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An Optimal Budgeting Model for Dynamic Network Systems Design Via Data Envelopment Analysis

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Abstract. For an optimal system design (OSD), data envelopment analysis (DEA) treats companies as black boxes disregarding their internal processes. Including the effects of these processes into companies, internal mechanisms promote optimal budgeting allocation. The internal processes can be defined as series and parallel networks or a combination of them. In the literature, DEA is utilized as a way for OSD in order to determine the optimal budget for companys activities in a system of series network production; but it is not yet provided a method for the optimal budgeting of dynamic systems. To fill this gap, a new model is presented in this paper to evaluate a company's optimal budgeting having dynamic network system and allocates the optimal budget to each of the internal processes of the Decision Making Units (DMUs), based on the efficiency of internal processes. The advantages of this model (which can be regarded as the first accomplishment) are expounded through the research body. Within the framework of the dynamic network systems, the optimal budgeting, the budget deficit and the budget congestion along with superior features of the proposed model are also investigated for dynamic systems. We have also used some theorems to theoretically examine the feasibility of the model. Two study cases on which the suggested model is implemented are further presented. In one application,

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the OSD network of a business entity with 5 DMUs is addressed; while in the other, a business venture with 24 DMUs is analyzed.

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1. Introduction

Data envelopment analysis (DEA) is a nonparametric method applied in diverse fields including economy, industrial engineering and operations research whose major goal is to estimate the production frontiers. It has been empirically utilized to measure the productive efficiency of Decision making units (DMUs) regarded as the entity in diverse areas of application responsible for converting inputs into outputs and whose performances are to be evaluated. Despite the strong link of DEA with production theories in economics, this approach could be used for benchmarking of operations management, where a set of measures is selected to benchmark the performance of manufacturing and service operations. Within DEA, predefined systems (DMUs) are considered through LP models so as to make a distinction between optimizing a predefined system and designing an optimal one [30] The introduced data envelopment analysis methods take the homogeneous functioning methods of organizations as granted. In fact, they are focused on calculating the initial inputs and the final outputs, that is, they fail to include the internal processes ([1], [4],[5], [9], [10], [11], [23], [31]). To address this issue, several models have been designated. Fare and Grosskopf published a paper entitled "network DEA" in 2000 where they explained the importance of this type of analysis. Castelli, Pesenti, and Ukovich [2] addressed the issue of efficiency evaluation of specialized and interdependent decision making subunits derived from bigger decision making units. Lewis and Sexton [24] worked on a two-stage DEA method to measure the efficiency of units manufacturing products. In [8], the researchers published an article entitled "Network DEA: Efficiency analysis of organizations with complex internal structures" which proposed a were still used by other sub-units. They formulated the network DEA for reverse output and input [19]. In the same year, Castelli, Pesenti, and Ukovich published a paper which evaluated the efficiency of hierarchically organized units. Prieto and Zofio [21] evaluated the potential technical efficiency via comparing appropriate technologies for different economies. Yu and Lin [28] explored the efficiency and effectiveness in railway performance using multi-activity network DEA model. The efficiency decomposition in the DEA network was explored by Kao in [13] which set up a relational network DEA model considering the interrelationship of the processes to simultaneously measure the efficiency of the system and processes. Indeed, since 1998, numerous researches have been dedicated to the measurement and analysis of the profit efficiency via DEA methods in the network systems ([8], [12], [14], [15], [16], [17], [22]). These methods, which require price data, have attracted a lot of attention. Wei and Chang [26], instead of measuring the profit efficiency of the existing network system, developed an optimal system design (OSD) of DEA model for the optimal design of DMUs with series network. They showed that by resorting to network OSD via DEA model, DMUs can face with the notorious economic phenomenon of budget congestion, implying the higher the injected budget, the lower the maximum profit gain will be. In order to verify the optimal budget and the budget congestion, Wei and Chang [25], developed a method for the accurate extraction of the DMU optimal budget which checks the existence of budget congestion in the OSD via DEA models. Their method was not however apt for the network OSD via DEA models. Fang [7] developed a new approach to derive the DMUs corresponding optimal budgets; it checked the existence of budget congestion not only for the OSD via DEA models but also for the OSD series network DEA models. Kou, M. et al [18] proposed a new formulation for the dynamic network DEA (DN-DEA) models based on systemic thinking to measure and decompose the overall efficiency of multi-period and multi-division systems (MPMDS). They adopted a general approach applicable to both radial and non-radial measures. Interestingly, it presents a weighted average decomposition of the overall efficiency score into component parts by a set of endogenous weight sets. These sets were the highly appropriate for measuring the overall efficiency of the MPMDS tested as a whole, ensuring the consistency in comparing the overall and the component efciency scores. S. Lozano and B. Adenso [20] considered a multiproduct supply network by network data envelopment analysis (NDEA) approach to assess the efficiency of the product flows in varying periods. S.H. Zegordi and A. Omid, [29], presented the DEA network to create a model for the complete analysis of the Iranian handmade carpet company (IHCC) at the same time. Bai-Chen et al. [27], applied the dynamic network slacks-based measure model to analyze the environmental efficiency of both the power systems and their divisions of provincial administrative areas in China. Moreover, Shih-Liang Chao et al. [3] took advantage of a dynamic network data envelopment analysis and proposed a model for decomposing the shipping service production for a container shipping company (CSC) into two processes: multiple evaluation periods linked through a carry-over process.

Budgeting of the dynamic networks has found extensive applications in industrial and service sectors and can cover a wider range of companies. Thus, solving these problems in these sectors is of crucial importance. A review of the literature reveals that no method has been so far proposed for budgeting of the dynamic systems. The present paper is thus the new attempt to develop a new method which can accomplish optimal budgeting for the DMUs in a dynamic network system which is one of the measure novelty of the paper: therefore, this paper presents a method for optimal budgeting in dynamical DMUs system based on DEA for the first time. In this regard, the feasibility of the model and the existence of budget congestion in network OSD (NOSD) via DEA models were investigated through some theorems (theoretically) and two applied examples. Moreover, owing to the simultaneous use of series and parallel networks, the proposed method is more effective as it reduced the amount of the requisite budget and the budget spoilage.

The rest of this paper is organized as follows: In the next section, the concept of NOSD DEA is explained. Section 3 discusses the optimal budgeting and budgeting congestion for the dynamic network OSD DEA models.

Two examples are provided in Section 4 to elucidate the notion of the proposed model. The final section is devoted to discussing the results and conclusions.

2. Optimal System Design in DEA Dynamic Network Models

The network internal processes can be defined as series and parallel networks or their combination [13]. Systems with combined processes (series and parallel) are called dynamic networks. The structures of a dynamic network carry out identical operations for a DMU in each stage where every two consecutive stages are connected via a carryover. Figure 1 depicts a dynamic network where the $\hat{Z}_{dj}^{k-1}$ carryover is used in the k-th input stage to produce the $\hat{Z}_{dj}^{k}$ carryover (the values employed in Figure 1 are provided in 2.1). In this research, the researchers developed a special mode of dynamic networks. As can be seen, this system is a series of horizontal systems where each process uses the carryover produced by the previous one to produce similar products for use in subsequent processes. It is also a parallel system vertically in which each parallel process independently produces a stage with similar inputs and outputs. The output of each process does not depend on the inputs or outputs of the other parallel process.

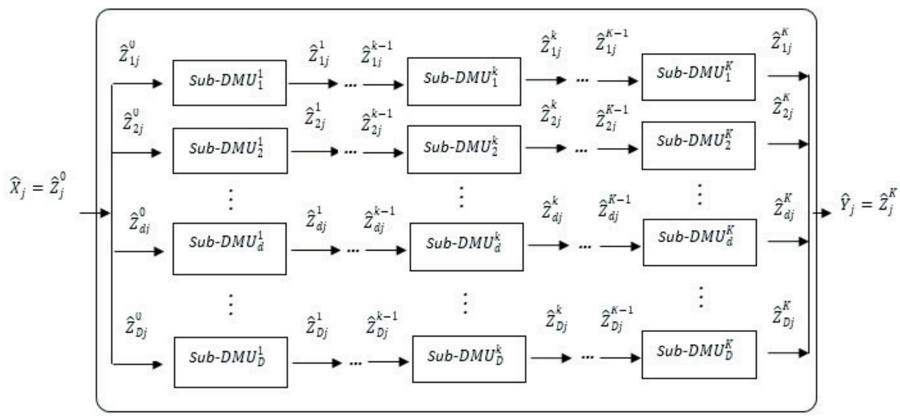

Figure 1. Dynamic systems DMU_js

2.1 Notation

Below is a list of the required variables used in representation of the dynamic network model in Figure 1 to determine the optimal budgeting:

n	number of DMUs ;
m	Total number of inputs;
s	Total number of outputs;
D	number of parallel systems used in each $DMU_j (j = 1, 2, \ldots, n)$;
K	number of series systems used in each $DMU_j (j = 1, 2, \ldots, n)$;
$\hat{Z}_{dj}^{k-1}$	number of independent inputs for DMU_d^k $(k = 1,\ 2,\ \ldots\ , K)$, $(d = 1, 2, \ldots, D)$;
$\hat{Z}_{dj}^{k}$	number of independent outputs for DMU_d^k $(k = 1, 2,\ \ldots\ , K)$, $(d = 1, 2, \ldots, D)$;
$\hat{X}$	$\hat{X} = \hat{Z}^0 = (\hat{Z}_1^0, \hat{Z}_2^0, \ldots, \hat{Z}_n^0) > 0$ the input vector of $m \times nD$ for the DMUs;
$\hat{Y}$	$\hat{Y} = \hat{Z}^K = (\hat{Z}_1^K, \hat{Z}_2^K, \ldots, \hat{Z}_n^K) > 0$ the input vector of $s \times nD$ for the DMUs;
$\hat{X}_j$	$\hat{X}_j = \hat{Z}_j^0 = (\hat{Z}_{1j}^0, \hat{Z}_{2j}^0, \ldots, \hat{Z}_{Dj}^0) > 0 (j = 1, 2, \ldots, n)$ the input vector for the $j - th$ DMU;
$\hat{Y}_j$	$\hat{Y}_j = \hat{Z}_j^K = (\hat{Z}_{1j}^K, \hat{Z}_{2j}^K, \ldots, \hat{Z}_{Dj}^K) > 0 (j = 1, 2, \ldots, n)$ the input vector for the $j - th$ DMU;
$\sum_{d=1}^{D} \widehat{Z}_{dj}^0$	number of input measures;
$\sum_{d=1}^{D} \widehat{Z}_{dj}^K$	number of output measures;
T	Production possibility sets;
T_d^k	Generalized dynamic system production possibility sets for DMU_j $(j = 1, 2, \ldots, K), (j = 1, 2, \ldots, K)$;
λ	$(nD) \times 1$ vector $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_D) \in \Re^{nD}, \lambda_d = (\lambda_d^1, \lambda_d^2, \ldots, \lambda_d^n)$ $\in \Re^n$;
C	$m \times 1$ vector $C^T = (C_1, C_2, \ldots, C_D)^T > 0$ representing the input;
P	$s \times 1$ vector $P^T = (P_1, P_2, \ldots, P_D)^T > 0$, representing the output prices;
B	Total accessible budget.

2.2 The proposed model: An explanation

Let a production system comprise n DMUs in which each $DMU_j(j = 1, 2, \ldots, n)$ has a dynamic system comprising $D \times K$ sub-DMUs; this network system has a K sub-process of series and a D sub-process of parallel independently, $DMU_d^k(d = 1, 2, \ldots, D), (k = 1, 2, \ldots, K)$ have k series sub-process and d parallel sub-processes.

Let us further assume that $\hat{X}_j = \hat{Z}_j^0 = (\hat{Z}_{1j}^0, \hat{Z}_{2j}^0, \ldots, \hat{Z}_{Dj}^0) > 0,\ \hat{Z}_j^k = (\hat{Z}_{1j}^k, \hat{Z}_{2j}^k, \ldots, \hat{Z}_{Dj}^k) > 0,\ \hat{Y}_j = \hat{Z}_j^K = (\hat{Z}_{1j}^K, \hat{Z}_{2j}^K, \ldots, \hat{Z}_{Dj}^K) > 0(j = 1, 2, \ldots, n)$ are the input, carryover and output vectors of $DMU_j(j = 1, 2, \ldots, n)$, respectively. Fig. 1 depicts the case of DMU_j where in each of its sub-DMUs, the total inputs consumed and outputs produced by the D sub-DMUs of DMU_j are $\hat{X}_j = \sum_{d=1}^{D} \widehat{Z}_{dj}^0$, and $\hat{Y}_j = \sum_{d=1}^{D} \widehat{Z}_{dj}^K(j = 1, 2, \ldots, n)$, respectively. Note that parallel systems do not have intermediate products for connecting different processes.[1]

To develop the dynamic network system via DEA model, first, the system production possibility set T and the generalized dynamic network system production possibility set $T_d^k$are presented as provided in([11], [25], [27]):

$$T = \{(x, y) | \hat{X}\lambda = x, \hat{Y}\lambda \geqslant y,\ e^T\lambda \leqslant 1, \lambda \geqslant 0,\ \lambda \in \mathbb{R}^n\}. \tag{1}$$

$$\begin{aligned} T_d^k = \{ \left(X_d^k, Y_d^k, Z_d^k\right) \mid \hat{X}_d^k\lambda_d^k = X_d^k,\ \hat{Y}_d^k\lambda_d^k \geqslant Y_d^k,\ \hat{Z}_d^{k-1}\lambda_d^{k-1} \leqslant Z_d^k, \\ \hat{Z}_d^k\lambda_d^k \geqslant Z_d^k, e^T\lambda_d^k \leqslant 1,\ \lambda_d^k \geqslant 0,\ \lambda_d^k \in \mathbb{R}^n\}. \end{aligned} \tag{2}$$

Let $C^T = (C_1, C_2, \ldots, C_D)^T > 0$ and $P^T = (P_1, P_2, \ldots, P_D)^T > 0$ be the unit price vector of inputs and outputs, respectively. If the DMUs total available budget B is known, then the following dynamic network system OSD via DEA model is suggested by maximizing the profit. Thus wise:

[1]Some [2], [8] investigated the best possible ways for allocating inputs X_j of a DMU_j to its different sub-DMUs to optimize the systems.

$$
\begin{aligned}
f(B) = Max\ (P^T \hat{Z}^K \lambda^k - C^T \hat{Z}^0 \lambda^1) \quad & (a) \\
S.\ to : \hat{Z}_d^{k-1} \lambda_d^k - \hat{Z}_d^{k-1} \lambda_d^{k-1} \leqslant 0, k = 2, 3, \ldots, K, d = 1, 2, \ldots, D; \quad & (b) \\
e^T \lambda_d^k \leqslant 1,\ k = 1, 2, \ldots, K,\ d = 1, 2, \ldots, D; \quad & (c) \\
C^T \hat{Z}^0 \lambda^1 \leqslant B; \quad & (d) \\
\lambda_d^k \geqslant 0,\ k = 1, 2, \ldots, K, d = 1, 2, \ldots, D, \quad & (3)
\end{aligned}
$$

where $e^T = (1, 1, \ldots, 1)^T \in R^n$, $\widehat{Z}_d^k = \left(\widehat{Z}_{d1}^k\ \widehat{Z}_{d2}^k, \ldots, \widehat{Z}_{dn}^k\right)$, $k = 0, 1, \ldots, K$, $d = 1, 2, \ldots, D$.

The objective function of problem (3) shows the model's maximum profit i.e. the difference between the revenues and the costs in the optimal solution. For calculating the revenue, the last output process and the cost, the first input of each DMU was considered. In model (3), for each parallel process, there are K-1 series constraints. The total number of constraints (3b) equals to $D \times (K - 1)$. Relation (3d) indicates the budget constraint; while relation (3c) characterizes the proposed model providing the optimal budgeting for similar parallel processes in DMUs. Indeed, this relation shows that for each $k = 0, 1, \ldots, K$, there are D number of parallel constraints. Constraint (3e) includes the variables of the model. It should be noted that the feasible set of (3) is not an empty set and has at least a trivial solution. Moreover, Relation (2) shows that we are faced with a variable return to the scale model.

Assuming $K = 1$ and$D = 1$, the model of (3) changes to OSD DEA with Equation (4) given by [25].

$$
\begin{aligned}
f(B) = Max\ (P^T \hat{Y} \lambda - C^T \hat{X} \lambda) \quad & (4) \\
S.\ to : e^T \lambda \leqslant 1; & \\
C^T \hat{Z} \lambda \leqslant B; & \\
\lambda \geqslant 0, &
\end{aligned}
$$

where $e^T = (1, 1, \ldots, 1)^T, \hat{X} = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n), \hat{Y} = (\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_n)$.

If we consider D=1, using Equation (3) of the series network OSD DEA model by Lei Fang [7], we will have:

$$\begin{aligned} f(B) = Max\ (P^T\hat{Z}^K\lambda^K - C^T\hat{Z}^0\lambda^1) \\ S.\ to : \hat{Z}^{k-1}\lambda^k - \hat{Z}^{k-1}\lambda^{k-1} \leqslant 0,\ k = 2,3,\dots,K; \\ e^T\lambda^k \leqslant 1,\ k = 1,2,\dots,K; \\ C^T\hat{Z}^0\lambda^1 \leqslant B; \\ \lambda^k \geqslant 0,\ k = 1,2,\dots,K, \end{aligned} \tag{5}$$

where $e^T = (1,1,\dots,1)^T \in R^n, \hat{Z}^k = (\hat{Z}_1^k, \hat{Z}_2^k, \dots, \hat{Z}_n^k), k = 0,1,\dots,K$. If $K = 1$, the parallel network OSD DEA model (3) provides:

$$\begin{aligned} f(B) = Max\ (P^T\hat{Z}\lambda - C^T\hat{Z}\lambda) \\ e^T\lambda_d \leqslant 1,\ d = 1,2,\dots,D; \\ C^T\hat{Z}\lambda \leqslant B; \\ \lambda_d \geqslant 0,\ d = 1,2,\dots,D, \end{aligned} \tag{6}$$

where $e^T = (1,1,\dots,1)^T \in R^n, \hat{Z}_d^k = (\hat{Z}_{d1}, \hat{Z}_{d2}, \dots, \hat{Z}_{dn}), d = 0,1,\dots,D$. The present paper assumes that the inputs of parallel processes are separate. For the cases with shared inputs, prior to modeling of the problem, the percentage of the shared input usage in each process should be determined using the previous data or some experts opinion [1].

3. Determining the Optimal Budget

Here, we try to show the optimal budget determination method and the budget congestion checking with regard to the dynamic network OSD DEA model. In order to determine the optimal solution for model (3), the following procedures were indicated. The procedure stages are presented and proven based on some theorems.

First, regardless of the budget constraints, the dynamic network of the OSD DEA model in (3) was set up and its relevant optimal solutions were obtained:

$$\begin{aligned} f(B) = Max\ & (P^T \hat{Z}^K \lambda^k - C^T \hat{Z}^0 \lambda^1) \\ S.\ to :& \hat{Z}^{k-1}\lambda^k - \hat{Z}^{k-1}\lambda^{k-1} \leqslant 0, k = 2,3,\ldots,K; \\ & e^T \lambda_d^k \leqslant 1,\ k = 1,2,\ldots,K,\ d = 1,2,\ldots,D; \\ & \lambda_d^k \geqslant 0,\ k = 1,2,\ldots,K,\ d = 1,2,\ldots,D. \end{aligned} \tag{7}$$

Let$f\left(B^{***}\right)$ be the optimal value for model (7). If model (7) has multiple optimal solutions, we solve models (8) and (9) to obtain the minimum budget and the maximum budget, respectively. From (8) and (9), the minimum cost B_L^* and the maximum cost B_U^* can be obtained based on the optimal value $f\left(B^{***}\right)$ from (7) (see the second constraint of (8) and (9)).

$$\begin{aligned} & Min\ B_L \\ S.\ to :& \hat{Z}^{k-1}\lambda^k - \hat{Z}^{k-1}\lambda^{k-1} \leqslant 0, \quad k = 2,3,\ldots,K, d = 1,2,\ldots,D; \\ & P^T \hat{Z}^K \lambda^K - C^T \hat{Z}^0 \lambda^1 = f\left(B^{***}\right); \\ & C^T \hat{Z}^0 \lambda^1 = B_L; \\ & e^T \lambda_d^k \leqslant 1,\ k = 1,2,\ldots,K,\ d = 1,2,\ldots,D; \\ & \lambda_d^k \geqslant 0,\ k = 1,2,\ldots,K,\ d = 1,2,\ldots,D; \end{aligned} \tag{8}$$

$$\begin{aligned} & Max\ B_U \\ S\ .to :& \hat{Z}^{k-1}\lambda^k - \hat{Z}^{k-1}\lambda^{k-1} \leqslant 0,\ k = 2,3,\ldots,K, d = 1,2,\ldots,D; \\ & P^T \hat{Z}^K \lambda^K - C^T \hat{Z}^0 \lambda^1 = f\left(B^{***}\right); \\ & C^T \hat{Z}^0 \lambda^1 = B_U; \\ & e^T \lambda_d^k \leqslant 1,\ k = 1,2,\ldots,K,\ d = 1,2,\ldots,D; \\ & \lambda_d^k \geqslant 0,\ k = 1,2,\ldots,K,\ d = 1,2,\ldots,D. \end{aligned} \tag{9}$$

Let us assume that B_L^* and B_U^* are the optimal values of objective functions of models (8) and (9) respectively; if it is assumed that the available budget B has been consumed completely, the dynamic network OSD

DEA of (3) can be expressed as:

$$
\begin{aligned}
f(B) = Max\ (P^T \hat{Z}^K \lambda^K - C^T \hat{Z}^0 \lambda^1) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (10)\\
S.\ to: \hat{Z}^{k-1}\lambda^k - \hat{Z}^{k-1}\lambda^{k-1} \leqslant 0,\ k = 2, 3, \ldots, K, d = 1, 2, \ldots, D;\\
e^T \lambda_d^k \leqslant 1,\ k = 1, 2, \ldots, K,\ d = 1, 2, \ldots, D;\\
C^T \hat{Z}^0 \lambda^1 = B;\\
\lambda_d^k \geqslant 0,\ k = 1, 2, \ldots, K,\ d = 1, 2, \ldots, D.
\end{aligned}
$$

In the following theorem, we are going to search among the allocated budget B to the model (10) in $[B_L^*, B_U^*]$ for different cases.

Theorem 3.1. *For the available budget B, if $B_L^* \leqslant B \leqslant B_U^*$, then the optimal value for model* (10) *is $f(B^{***})$.*

Proof. we follow [7] and for $B_L^* \leqslant B \leqslant B_U^*$, we can define $B = B_L^* + \alpha(B_U^* - B_L^*), 0 \leqslant \alpha \leqslant 1$. Let $\lambda_L^{1*} = (\lambda_{1L}^{1*}, \lambda_{2L}^{1*}, \ldots, \lambda_{DL}^{1*})$, and $\lambda_U^{1*} = (\lambda_{1U}^{1*}, \lambda_{2U}^{1*}, \ldots, \lambda_{DU}^{1*})$ be the optimal solutions for models (8) and (9) , respectively. We define $\lambda^1 = \lambda_L^{1*} + \alpha(\lambda_U^{1*} - \lambda_L^{1*})$, $0 \leqslant \alpha \leqslant 1$; and the following obtains:

$$
\begin{aligned}
C^T \hat{Z}^0 \lambda^1 =& C^T \hat{Z}^0 (\lambda_L^{1*} + \alpha(\lambda_U^{1*} - \lambda_L^{1*}))\\
&= \alpha C^T \hat{Z}^0 \lambda_U^{1*} + (1-\alpha) C^T \hat{Z}^0 \lambda_L^{1*}\\
&= \alpha B_U^* + (1-\alpha) B_L^* = B;\\
e^T \lambda_d^{k*} =& e^T (\lambda_{dL}^{k*} + \alpha(\lambda_{dU}^{k*} - \lambda_{dL}^{k*}))\\
&= \alpha e^T \lambda_{dU}^{k*} + (1-\alpha) e^T \lambda_{dL}^{k*} \leqslant 1, d = 1, 2, \ldots, D,\ k = 1, 2, \ldots, K.
\end{aligned}
$$

Also we have:

$$
\begin{aligned}
&\hat{Z}_d^{k-1} \lambda_d^{k*} - \hat{Z}_d^{k-1} \lambda_d^{(k-1)*}\\
&= \hat{Z}_d^{k-1} (\lambda_{dL}^{k*} + \alpha(\lambda_{dU}^{k*} - \lambda_{dL}^{k*})) - \hat{Z}_d^{k-1} (\lambda_{dL}^{(k-1)*} + \alpha(\lambda_{dU}^{(k-1)*} - \lambda_{dL}^{(k-1)*}))\\
&= \alpha(\hat{Z}_d^{k-1} \lambda_{dU}^{k*} - \hat{Z}_d^{k-1} \lambda_{dU}^{(k-1)*}) + (1-\alpha)(\hat{Z}_d^{k-1} \lambda_{dL}^{k*} - \hat{Z}_d^{k-1} \lambda_{dL}^{(k-1)*}) \leqslant 0,\\
&\ d = 1, 2, \ldots, D,\ k = 1, 2, \ldots, K.
\end{aligned}
$$

Thus, λ^1 is a feasible solution for model (3), and:

$$\begin{aligned} f(B) =& P^T \hat{Z}^K \lambda^K - C^T \hat{Z}^0 \lambda^1 \\ &= P^T \hat{Z}^k (\lambda_L^{K*} + \alpha(\lambda_U^{K*} - \lambda_L^{K*})) - C^T \hat{Z}^0 (\lambda_L^{1*} + \alpha(\lambda_U^{1*} - \lambda_L^{1*})) \\ &= \alpha(P^T \hat{Z}^K \lambda_U^{K*} - C^T \hat{Z}^0 \lambda_U^{1*}) + (1-\alpha)(P^T \hat{Z}^K \lambda_L^{K*} - C^T \hat{Z}^0 \lambda_L^{1*}) \\ &= \alpha f(B^{***}) + (1-\alpha) f(B^{***}) = f(B^{***}). \end{aligned}$$

According to the above relation, when $f(B)$ equals $f(B^{***})$, λ^1 is an optimal solution for model (3) , and $f(B) = f(B^{***})$ for $B_L^* \leqslant B \leqslant B_U^*$. □

Theorem 3.2. *When $B < B_L^*$, the optimal value for model* (10) *, $f(B)$ is a monotonically increasing function of B.*

Proof. we follow [7] assuming that$B_1 \leqslant B_2 \leqslant B_L^*$; let $\lambda_{B_1}^{k*}$ and $\lambda_{B_L^*}^{k*}$ be the optimal solutions for model (10) subject to the budget equaling$B_1$and B_L^* , respectively. In case of $B_1 \leqslant B_2 \leqslant B_L^*$, determine $B_2 = B_1 + \beta(B_2 - B_1)$,$0 < \beta < 1$. Specify $\lambda_{B_2}^{K*} = \lambda_{B_1}^{K*} + \beta(\lambda_{B_L^*}^{K*} - \lambda_{B_1}^{K*})$; then, the verification that $\lambda_{B_2}^{K*}$ is a feasible solution for model (10) will be straight forward.

$$\begin{aligned} f(B_2) =& P^T \hat{Z}^K \lambda_{B_2}^{K*} - C^T \hat{Z}^0 \lambda_{B_2}^{1*} \\ &= P^T \hat{Z}^k (\lambda_{B_1}^{K*} + \beta(\lambda_{B_L^*}^{K*} - \lambda_{B_1}^{K*})) - C^T \hat{Z}^0 (\lambda_{B_1}^{1*} + \beta(\lambda_{B_L^*}^{1*} - \lambda_{B_1}^{1*})) \\ &= \beta(P^T \hat{Z}^K \lambda_{B_L^*}^{K*} - C^T \hat{Z}^0 \lambda_{B_1^*}^{1*}) + (1-\beta)(P^T \hat{Z}^K \lambda_{B_L^*}^{K*} - C^T \hat{Z}^0 \lambda_{B_1}^{1*}) \\ &= \beta f(B_L^*) + (1-\beta) f(B_1). \end{aligned}$$

Indubitably, $f(B_L^*) > f(B_1)$; then, $f(B_2) = \beta f(B_1) + (1-\beta) f(B_1) = f(B_1)$; hence $f(B)$ is a function of B which monotonically increases for $B < B_L^*$. □

Theorem 3.3. *When $B \geqslant B_U^*$, the optimal value for model* (10)*, $f(B)$ is a monotonically decreasing function of B.*

Proof. The proof is similar to that of Theorem 3.2. □

If $f(B)$ is defined via the entire domain $\Omega = \{B | B \geqslant 0, B \in \mathbb{R}\}$, then the optimal budget (3) is defined as:

Definition 3.4. *Let $B^{**} \in \Omega$. If $f(B) \leqslant f(B^{**})$ for any$B \in \Omega$, that is, $Max_{B \in \Omega} f(B) = f(B^{**})$, the $B^{(**)}$ is called the optimal budget for* (3)*.*

Theorem 3.5. *$f(B)$ is a monotonically non-decreasing concave function of B over domain Ω, here $f(B) \geqslant 0$, and $f(0) = 0$.*

Proof. As indicated by [25], it is not a difficult to verify that $f(B)$ is a monotonically non-decreasing function i.e., $f(B) \geqslant 0$, and $f(0) = 0$. Thus, all we need to show is that$f(B)$ is a concave function. The dual problem (11) of (3) can be stated as:

$$f(B) = Min(wB + \sum_{d=1}^{D}\sum_{k=1}^{K} u_d^k) \tag{11}$$

$$\begin{aligned} S.\,to: & -(v_d^1)^T \hat{Z}_d^1 + u_d^1 e^T + wC^T \hat{Z}^0 \geqslant -C^T \hat{Z}^0, d = 1, 2, \ldots, D; \\ & (v_d^{k-1})^T \hat{Z}_d^{k-1} - (v_d^k)^T \hat{Z}_d^k + u_d^k e^T \geqslant 0, \quad k = 2, 3, \ldots, K-1, \\ & d = 1, 2, \ldots, D; \\ & (v_d^{K-1})^T \hat{Z}^{K-1} + u_d^K e^T \geqslant P^T \hat{Z}_d^K, d = 1, 2, \ldots, D; \\ & u_d^1, u_d^2, \ldots, u_d^K \geqslant 0, v_d^1, v_d^2, \ldots, v_d^{K-1} \geqslant 0, w \geqslant 0, d = 1, 2, \ldots, D, \end{aligned}$$

where $u_d^k \in \Re, w \in \Re, v_d^k \in \Re^k$. R is defined as a feasible set of (11), specifically,

$$\begin{aligned} R = & \{(u_1, u_2, \ldots, U_K, v_1, v_2, \ldots, v_{K-1}, w) | \\ & -(v_d^1)^T \hat{Z}_d^1 + u_d^1 e^T + wC^T \hat{Z}^0 \geqslant -C^T \hat{Z}^0, d = 1, 2, \ldots, D, \\ & (v_d^{k-1})^T \hat{Z}_d^{k-1} - (v_d^k)^T \hat{Z}_d^k + u_d^k e^T \geqslant 0, k = 2, 3, \ldots, K-1, d = 1, 2, \ldots, D, \\ & (v_d^{K-1})^T \hat{Z}_d^{K-1} + u_d^K e^T \geqslant P^T \hat{Z}_d^K, d = 1, 2, \ldots, D, \\ & u_d^1, u_d^2, \ldots, u_d^K \geqslant 0, v_d^1, v_d^2, \ldots, v_d^{K-1} \geqslant 0, w \geqslant 0, d = 1, 2, \ldots, D\}. \end{aligned}$$

Then, for any $B_1 \geqslant 0$, $B_2 \geqslant 0$, $\alpha \in [0, 1]$, we have:

$$
\begin{aligned}
&f\left(\alpha B_1+(1-\alpha)B_2\right)= \\
&Min_R[w(\alpha B_1+(1-\alpha)B_2)+\sum_{d=1}^{D}\sum_{k=1}^{K}u_d^k]= \\
&Min_R[\alpha(wB_1+\sum_{d=1}^{D}\sum_{k=1}^{K}u_d^k)+(1-\alpha)(wB_2+\sum_{d=1}^{D}\sum_{k=1}^{K}u_d^k)]\geqslant \\
&Min_R[\alpha(wB_1+\sum_{d=1}^{D}\sum_{k=1}^{K}u_d^k)]+Min_R[(1-\alpha)(wB_2+\sum_{d=1}^{D}\sum_{k=1}^{K}u_d^k]= \\
&\alpha f(B_1)+(1-\alpha)f(B_2).
\end{aligned}
$$

hence, $f(B)$ is a concave function. □

Now we can illustrate the relationship between the dynamic networks OSD DEA model (3) and models (8)and (9) , as well as Theorems 3.1, 3.2 and 3.2 as follow:

(i) If $B < B_L^*$, then according to Theorem 3.2, the optimal value for model (10), f(B), is a monotonically increasing function based on B; then $f(B) < f(B_L^*)$. On the other hand, based on model (8), $f(B_L^*) = f(B^{***})$; therefore, $f(B) < f(B^{***})$. In addition, since the objective function f(B) of model (10) is a monotonically increasing function of B, according to Theorem 3.2, the constraint inequity $C^T\hat{Z}^0\lambda^1 \leqslant B$, which is related to model (3), is an active constraint which makes the models (3) and (8) equivalent. In this case, the maximum profit for model (3) is $f(B) < f(B^{***})$.

(ii) If $B_L^* \leqslant B \leqslant B_U^*$, then, based on Theorem 3.1, the optimal value for model (10) is$f(B_L^*) = f(B^{***})$; thus $f(B) \geqslant f^(B) = f(B^{***})$. Moreover, $f(B^{***})$ is the optimal value for model (7) in dynamic network OSD DEA model (3) without the budget constraint, then $f(B) \geqslant f(B^{***})$. Thus, $f(B) = f(B^{***})$ and according to model (8), $B_L^* = min\{B|f(B) = f(B^{**})\}$; therefore, in this case, the optimal budget for model (3) is B_L^*, whose maximum profit is $f(B^{***})$.

(iii) If $B > B_U^*$, then, based on Theorem 3.3, the optimal value of $f(B)$ for model (3) is a monotonically non-decreasing function and $f(B) >$

$f(B_U^*)$. For model (9), $f(B_U^*) = f(B^{***})$ and $f(B) \geqslant f(B^{***})$. Additionally, since $f(B^{***})$ is the optimal value for (7), which is indeed the dynamic network OSD DEA model (3) without the budget constraints, $f(B) \leqslant f(B^{***})$, thus, $f(B) = f(B^{***})$. Based on model (8), $B_L^* = min\{B|f(B) = f(B^{**})\}$; in this case, the optimal budget for model (3) is $B^*{}_L$. Based on Theorem 3.3, the optimal value of model (10) is a monotonically decreasing function in terms of B where $f(B) < f(B_U^*) = f(B^{***})$. Thus, DMU will face with budget congestion when $B > B_U^*$.

In accordance with the above analyses, the following corollary can be obtained.

Corollary 3.6. *The optimal budget for model* (3) *is* B_L^* *and the maximum profit is* $f(B^{***})$. *Also, when*$B > B_U^*$, *DMU faces with budget congestion.*

Theorem 3.7. *Let the optimal solution* $\lambda^* = (\lambda^{1*}, \lambda^{2*}, \ldots, \lambda^{K*})$ *of (3) satisfy* $C^T\hat{Z}^0\lambda^{1*} < B$ *, and let* $B^{**} = C^T\hat{Z}^0\lambda^{1*}$. *Then, for any* $\overline{B} \geqslant B^{**}$, *we have* $f\left(\overline{B}\right) = f\left(B^{**}\right) = Max(P^T\hat{Z}^K\lambda^{K*} - C^T\hat{Z}^0\lambda^{*1})$. *Thus,* B^{**} *is the optimal budget of the target organization.*

Proof. Following [25], let (u^*, v^*, w^*) be the optimal solution to (11). Then, the following complementary conditions hold:

$$
\begin{aligned}
&(v_d^{k-1})^T(\hat{Z}_d^{k-1}\lambda_d^{k*} - \hat{Z}_d^{k-1}\lambda_d^{(k-1)*}) = 0, k = 2,3,\ldots,K,\ d = 1,2,\ldots,D;\\
&u_d^{k*}(e^T\lambda_d^{k*} - 1) = 0, k = 1,2,\ldots,K,\ d = 1,2,\ldots,D;\\
&w^*(C^T\hat{Z}^0\lambda^{1*} - B) = 0;\\
&(-(v_d^{1*})^T\hat{Z}_d^1 + u_d^{1*}e^T + w^*C^T\hat{Z}^0 + C^T\hat{Z}^0)\lambda_d^{1*} = 0,\ d = 1,2,\ldots,D;\\
&((v_d^{(k-1)*})^T\hat{Z}_d^{k-1} - (v_d^{k*})^T\hat{Z}_d^k + u_d^{k*}e^T)\lambda_d^{k*} = 0,\ k = 2,3,\ldots,K-1,\ d = \\
&\ 1,2,\ldots,D;\\
&((v_d^{(K-1)*})^T\hat{Z}_d^{K-1} + u_d^{K*}e^T - P^T\hat{Z}_d^K)\lambda_d^{K*} = 0,\ d = 1,2,\ldots,D.
\end{aligned}
$$

Since $C^T\hat{Z}^0\lambda^{1*} < B$, thus by the third equations above, we have $w^* = 0$. Now consider the linear programming problem of (12) and its dual problem (13) with a total available budget of $\bar{B}$:

$$f(\bar{B}) = Max(P^T\hat{Z}^K\lambda^K - C^T\hat{Z}^0\lambda^1) \tag{12}$$
$$S.\ to: \hat{Z}_d^{k-1}\lambda_d^k - \hat{Z}_d^{k-1}\lambda_d^{k-1} \leqslant 0, k=2,3,\ldots,K,\ d=1,2,\ldots,D;$$
$$e^T\lambda_d^k \leqslant 1,\ k=1,2,\ldots,K,\ d=1,2,\ldots,D;$$
$$C^T\hat{Z}^0\lambda^1 \leqslant \bar{B};$$
$$\lambda_d^k \geqslant 0,\ k=1,2,\ldots,K,\ d=1,2,\ldots,D.$$

$$f(\bar{B}) = Min(w\bar{B} + \sum_{d=1}^{D}\sum_{k=1}^{K} u_d^k) \tag{13}$$
$$(-(v_d^1)^T\hat{Z}_d^1 + u_d^1 e^T + wC^T\hat{Z}^0 \geqslant -C^T\hat{Z}^0,\ d=1,2,\ldots,D;$$
$$(v_d^{k-1})^T\hat{Z}_d^{k-1} - (v_d^k)^T\hat{Z}_d^k + u_d^k e^T \geqslant 0, k=2,3,\ldots,K,\ d=1,2,\ldots,D;$$
$$(v_d^{K-1})^T\hat{Z}_d^{K-1} + u_d^K e^T \geqslant P^T\hat{Z}_d^K,\ d=1,2,\ldots,D;$$
$$u_d^1, u_d^2, \ldots, u_d^K \geqslant 0, v_d^1, v_d^2, \ldots, v_d^{K-1} \geqslant 0, w \geqslant 0,\ d=1,2,\ldots,D.$$

Since $C^T\hat{Z}^0\lambda^{1*} \leqslant \bar{B}$, and λ^* is the optimal solution of (3) , then, λ^* is a feasible solution to (12) and (u^*, v^*, w^*) is a feasible solution to (13). In addition, since $w^* = 0$, the following complementary conditions hold:

$$(v_d^{k-1})^T(\hat{Z}_d^{k-1}\lambda_d^{k*} - \hat{Z}_d^{k-1}\lambda_d^{(k-1)*}) = 0, k=2,3,\ldots,K,\ d=1,2,\ldots,D;$$
$$u_d^{k*}(e^T\lambda_d^{k*} - 1) = 0, k=1,2,\ldots,K,\ d=1,2,\ldots,D;$$
$$w^*(C^T\hat{Z}^0\lambda^{1*} - \bar{B}) = 0;$$
$$(-(v_d^{1*})^T\hat{Z}_d^1 + u_d^{1*}e^T + w^*C^T\hat{Z}^0 + C^T\hat{Z}^0)\lambda_d^{1*} = 0,\ d=1,2,\ldots,D;$$
$$((v_d^{(k-1)*})^T\hat{Z}_d^{k-1} - (v_d^{k*})^T\hat{Z}_d^k + u_d^{k*}e^T)\lambda_d^{k*} = 0,\ k=2,3,\ldots,K-1,\ d=1,2,\ldots,D;$$
$$((v_d^{(K-1)*})^T\hat{Z}_d^{K-1} + u_d^{K*}e^T - P^T\hat{Z}_d^K)\lambda_d^{K*} = 0,\ d=1,2,\ldots,D.$$

Therefore, λ^* is also the optimal solution to (12), and thus $f(\bar{B}) = P^T\hat{Z}^K\lambda^{K*} - C^T\hat{Z}^0\lambda^{1*} = f(B^{**})$ for any $\bar{B} \geqslant C^T\hat{Z}^0\lambda^{1*} = B^{**}$. Besides,

$f(\hat{B}) \leqslant f(B^{**})$ for any $\hat{B} \in \Omega, \hat{B} \leqslant B^{**}$ since $f(B)$ is a monotonically non-decreasing function by Theorem 3.5. In this regard, $B^{**} = C^T \hat{Z}^0 \lambda^{1*}$ can be considered as the optimal budget. □

Using Theorem 3.7, it can be concluded that the optimal budget B^{**} for the target DMU and the optimal values of inputs and outputs are the expected results of (3) regardless of the budget constraints. Incidentally, the procedure of finding the optimal budget is shown through a flowchart in the Appendix.

4. Empirical Illustration

Two numerical cases were used to illustrate model (3). Input-based DEA techniques were used for the dynamic network OSD DEA model. The first example included 5 DMUs and the second one was an applied case containing 24 DMUs. In order to solve the relevant problems, LINGO 10.0 on a 2.20 GHz Intel Core Dual E2200 PC having a 1.98 GB RAM was employed. The CPU time for solving the problem given below was 1 second.

4.1 Example 1

Consider 5 DMU data from Kao [15]. The relevant data are listed in Table 1.

Table 1: Dynamic network system

DMU	Process 1							Process 2						
	$\hat{z}_0^1$			$\hat{z}_1^1$		$\hat{z}_2^1$		$\hat{z}_0^2$			$\hat{z}_1^2$		$\hat{z}_2^2$	
	$\hat{x}_1^1$	$\hat{x}_2^1$	$\hat{x}_3^1$	$\hat{z}_1^1$	$\hat{z}_2^1$	$\hat{y}_1^1$	$\hat{y}_2^1$	$\hat{x}_1^2$	$\hat{x}_2^2$	$\hat{x}_3^2$	$\hat{z}_1^2$	$\hat{z}_2^2$	$\hat{y}_1^2$	$\hat{y}_2^2$
1	9	50	1	20	10	100	35	7	35	2	35	10	90	35
2	10	40	4	10	15	80	10	7	30	5	10	25	100	25
3	9	30	3	8	20	96	30	12	40	4	20	25	120	10
4	8	25	1	10	10	60	15	9	25	2	10	10	110	15
5	10	40	5	15	20	85	25	10	50	1	20	15	80	20

If, for example, $C = (0.5, 0.1, 0.5)$, $P = (0.2, 1)$ and $B = 100$ are the cost and price vector coefficients, then the dynamic network OSD DEA model for this example includes $K = 2$, $D = 2$, $m = 6$, $s = 4$. Fig 2 shows the information. In the exemplified case, the optimal budgeting for the dynamic network system related to the data from 5 DMUs was conducted; each of which contained two parallel processes in which each parallel process included two series processes.

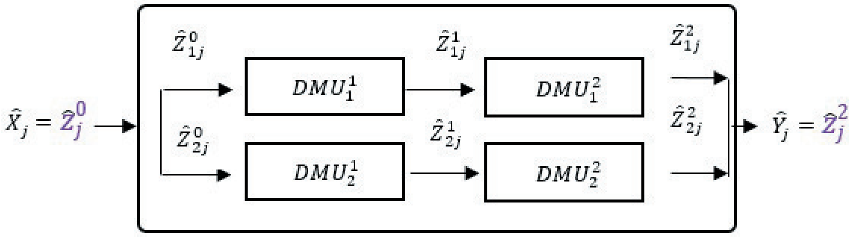

Figure 2. Dynamic Systems DMU_j

As for this case, model (14) was executed using Equation (3).

$$
\begin{aligned}
f(B) = Max\ &(P^T\hat{Z}^2\lambda^2 - C^T\hat{Z}^0\lambda^1) \qquad (14)\\
S.\ to: &\ \hat{Z}^{k-1}\lambda_d^k - \hat{Z}^{k-1}\lambda_d^{k-1} \leqslant 0,\ k = 2,\ d = 1,2;\\
&\ e^T\lambda_d^k \leqslant 1,\ k = 1,2,\ d = 1,2;\\
&\ C^T\hat{Z}^0\lambda^1 \leqslant B;\\
&\ \lambda_d^k \geqslant 0,\ k = 1,2,\ d = 1,2.
\end{aligned}
$$

Table 2 summarizes the results obtained by solving the problem. Some of the results are analyzed in the following.

Table 2: Optimal solutions using model (3) for 5 DMUs

Dynamic model $B = 100$					
Row	Process 1	Process 2	$F(B)$	B^{**}	$B - B^{**}$
1	$\lambda_{13}^1 = 0/7142857$, $\lambda_{15}^1 = 0/2857143$, $\lambda_{11}^2 = 1$,	$\lambda_{21}^2 = 1$, $\lambda_{21}^1 = 1$,	90/28571	17/7143	82/2857
2	$\lambda_{11}^2 = 1$, $\lambda_{11}^1 = 1$,	$\lambda_{22}^2 = 1$, $\lambda_{22}^1 = 1$,	81/00000	19	63/2857
3	$\lambda_{14}^2 = 1$, $\lambda_{14}^1 = 1$,	$\lambda_{24}^2 = 1$, $\lambda_{23}^1 = 0/4444$,	51/66667	12/3333	50/9524
4	$\lambda_{12}^2 = 1$, $\lambda_{12}^1 = 1$,	$\lambda_{24}^2 = 1$, $\lambda_{25}^1 = 0/5714286$,	46/00000	17	33/9524
5	This process has no optimal solution.	$\lambda_{24}^2 = 1$, $\lambda_{24}^1 = 1$,	29/00000	8	25/9524

Some of the advantages of the proposed dynamic network OSD DEA for the exemplified case are:

1- This is the first attempt to allocate budget to the dynamic network systems. The relevant results are summarized in Table 2.

2- Optimal budgeting in a dynamic network system was considered in the allocation of the budget to DMUs which contained both parallel and series processes. This model provides a visual impression of the effects of the parallel processes and the intermediate processes on the series processes in the budgeting. As Table 2 suggests, the first budgeting priority had the profit equal to $f(B) = 90.28571$ and a budget consumption of $B^{**} = 17.7143$ for the first processes of DMUs 3 and 5 and the second process of DMU1. If the conventional model (4) were used, only one DMU would receive the budget, and the intermediate and the parallel processes, which might play a significant role in the optimal budgeting, would be most likely overlooked.

3- Due to the inherent constraints associated with Equations (3b) and (3c) in the optimal budgeting method for the dynamic networks, it can easily be recognized in case of one sub-process incurs losses.

4- If there is a limited budgeting, the budget can be allocated specifically for the parallel processes in one DMU or in the series processes depending on the intermediate procedures. In this case, the optimal budgets are allocated based on the data provided in rows in Table 2.

4.2 Example 2

Suppose a service company with two counts of investments on two parallel independent business ventures: the international tourist hotels and non-life insurance companies.

The data presented in Table 3 comes from Hsieh and Lin [10] and are related to the first parallel process comprising 24 Taiwan's international tourist hotels. Their relevant inputs which are also inputs to the first stage are: accommodation costs (X_1), employees of the accommodations department (X_2), catering costs (X_3) and employees of the catering department (X_4). The outputs of the system (which are also the outputs of the second stage) are: the revenue of the accommodations (Y_1) and the catering departments (Y_2). The two intermediate products in the system (the outputs of the first stage as well as the inputs for the second stage) are: the rooms (Z_1) and the catering floors (Z_2). The price vectors for the inputs and the outputs are $C = (1, 0.24, 1, 0.48)$ and $p = (1, 1)$, respectively.

On the other hand, the second parallel process consisted of 24 non-life insurance companies. The data of these companies were derived from Hwang and Kao [17] as listed in Table 3. Their respective inputs are as follows: operation expenses (X_1): salaries of the employees and various types of costs occurring during their daily operations; insurance expenses (X_2) are the expenses paid to the agencies, brokers, and solicitors and other expenses associated with marketing the insurance service. The outputs of the system (also the outputs to the second stage) are underwriting profit (Y_1): profit earned from the insurance business and investment profit (Y_2): profit earned from the investment portfolio. There are also two intermediate products in the system (constituting the outputs of the marketing sub-process as well as the inputs to the investment sub-process): direct written premiums (Z_1) (premiums received from the insured clients) and reinsurance premiums (Z_2) (premiums received from the ceding companies).

The efficiency of the first stage measures the performance in marketing the insurance service, while the efficiency of the second stage evaluates the performance in generating profit from the premiums. The product of the efficiencies of the two sub-processes is the total efficiency of the process. These data are also presented in Table 3. The data were taken from Hwang and Kao [11], and the averages were obtained between 2001 and 2002. The price vectors for the inputs and outputs are $C = (1, 1)$ and $P = (1, 1)$, respectively.

The total available budget for the 24-DMU-unit sector was predicted to be $B = \$48,000,000$. The purpose of the dynamic system optimal budgeting is defined as the data envelopment analysis of the dynamic network OSD DEA via model (3). The optimum budget outcomes as well as the initial budget of $\$25,000,000$ to $\$55,000,000$ are presented in Table 4. This table also shows the range of optimal budget, or the range in which it has a deficit or congestion.

Table 3: Input, intermediate and output data for dynamic systems

	Process 1								Process 2					
DMU	$\hat{x}1$	$\hat{x}_2$	$\hat{x}_3$	$\hat{x}_4$	$\hat{z}_1$	$\hat{z}_2$	$\hat{y}_1$	$\hat{y}_2$	$\hat{x}_1$	$\hat{x}_2$	$\hat{z}_1$	$\hat{z}_2$	$\hat{y}_1$	$\hat{y}_2$
1	251708/2	190000	688564/1	451000	873000	193500	1332495	1044862	1,178,744	673,512	7,451,757	856,735	984,143	681,687
2	215692/4	260000	621045/2	512000	686000	232000	799749/3	989555/8	1,381,822	1,352,755	10,020,274	1,812,894	1,228,502	834,754
3	134354/8	214000	491793/9	298000	422000	270000	720400/8	720999/8	1,177,494	592,790	4,776,548	560,244	293,613	658,428
4	474292/1	182000	837118/6	342000	569000	230100	757776/2	1032715	601,320	594,259	3,174,851	371,863	248,709	177,331
5	277457/9	214000	604081/8	411000	606000	152000	637248/3	656468	6,699,063	3,531,614	37,392,862	1,753,794	7,851,229	3,925,272
6	112866/8	134000	340400/4	230000	288000	133400	520123/6	570376	2,627,707	668,363	9,747,908	952,326	1,713,598	415,058
7	398321/1	211000	603983/1	236000	405000	372700	413969/2	580752/9	1,942,833	1,443,100	10,685,457	643,412	2,239,593	439,039
8	219183/2	146000	540573/8	237000	432000	115200	388553/8	642622/2	3,789,001	1,873,530	17,267,266	1,134,600	3,899,530	622,868
9	173001/5	110000	412204/5	295000	250000	85000	188208/8	435090/2	1,567,746	950,432	11,473,162	546,337	1,043,778	264,098
10	180956/1	751000	240134/8	211000	345000	56900	465302/3	301790/6	1,303,249	1,298,470	8,210,389	504,528	1,697,941	554,806
11	82228/3	69000	143161/9	81000	220000	46400	146915	160459/4	1,962,448	672,414	7,222,378	643,178	1,486,014	18,259
12	102707/6	86000	263655/4	200000	268000	64500	251241/7	347052	2,592,790	650,952	9,434,406	1,118,489	1,574,191	909,295
13	69167/55	41000	185515/8	136000	209000	47700	223481/6	234118/8	2,609,941	136,802	13,921,464	811,343	3,609,236	223,047
14	118717/8	83000	189168/4	93000	336000	115000	188142/4	187872/1	1,396,002	988,888	7,396,396	465,509	1,401,200	332,283
15	62196/25	69000	179051/2	133000	202000	69400	238299/8	208562/8	2,184,944	651,063	10,422,297	749,893	3,355,197	555,482
16	153422	75000	172270/9	98000	388000	137100	336526/1	206683/8	1,211,716	415,071	5,606,013	402,881	854,054	197,947
17	120982/8	79000	123667/9	101000	228000	41000	195102	106923/6	1,453,797	1,085,019	7,695,461	342,489	3,144,484	371,984
18	79332/69	83000	95047/8	79000	287000	84000	121322/7	89347/36	757,515	547,997	3,631,484	995,620	692,731	163,927
19	95699/36	53000	103608/7	79000	215000	70000	121322/7	113419/5	159,422	182,338	1,141,950	483,291	519,121	46,857
20	106232/6	75000	80618/43	65000	304000	75700	104180/2	65479/56	145,442	53,518	316,829	131,920	355,624	26,537
21	77493/05	68000	46489/84	45000	243000	26500	147662/1	42248/13	84,171	26,224	225,888	40,542	51,950	6,491
22	52646/63	46000	67006/04	44000	201000	102300	92087.87	74517/39	15,993	10,502	52,063	14,574	82,141	4,181
23	31923/68	33000	8904/455	11000	97000	4800	34923/62	8376/722	54,693	28,408	245,910	49,864	0	18,980
24	258743/5	173000	727949/1	383000	436000	301000	444194/9	742426/4	163,297	235,094	476,419	644,816	142,370	16,976

The optimal budgeting of the DEA dynamic network for all DMUs is summarized in Table 5 as follows.

The first column of Table 5 (left to right) shows the budget allocation priorities. The second column presents the optimal solution $\lambda_{1j}^{*k}, k = 1,2, j = 1,2,\ldots,24$ for the first process (24-DMU Taiwan's international tourist hotels). The third column shows the optimal solution$\lambda_{2j}^{*k}, k = 1,2, j = 1,2,\ldots,24$ for the second process (24-DMU non-life insurance companies) with $\lambda_{1j}^{*1}, \lambda_{1j}^{*2}, \lambda_{2j}^{*1}$ and λ_{2j}^{*2} where the first two $(\lambda_{1j}^{*1}, \lambda_{1j}^{*2})$ respectively relate to optimal budget and revenue of the first process (column 2), while the second two $(\lambda_{2j}^{*1}, \lambda_{2j}^{*2})$ respectively relate to optimal budget and revenue of the second process(column 3). In the first priority, according to theorems, we have $\lambda_{101}^{*1} = 1$ and $\lambda_{213}^{*1} = 1$, i.e. the optimal budget is allocated to the first process of DMU1 and the second process of DMU13. The last five columns illustrate $f(B^{**}), B_L, B_U, B^{**}$ and $B - B^{**}$, which in the order mentioned relate to model the optimal profit, minimum budget, maximum budget, optimal budget and congestion budget.

Table 4: Optimal value w.r.t. different values for budget B

5500000	5000000	4500000	4000000	3949095	3949095	3540000	3000000	2500000	2000000	B
3514810	3581722	3648634	3715546	3722290	3722290	3587084	3395668	3183574	2971480	$f(B)$

In this regard, the first process is the hoteling sector which had the first priority with an allocated budget of \$1202352 and a profit of \$1174970; the second process is the non-life insurance industry recognized as the first priority with an allocated budget of \$2746743 and a profit of \$2547320.

Table 5: Dynamic Network OSD DEA results using model (3) for 24 DMUs

Parallel model $B =$ \$48000000							
ROW	Process 1	Process 2	$F(B)$	B_L	B_U	B^{**}	$B-B^{**}$
	$\lambda^1_{101}=1$	$\lambda^1_{213}=1$	3722290	3949095	3949099	3949095	44050905
1	$\lambda^2_{101}\ = 0.9999713$	$\lambda^2_{205}=0/2152085$	3722290	3949095	3949099	3949095	44050905
		$\lambda^2_{217}=0/7847915,$	3722290	3949095	3949099	3949095	44050905
	$\lambda^1_{116}=1$	$\lambda^1_{209}=0/9999998$	2947431	2908910	2908916	2908910	41141994
2	$\lambda^2_{104}=0.2745565$	$\lambda^2_{205}=0/1279882$	2947431	2908910	2908916	2908910	41141994
	$\lambda^2_{106}=0.7254435$	$\lambda^2_{217}=0/8720118$	2947431	2908910	2908916	2908910	41141994
	$\lambda^1_{120}=0/2208805$	$\lambda^1_{202}=1$	2635943	3678728	3678728	3678728	37463266
3	$\lambda^1_{102}=0/7791195$	$\lambda^2_{205}=0/1219985$	2635943	3678728	3678728	3678728	37463266
	$\lambda^2_{104}=1$	$\lambda^2_{217}=0/8780015$	2635943	3678728	3678728	3678728	37463266
	$\lambda^1_{114}=0/9999999$	$\lambda^1_{201}=0/9999999$	2508958	2224702	2224717	2224702	35238564
4	$\lambda^2_{104}=0/0783691$	$\lambda^2_{205}=0/00869663$	2508958	2224702	2224717	2224702	35238564
	$\lambda^2_{106}=0/9216309$	$\lambda^2_{217}=0/9913034$	2508958	2224702	2224717	2224702	35238564
	$\lambda^1_{103}=1$	$\lambda^1_{205}=0/9999997$	2317277	11051223	11051323	11051223	24187342
5	$\lambda^2_{104}=0/7164416$	$\lambda^2_{205}=0/9999997$	2317277	11051223	11051323	11051223	24187342
	$\lambda^2_{106}=0/2835584,$		2317277	11051223	11051323	11051223	24187342
	$\lambda^1_{107}=0/1228890$	$\lambda^1_{215}=0/6717024$	2241727	2364151	2364152	2364151	21823191
6	$\lambda^1_{118}=0/8771110$	$\lambda^1_{219}=0/3282976$	2241727	2364151	2364152	2364151	21823191
	$\lambda^2_{106}=0/9990009$	$\lambda^2_{217}=1$	2241727	2364151	2364152	2364151	21823191
	$\lambda^1_{105}=0/2578919$	$\lambda^1_{208}=0/1637263$	1913133	2691670	2691675	2691670	19131520
7	$\lambda^1_{122}=0/7421081$	$\lambda^1_{216}=0/8362737$	1913133	2691670	2691675	2691670	19131520
	$\lambda^2_{106}=0/9980148$	$\lambda^2_{217}=1$	1913133	2691670	2691675	2691670	19131520
	$\lambda^1_{121}=0/2578919$	$\lambda^1_{207}=0/1637263$	2017444	2587359	2587365	2587359	16544161
8	$\lambda^1_{124}=0/7421081$	$\lambda^1_{218}=0/8362737$	2017444	2587359	2587365	2587359	16544161
	$\lambda^2_{106}=0/9980148$	$\lambda^2_{217}=1$	2017444	2587359	2587365	2587359	16544161
	$\lambda^1_{104}=0/2590584$	$\lambda^1_{210}=0/9195919$	1620666	2977973	2977980	2977973	13566188
9	$\lambda^1_{119}=0/7409416$	$\lambda^1_{223}=0/080408$	1620666	2977973	2977980	2977973	13566188
	$\lambda^2_{106}=0/9923633$	$\lambda^2_{217}=1$	1620666	2977973	2977980	2977973	13566188
	$\lambda^1_{106}=0.9949648$	$\lambda^1_{212}=0.0654201$	1569631	3035317	3035319	3035317	10530871
10	$\lambda^1_{111}=0.0.005035$	$\lambda^1_{214}=0.9345798$	1569631	3035317	3035319	3035317	10530871
	$\lambda^2_{106}=0.9981479$	$\lambda^2_{217}=1$	1569631	3035317	3035319	3035317	10530871
11	$\lambda^1_{117}=1$	$\lambda^1_{206}=0/7511937$	1241821	2788078	2788081	2788078	7742793
	$\lambda^2_{110}=0/6693207$	$\lambda^2_{217}=0/9999998$	1241821	2788078	2788081	2788078	7742793
12	$\lambda^1_{115}=1$	$\lambda^1_{217}=0/9999999$	1174016	2860463	2860464	2860463	4882330
	$\lambda^2_{110}=0/6752924$	$\lambda^2_{217}=0/9999999$	1174016	2860463	2860464	2860463	4882330
13	$\lambda^1_{113}=0/9999029$	$\lambda^1_{203}=0/0840859$	393441	478627	478628	478627	4403703
	$\lambda^2_{110}=0/6386541$	$\lambda^2_{220}=1$	393441	478627	478628	478627	4403703
	$\lambda^1_{112}=0/9932118$	$\lambda^1_{211}=0/0570524$	383387	630416	630421	630416	3773287
14	$\lambda^1_{123}=0/0067819$	$\lambda^2_{220}=1$	383387	630416	630421	630416	3773287
	$\lambda^2_{110}=0/8234236$		383387	630416	630421	630416	3773287
15	$\lambda^1_{108}=0/9996235$	$\lambda^1_{204}=0/1265253$	353463	1059486	1059486	1059486	2713801
	$\lambda^2_{108}=0/9996235$	$\lambda^2_{220}=1$	353463	1059486	1059486	1059486	2713801
16	$\lambda^1_{110}=0/9988397$	$\lambda^2_{220}=1$	287121	861243	861243	861244	1852559
	$\lambda^2_{110}=0/9988397$	$\lambda^2_{220}=1$	287121	861243	861243	861244	1852559
17	—-	$\lambda^1_{220}=1$	183201	198960	198960	198960	1653599
	—-	$\lambda^2_{220}=1$	183201	198960	198960	198960	1653599
18	—	$\lambda^1_{222}=1$	59827	26495	26495	26495	1627104
	—	$\lambda^2_{222}=1$	59827	26495	26495	26495	1627104

After allocating optimal budget to the first priority of the first process from DMU1 and the second process from DMU13, these two processes are removed from the latter DMUs and re-budgeting $(B - B^{**})$ is executed for other remaining DMUs. In this second priority, the optimal budget $B^{**} = \$2908910$ is allocated to the first process of the DMU16 and the second process of the DMU9. In the last rows, only the second process of DMU22 is allocated a budget of \$26495. Besides the advantages gained through model (3) in Example 4.1, it is worth mentioning that despite available budget, since the first process of DMU9 and the second process of DMU21 are not profit making, no budget is allocated to them.

Using a dynamic network OSD DEA model in (3), in this example the budget allocation was simultaneously affected by two parallel processes for a company displaying a dynamic system of 24 DMUs. From a total predicted budget of \$48000000, \$1627104 was surplus. If the allocated budget is less than \$48372896, the company will naturally face deficit and budget allocation priorities are exhibited in Table 5. Accordingly, the model is applicable to any numbers of DMUs that have a dynamic network system.

5. Concluding Remarks

In order to solve the problems related to the dynamic network systems, it is impracticable to resort to series or parallel problem solving methods. In case of using conventional methods, these can be interpreted as disregarding series or parallel processes which can play crucial roles in the optimal budgeting. To address the issue, the present study introduced a new model of OSD modeling adopting DEA framework, accompanied by a new technique promoting the optimal design of the DMUs network systems. To establish the model, the following was implemented: optimal design of the network system, optimal budgeting of the DMUs networks via entering series or parallel processes of each DMU and optimal allocation of the budget at the process level based on the limited budget of the organization. Meanwhile, a new technique for budgeting of the dynamic DMUs organization network is presented based on DEA. Besides that, by prioritizing the DMUs in allocating the budget, the proposed model managed to recognize the budget congestions and deficits of the DMUs. The model was concurrently carried out considering series and parallel processes of budgeting. Using the propounded dynamic network OSD DEA model, the budgeting of series or parallel processes was also accomplished. In each DMU, the process with higher returns was allocated with the optimal budget, while suggestions were made to eliminate or increase the efficiency of the loss-making processes. The presented two numerical and applied cases were also employed to illustrate both the feasibility and the efficiency of the proposed model.

References

[1] J. E. Beasley, Determining teaching and research efficiencies, *Journal of the operational research society*, 46 (1995), 441-452.

[2] L. Castelli, R. Pesenti, and W. Ukovich, DEA-like models for efficiency evaluations of specialized and interdependent units,*European journal of operational research*, 132 (2) (2001), 274-286.

[3] S. L. Chao, M. M. Yu, and W. F. Hsieh, Evaluating the efficiency of major container shipping companies: A framework of dynamic network DEA with shared inputs, *Transportation Research Part A: Policy and Practice*, 117 (2018), 44-57.

[4] A. Charnes, W. W. Cooper, and E. Rhodes, Measuring the efficiency of decision making units.*European Journal of Operational Research*, 2 (6) (1978), 429-444.

[5] W. W. Cooper, L. M. Seiford, and K. Tone, *Introduction to Data Envelopment Analysis and its Uses: With DEA-Solver Software and References.* New York, (2007).

[6] Fakharzadeh, Mojtabaei, and Haghguyan, Optimal Budgeting by Using a Generalized Optimal System Design DEA Model and the Effect of a New DMU's Budget in the Model, *The Journal of Organizational Behavior Research*, 4 (2019), 1-16.

[7] L. Fang, Optimal budget for system design series network DEA model. *Journal of the Operational Research Society*, 65 (12) (2014), 1781-1787.

[8] R. Fare, S. Grosskopf, and G. Whittaker, Network dea Modeling data irregularities and structural complexities in data envelopment analysisl, *Springer*, (2007), 209-240.

[9] M. J. Farrell, The measurement of productive efficiencyl, *Journal of the Royal Statistical Society: Series A (General)*, 120 (3) (1957), 253-281.

[10] L. F. Hsieh and L. H. Lin, A performance evaluation model for international tourist hotels in TaiwanAn application of the relational network DEA. *International Journal of Hospitality Management*, 29 (1) (2010), 14-24.

[11] S. N. Hwang and K. Tong-Liang, Measuring managerial efficiency in non-life insurance companies: an application of two-stage data envelopment analysis, *International Journal of Management*, 23 (3) (2006), 699-700.

[12] M. O. Jo and M. G. Yoon, Developing the Competitive Advantage Evaluation Model (CAEM) Using Network DEA, *JKORMS*, 43 (1) (2018), 57-70.

[13] C. Kao, Efficiency decomposition in network data envelopment analysis: A relational model, *European journal of operational research*, 192 (2009), 949-962 .

[14] C. Kao, Efficiency decomposition in network data envelopment analysis with slacks-based measures,*Omega*, 45 (2014a), 1-6.

[15] C. Kao, Network data envelopment analysis: A review, *European Journal of Operational Research*, 239 (1) (2014b), 1-16.

[16] C. Kao, Efficiency Measurement in Special Production Stages Network Data Envelopment Analysis, *Springer*, (2017), 89-112.

[17] C. Kao and S. N. Hwang, Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan,*European Journal of Operational Research*, 185 (1) (2008), 418-429.

[18] M. Kou, Measuring efficiencies of multi-period and multi-division systems associated with DEA: An application to OECD countries national innovation systems, *Expert Systems with Applications*, 46 (2016), 494-510.

[19] H. F. Lewis and T. R. Sexton, Network DEA: efficiency analysis of organizations with complex internal structure, *Computers and Operations Research*, 31 (9) (2004), 1365-1410.

[20] S. Lozano and B. Adenso-Diaz, Increasing Sustainability of Logistic Networks by Reducing Product Losses: A Network DEA Approach, *Mathematical Problems in EngineeringVolume*, (2018), Article ID 3479251, 21 pages.

[21] A. M. Prieto and J. L. Zofo, Network DEA efficiency in inputoutput models: with an application to OECD countries, *European Journal of Operational Research*, 178 (1) (2007), 292-304.

[22] B. K. Sahoo, J. Zhu, K. Tone, and B. M. Klemen, Decomposing technical efficiency and scale elasticity in two-stage network DEA, *European Journal of Operational Research*, 233 (3) (2014), 584-594.

[23] L. M. Seiford and J. Zhu, Profitability and marketability of the top 55 US commercial banks. *Management Science*, 45 (9) (1999), 1270-1288.

[24] T. R. Sexton and H. F. Lewis, Two-stage DEA: An application to major league baseball, *Journal of Productivity Analysis*, 19 (2003), 227-249.

[25] Q. Wei and T. S. Chang, DEA-type models for designing optimal systems and determining optimal budgets, *Mathematical and Computer Modelling*, 54 (2011), 2645-2658 .

[26] Q. Wei and T. S. Chang, Optimal system design series-network DEA models,*Journal of the Operational Research Society*, 62 (2011), 1119-112 .

[27] B. C. Xie, J. W. Wang, J. J. Ma, and N. Duan, Efficiency evaluation of China's provincial power systems based on the dynamic network slacks-based measure model,*Journal of Cleaner Production*, 174 (2018), 650-660.

[28] M. M. Yu and E. T. Lin, Efficiency and effectiveness in railway performance using a multi-activity network DEA model, *Omega*, 36 (2008), 1005-1017.

[29] S. Zegordi and A. Omid, *Efficiency Assessment of Iranian Handmade Carpet Company by Network DEA,* (2018).

[30] M. ZMELEN, Optimizing given systems vs. designing optimal systems: The De Novo programming approach, *International Journal of General Systems*, 17 (4) (1990), 295-307.

[31] J. Zhu, Multi-factor performance measure model with an application to Fortune 500 companies, *European Journal of Operational Research*, 123 (1) (2000), 105-124.

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6 Appendix

For optimal budgeting of the dynamic network in each priority, the following flowchart is executed. After determining the optimal budgeting, the processes which have been allocated the budget, are removed and the flowchart is performed again [6].