otimes N :_{F\otimes M} a^n) = F \otimes M$. In this case, $F \otimes (N :_M a^n) = (F \otimes N :_{F\otimes M} a^n) = F \otimes M$. Hence $0 \to F \otimes (N :_M a^n) \to F \otimes M \to 0$ is an exact sequence. Since F is faithfully flat so $0 \to (N :_M a^n) \to M \to 0$ is exact i.e., $(N :_M a^n) = M$. Consequently, N is semi-n-absorbing in M. $\square$

4. Conclusion

The class of semi-n-absorbing submodules is a new one which is comparable with the class of prime submodules. In fact, every prime submodule is semi-n-absorbing, for each positive integer n greater than 1. But the converse is not true, in general. The definition of semi-n-absorbing submodules is a tool which gives many good information.

Acknowledgements
The authors would like to thank the referees for their valuable comments and suggestions.

References

[1] M. S. Abbas, *On Fully Stable Modules*, Ph.D Thesis, University of Baghdad, (1990).

[2] H. Ansari-Toroghy and F. Farshadifar, Quasi 2-absorbing second modules, Commun. *Fac. Sci. Univ. Ank. Ser. Math. Stat.*, 68 (1) (2019), 1090-1096.

[3] A. Azizi, Weakly prime submodules and prime submodules, *Glasgow Math. J.,* 48 (2006), 343-346.

[4] P. E. Bland, *Rings and Their Modules*, Walter de Gruyter GmbH and Co. KG, Berlin New York, (2011).

[5] Z. El-Bast, Multiplication modules, *Comm. in Alg.,* 16 (4) (2007), 755-779.

[6] I. M. Hadi and A. Harfash, Semi-2-Absorbing Submodules and Semi-2-Absorbing modules, *International Journal of Advanced Scientific and Technical Research,* 5 (3) (2015), 521-530.

Shamsolmolouk Khoshdel
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University
Tehran, Iran
E-mail: khoshdel@pnu.ac.ir

Mansooreh Maani-Shirazi
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University
Tehran, Iran
E-mail: maani@pnu.ac.ir