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Tauberian Theorems For Statistical Limit and Statistical Summability by Weighted Means of Continuous Fuzzy Valued Functions

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Abstract. We introduce the notion of statistical limit of continuous fuzzy number valued functions at infinity and compare its relationship with ordinary limit. We obtain ordinary limit from statistical limit at infinity in terms of a slowly oscillating-like condition. We also establish a Tauberian theorem for statistical summability by weighted means of continuous fuzzy number valued functions.

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1. Introduction

Since 1965 when Zadeh [42] introduced the fuzzy set theory, several applications of this theory have been investigated in many disciplines to handle uncertainties. Fuzzy set is any set whose elements have degrees of membership, as opposed to crisp membership or non-membership in classical sets. Fuzzy analysis is based on the notion of fuzzy numbers which is a particular fuzzy set of real numbers. Dubois and Prade [12] introduced the concept of fuzzy numbers and a modified definition was also presented by Goetschel and Voxman [18]. This idea provided considerably the development of theories concerning the sequences of fuzzy numbers and fuzzy number valued functions. The notion of sequences of fuzzy numbers and some of its properties including convergence, boundedness etc. was introduced and studied by Matloka [19] and Nanda [23].

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The idea of statistical convergence of a sequence was introduced by Fast [15] and Steinhaus [30]. Nuray and Savaş [24] extented this notion for sequences of fuzzy numbers. In recent years, special summability methods of sequences of fuzzy numbers, including Cesàro, weighted mean (or Riesz), Borel, Hölder, Abel and Euler methods, have been studied by several authors (*cf.* [5, 6, 7, 8, 9, 10, 25, 37, 38, 39]). All of these works include some Tauberian theorems which allow to deduce the classical (ordinary) convergence from aforementioned methods in the setting of fuzzy analysis. Moreover, some fuzzy analogues of statistical extensions of some Tauberian theorems are established in [1, 27, 31, 32, 41]. The readers should see [33, 34, 35] for more information in the theory of summability of sequences of fuzzy numbers.

Yavuz et al. [40] introduced Cesàro summability of integrals of fuzzy number valued functions and presented one-sided Tauberian conditions under which convergence of improper fuzzy Riemann integrals follows from Cesàro summability. Önder and Çanak [26] and also Belen [4] studied summability of Riemann integrals of fuzzy valued functions with respect to a weight function $q(t)$, shortly $\left(\overline{N},q\right)$ summability, and they recovered convergence of an improper fuzzy Riemann integral from its $\left(\overline{N},q\right)$ summability.

Motivated by Móricz's idea of statistical limit of measurable functions at infinity (see [21]), we define the statistical limit of continuous fuzzy valued functions at infinity and then we examine relationship between statistical and ordinary limits.

Secondly we present a necessary and sufficient condition under which statistical limit of integrals of continuous fuzzy valued functions follows from its statistical $\left(\overline{N},q\right)$ summability.

2. Preliminaries

In this section we first recall some basic facts on fuzzy numbers and fuzzy number valued functions. If X is a collection of objects, then a fuzzy set u in X is ordered pairs

$$u = \{(x, \mu_u(x)) : x \in X\},$$

where $\mu_u(x)$ is membership function for the fuzzy set u that maps each element of X to a membership grade between 0 and 1. We note that the terms membership function and fuzzy set is used interchangeably and so we prefer to write $u(x)$ instead of $\mu_u(x)$.

Let $\mathbb{R}$ denote the set of all real numbers. A map $u : \mathbb{R} \to [0,1]$ is called a fuzzy number with the following properties (see e.g. [12, 18]):

(i) u is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda) y) \geqslant \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi continuous, i.e. $\{x : u(x) \geqslant \alpha\}$ is closed for every α;

(iv) The support of u denoted by $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is a compact set, where $\overline{A}$ denotes the closure of the set A in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers by $\mathbb{R}_{\mathcal{F}}$ and call it fuzzy number space.

α-level set $[u]_\alpha$ of $u \in \mathbb{R}_{\mathcal{F}}$ is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geqslant \alpha\}, & 0 < \alpha \leqslant 1 \\ \overline{\{x \in \mathbb{R} : u(x) > \alpha\}}, & \alpha = 0. \end{cases}$$

Note that $[u]_0$ is called the support of $u \in \mathbb{R}_{\mathcal{F}}$. Properties (i)-(iv) imply that $[u]_\alpha$ is non-empty closed, bounded and convex subset of $\mathbb{R}$ defined by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$.

Each $a \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{a}$ defined by

$$\overline{a}(x) = \chi_{\{a\}}(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a. \end{cases}$$

If $u, v \in \mathbb{R}_{\mathcal{F}}$, $0 \leqslant \alpha \leqslant 1$, $\lambda \in \mathbb{R}$, then the addition and product with real scalars in $\mathbb{R}_{\mathcal{F}}$ are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \left[u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+\right]$$

and

$$[\lambda u]_\alpha = \lambda [u]_\alpha = \left[\lambda u_\alpha^-, \lambda u_\alpha^+\right] \ (\lambda \geqslant 0) \text{ or } \left[\lambda u_\alpha^+, u_\alpha^-\right] \ (\lambda < 0).$$

Note that $1u = u1 = u$ and $u + \overline{0} = \overline{0} + u$, that is, $\overline{0}$ is neutral element in $\mathbb{R}_{\mathcal{F}}$ with respect to the $+$. Also for any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$ we have $u + v = v + u$ and $\lambda(u+v) = \lambda u + \lambda v$ (see e.g. [2, 14]).

If we define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to [0, +\infty)$ by

$$D(u,v) = sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) = sup_{\alpha \in [0,1]} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}, \tag{1}$$

then we have the following.

Lemma 2.1. $(\mathbb{R}_{\mathcal{F}}, D)$ *is a complete metric space and also* *(i)* $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ *for any* $u, v \in \mathbb{R}_{\mathcal{F}}$ *and* $\lambda \in \mathbb{R}$;

(ii) $D(u+w, v+w) = D(u,v)$ *for any* $u, v, w \in \mathbb{R}_{\mathcal{F}}$;

(iii) $D(u+v, w+z) \leqslant D(u,w) + D(v,z)$ for any $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$ (see e.g. [20]).

We say that $\widetilde{f}(x)$ is a fuzzy number valued function if $\widetilde{f} : A \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. A fuzzy number valued function $\widetilde{f}(x)$ is said to be bounded if there exists a $M \in \mathbb{R}$ such that $D\left(\widetilde{f}(x), \overline{0}\right) \leqslant M$ for all $x \in A$. Also continuity of a fuzzy valued function at a point can be described with the help of metric defined by (1).

A fuzzy number valued function $\widetilde{f} : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is Riemann integrable on $[a,b]$ if there exists $I \in \mathbb{R}_{\mathcal{F}}$ with the property: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition of $[a,b]$, $P : a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of norm $|P| < \delta$, and for any points $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$, we have

$$D\left(\sum_{i=0}^{n-1} \widetilde{f}(\xi_i)(x_{i+1} - x_i), I\right) < \varepsilon.$$

In this case we write $I = \int_a^b \widetilde{f}(x)\,dx$.

Note that if $\widetilde{f} : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is continuous then $\widetilde{f}$ is fuzzy Riemann integrable on $[a,b]$ (see [17]).

Lemma 2.2. *[3] Let* $\widetilde{f}, \widetilde{g} : [a,b] \to \mathbb{R}_{\mathcal{F}}$ *be continuous functions. Then*
(i) The function $F : [a,b] \to [0,\infty)$ *defined by* $F(x) = D\left(\widetilde{f}(x), \widetilde{g}(x)\right)$ *is continuous on* $[a,b]$ *and*

$$D\left(\int_a^b \widetilde{f}(x)\,dx, \int_a^b \widetilde{g}(x)\,dx\right) \leqslant \int_a^b D\left(\widetilde{f}(x), \widetilde{g}(x)\right) dx.$$

(ii)

$$\widetilde{s}(x) = \int_a^x \widetilde{f}(t)\,dt$$

is a continuous fuzzy number valued function in $x \in [a,b]$.

Now we deal with the concept of fuzzy Riemann-Stieltjes integral introduced by Ren and Wu [29].

Let $\widetilde{f} : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a bounded function, q be an increasing real function on $[a,b]$ and $w \in \mathbb{R}_{\mathcal{F}}$. Also let P be any partition of $[a,b]$ such that $P : a = x_0 <$

$x_1 < \cdots < x_n = b$. Choose any point $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$, and form the fuzzy summation

$$\widetilde{s}_T = \sum_{i=0}^{n-1} \widetilde{f}(\xi_i)\left[q(x_{i+1}) - q(x_i)\right].$$

Then we say that w is the Riemann-Stieltjes integral of $\widetilde{f}$ with respect to the function q if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every partition P with $|P| := \max_{0 \leqslant i \leqslant n-1}(x_{i+1} - x_i) < \delta(\varepsilon)$, and for every choice of points ξ_i, we have $D(w, \widetilde{s}_T) < \varepsilon$. In this case we write $w = \int_a^b \widetilde{f} dq$. If the Riemann-Stieltjes integral of $\widetilde{f}$ with respect to the function q exists, then we write $\left(\widetilde{f}, q\right) \in \mathcal{FRS}[a, b]$. Some important properties of fuzzy Riemann-Stieltjes integral can be listed as follows:

Lemma 2.3. *[29]*
(i) If $\int_a^b \widetilde{f} dq$ *exists and* c *is a positive constant then* $\int_a^b \left(c\widetilde{f}\right) dq$ *exists and* $\int_a^b \left(c\widetilde{f}\right) dq = c\int_a^b \widetilde{f} dq$.
(ii) If $\widetilde{f}(x) = u \in \mathbb{R}_{\mathcal{F}}$ *for all* $x \in [a, b]$, *then* $\left(\widetilde{f}, g\right) \in \mathcal{FRS}[a, b]$ *and*

$$\int_a^b \widetilde{f} dq = u\left(q(b) - q(a)\right).$$

(iii) If $\widetilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ *is continuous and* q *is an increasing real function on* $[a, b]$, *then* $\left(\widetilde{f}, q\right) \in \mathcal{FRS}[a, b]$.
(iv) If $\left(\widetilde{f}, q\right) \in \mathcal{FRS}[a, b]$, *then for any* $c \in (a, b)$, *we have* $\left(\widetilde{f}, q\right) \in \mathcal{FRS}[a, c]$, $\left(\widetilde{f}, q\right) \in \mathcal{FRS}[c, b]$ *and*

$$\int_a^b \widetilde{f} dq = \int_a^c \widetilde{f} dq + \int_c^b \widetilde{f} dq.$$

Let $\widetilde{f}, \widetilde{g} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ are continuous and g is an increasing real function on $[a, b]$. Then we have (see, [4])

$$D\left(\int_a^b \widetilde{f} dq, \int_a^b \widetilde{g} dq\right) \leqslant \int_a^b D\left(\widetilde{f}, \widetilde{g}\right) dq \tag{2}$$

where the right-hand side integral exists in usual Riemann-Stieltjes sense since $F(x) = D\left(\widetilde{f}(x), \widetilde{g}(x)\right)$ is continuous and q is increasing real functions on $[a, b]$.

In the rest of this section we deal with the concept of statistical limit of measurable functions at ∞ introduced by Móricz (see [21]).

Definition 2.4. *Let $f(x)$ be a real valued measurable function (in Lebesgue's sense) on any interval $[0,\infty)$. We say that $f(x)$ has statistical limit as $x\to\infty$ if there exists a $l\in\mathbb{R}$ such that for each $\varepsilon>0$*

$$\lim_{a\to\infty}\frac{1}{a}\left|\{x\in[0,a):|f(x)-l|>\varepsilon\}\right|=0, \tag{3}$$

where by $|\{.\}|$, we denote the Lebesgue measure of the set $\{.\}$. In this case we write st-$\lim\limits_{x\to\infty} f(x)=l$.

Remark 2.5. (i) *According to definition of a measurable function we can replace $|f(x)-l|>\varepsilon$ with $|f(x)-l|\geqslant\varepsilon$ in (3). Also, the interval $[0,a)$ can be replaced with $[0,a]$.*

(ii) *Let $A\subset\mathbb{R}$ be a measurable set. If*

$$\lim_{a\to\infty}\frac{|A\cap[0,a)|}{a}=0$$

then the set A is said to has zero density. Hence Definition 2.4 can be characterized as follows:
st-$\lim\limits_{x\to\infty} f(x)=l\Leftrightarrow$ There exists a set $A\subset\mathbb{R}$ of zero density such that

$\lim\limits_{x\to\infty,\ x\in[0,\infty)\setminus A} f(x)=l$ *(*cf. *[28]).*

(iii) *If $\lim\limits_{x\to\infty} f(x)=l$ then for each $\varepsilon>0$ there exist a $\delta>0$ such that for all $x\in[0,\infty)\setminus[0,\delta]$ we have $|f(x)-l|<\varepsilon$. Since the interval $[0,\delta]$ has zero density we obtain that st-$\lim\limits_{x\to\infty} f(x)=l$. Thus $\lim\limits_{x\to\infty} f(x)=l$ implies st-$\lim\limits_{x\to\infty} f(x)=l$ but the converse statement is not true in general. For instance, consider the measurable function*

$$f(x)=\chi_{[2^n,2^n+1)}(x)=\begin{cases}1, & x\in[2^n,2^n+1),\ n=1,2,3,...\\ 0, & \text{otherwise.}\end{cases}$$

Then st-$\lim\limits_{x\to\infty} f(x)=0$ but the limit $\lim\limits_{x\to\infty} f(x)$ does not exist (cf. *[16]).*

Let ϕ be real valued measurable function on $[0,\infty)$. Supremum of numbers $\beta\in\mathbb{R}$ such that

$$\lim_{a\to\infty}\frac{1}{a}\left|\{x\in[0,a]:\phi(x)>\beta\}\right|\neq0$$

is called the statistical limit superior of ϕ as $x\to\infty$ and is denoted by *st*-$\limsup\limits_{x\to\infty}\phi(x)$. Similarly, infimum of numbers $\alpha\in\mathbb{R}$ such that

$$\lim_{a\to\infty}\frac{1}{a}\left|\{x\in[0,a]:\phi(x)<\alpha\}\right|\neq0$$

is called the statistical limit inferior of ϕ as $x \to \infty$ and is denoted by st-$\liminf_{x\to\infty} \phi(x)$ (*cf.* [21]).

Remark 2.6. *[21](i)* $st\text{-}\limsup_{x\to\infty} \phi(x) = -st\text{-}\liminf_{x\to\infty} (-\phi(x))$.

(ii) If $\phi(x) > 0$ *for all* $x \geqslant 0$ *then* $st\text{-}\limsup_{x\to\infty} \phi(x) = \left[st - \liminf_{x\to\infty} \frac{1}{\phi(x)}\right]^{-1}$

(iii) ϕ *is said to be statistically bounded if there exist* $K \in \mathbb{R}$ *such that*

$$\lim_{a\to\infty} \frac{1}{a} \left|\{x \in [0,a] : |\phi(x)| > K\}\right| = 0.$$

If ϕ *is statistically bounded, then* $st\text{-}\lim_{x\to\infty} \phi(x) = l$ *if and only if* $st\text{-}\limsup_{x\to\infty} \phi(x) = st\text{-}\liminf_{x\to\infty} \phi(x) = l$ *(*cf. *).*

3. Main Results

First we adapt the Definition 2.4 to the fuzzy valued functions. We know that if $\widetilde{f} : [0,a] \to \mathbb{R}_{\mathcal{F}}$ is continuous and $\mu \in \mathbb{R}_{\mathcal{F}}$ then the real valued function $F(x) = D\left(\widetilde{f}(x), \mu\right)$ is continuous on $[0,a]$ and so measurable in Lebesgue's sense.

Definition 3.1. *Let* $\widetilde{f} : [0,\infty) \to \mathbb{R}_{\mathcal{F}}$ *be a continuous fuzzy number valued function. If there exist* $\mu \in \mathbb{R}_{\mathcal{F}}$ *such that for all* $\varepsilon > 0$

$$\lim_{a\to\infty} \frac{1}{a} \left|\left\{x \in [0,a] : D\left(\widetilde{f}(x), \mu\right) \geqslant \varepsilon\right\}\right| = 0,$$

then we say that $\widetilde{f}$ *has statistical limit* μ *as* $x \to \infty$. *If this is the case, we write* $st\text{-}\lim_{x\to\infty} \widetilde{f}(x) = \mu$ *or* $\widetilde{f}(x) \xrightarrow{st} \mu$.

As in Remark 2.5 the statement

$$\widetilde{f}(x) \to \mu \Rightarrow \widetilde{f}(x) \xrightarrow{st} \mu \tag{4}$$

is true. But the converse statement of (4) is not necessarily true in general, follows from example given below.

Example 3.2. Consider the fuzzy number valued function $\widetilde{f} : [0,\infty) \to \mathbb{R}_{\mathcal{F}}$ defined by

$$\widetilde{f}(x)(u) = \begin{cases} \eta(x)(u), & x \in [0,1] \\ \kappa(x)(u), & \text{otherwise} \end{cases}$$

where

$$\eta(x)(u)=\begin{cases} u-x+1, & u\in[x-1,x],\\ -u+x+1, & u\in(x,x+1]\\ 0, & \text{otherwise}\end{cases}$$

and

$$\kappa(x)(u)=\begin{cases} u-\frac{1}{x+1}+1, & u\in\left[\frac{1}{x+1}-1,\frac{1}{x+1}\right],\\ -u+\frac{1}{x+1}+1, & u\in\left(\frac{1}{x+1},\frac{1}{x+1}+1\right]\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$\mu(u)=\begin{cases} u+1, & u\in[-1,0],\\ -u+1, & u\in(0,1]\\ 0, & \text{otherwise.}\end{cases}$$

The interval $[0,1]$ has zero density and for all $x\notin[0,1]$ we have

$$\begin{aligned} D\left(\widetilde{f}(x),\mu\right) &= sup\max\left\{\left|\kappa_\alpha^-(x)-\mu_\alpha^-\right|,\left|\kappa_\alpha^+(x)-\mu_\alpha^+\right|\right\}\\ &= sup\max\left\{\left|\frac{1}{x+1}+\alpha-1-(\alpha-1)\right|,\right.\\ &\left.\left|\frac{1}{x+1}+1-\alpha-(1-\alpha)\right|\right\}=\frac{1}{x+1}\to 0\qquad(x\to\infty).\end{aligned}$$

Hence $\widetilde{f}(x)\xrightarrow{st}\mu$ but $\widetilde{f}(x)\not\to\mu$.

The following is a Tauberian theorem from $\widetilde{f}(x)\xrightarrow{st}\mu$ to $\widetilde{f}(x)\to\mu$ $(x\to\infty)$. Note that the conditions (5) and (6) are fuzzy analogues of the conditions given by Chen and Chang (*cf.* Theorem 3.1 in [11]).

Theorem 3.3. *Let $\widetilde{f}:[0,\infty)\to\mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy number valued function and st-$\lim_{x\to\infty}\widetilde{f}(x)=\mu$. If one of the conditions*

$$\inf_{\lambda>1}\left\{\limsup_{x\to\infty}\left(sup_{x<u<\lambda x}D\left(\widetilde{f}(u),\widetilde{f}(x)\right)\right)\right\}=0 \tag{5}$$

or

$$\inf_{0<\lambda<1}\left\{\limsup_{x\to\infty}\left(sup_{\lambda x<u<x}D\left(\widetilde{f}(u),\widetilde{f}(x)\right)\right)\right\}=0 \tag{6}$$

holds then $\lim_{x\to\infty}\widetilde{f}(x)=\mu$.

Proof. Conditions (5) and (6) are equivalent to each other since $\lambda>1\Leftrightarrow\frac{1}{\lambda}<1$. Hence, it is sufficient to prove the case of (5). Let $\varepsilon>0$. By (5) we can find $\lambda>1$ and $x_1\geqslant 0$ such that

$$x\geqslant x_1\Rightarrow sup_{x<u<\lambda x}D\left(\widetilde{f}(u),\widetilde{f}(x)\right)<\varepsilon. \tag{7}$$

By the assumption of $st\text{-}\lim_{x\to\infty} \widetilde{f}(x) = \mu$ there exists $x_2 \geqslant 0$ such that for $a \geqslant x_2$,

$$\frac{1}{a}\left|\left\{u \in [0,a] : D\left(\widetilde{f}(u), \mu\right) \geqslant \varepsilon\right\}\right| < 1 - \frac{1}{\lambda}. \tag{8}$$

Let $x_0 = \max\{x_1, x_2\}$. Then for $x \geqslant x_0$ we have $\lambda x \geqslant x_2$ and so by (7)

$$\left|\left\{u \in [0,\lambda x] : D\left(\widetilde{f}(u), \mu\right) \geqslant \varepsilon\right\}\right| < \left(1 - \frac{1}{\lambda}\right)\lambda x = (\lambda x - x).$$

Hence we can find $u^* \in (x, \lambda x)$ such that $D\left(\widetilde{f}(u^*), \mu\right) < \varepsilon$. Combining this with (7) we get

$$\begin{aligned} D\left(\widetilde{f}(x), \mu\right) &= D\left(\widetilde{f}(x) + \widetilde{f}(u^*), \mu + \widetilde{f}(u^*)\right) \\ &\leqslant D\left(\widetilde{f}(u^*), \mu\right) + supD\left(\widetilde{f}(u), \widetilde{f}(x)\right) < 2\varepsilon. \end{aligned}$$

Thus we have $\lim_{x\to\infty} \widetilde{f}(x) = \mu$. $\square$

We note that the conditions (5) and (6) can be replaced with

$$\inf_{\lambda>1}\left\{\limsup_{x\to\infty}\left(sup_{x\leqslant u\leqslant\lambda x} D\left(\widetilde{f}(u), \widetilde{f}(x)\right)\right)\right\} = 0 \tag{9}$$

and

$$\inf_{0<\lambda<1}\left\{\limsup_{x\to\infty}\left(sup_{\lambda x\leqslant u\leqslant x} D\left(\widetilde{f}(u), \widetilde{f}(x)\right)\right)\right\} = 0 \tag{10}$$

respectively. If the condition (9) or (10) holds then we say that the function $\widetilde{f}$ is slowly oscillating (see e.g. [22]).

Now we present the idea of statistically $\left(\overline{N}, q\right)$ summability for continuous fuzzy valued functions. From now on let Q be class of all increasing functions $0 \neq q : [0,\infty) \to [0,\infty)$ such that $q(0) = 0$, $q(t) \to \infty$ as $t \to \infty$. Let

$$st\text{-}\liminf_{t\to\infty} \frac{q(\lambda t)}{q(t)} > 1 \text{ for all } \lambda > 1 \tag{11}$$

Suppose $\widetilde{f} : [0,\infty) \to \mathbb{R}_{\mathcal{F}}$ is a continuous fuzzy number valued function function and let

$$\widetilde{s}(t) = \int_0^t \widetilde{f}(u)\, du \quad \text{and} \quad \widetilde{\sigma}(t) = \frac{1}{q(t)}\int_0^t \widetilde{s}(x)\, dq(x),$$

provided $q(t) > 0$. Note that the second integral exists in the fuzzy Riemann-Stieltjes sense.

The fuzzy number valued function $\widetilde{s}(t)$ is said to be statistically summable to a fuzzy number $l \in \mathbb{R}_{\mathcal{F}}$ with respect to weight function q, in short statistically $(\overline{N}, q)$ summable to $l \in \mathbb{R}_{\mathcal{F}}$, if

$$st\text{-}\lim_{t\to\infty} \widetilde{\sigma}(t) = l$$

or equivalently

$$st\text{-}\lim_{t\to\infty} D(\widetilde{\sigma}(t), l) = 0.$$

In this case we write $\widetilde{s}(t) \xrightarrow{st} l \ (\overline{N}, q)$. As in the ordinary case (see, Fekete [16]) the implication

$$\widetilde{s}(t) \xrightarrow{st} l \Longrightarrow \widetilde{s}(t) \xrightarrow{st} l \ (\overline{N}, q)$$

is true but its converse is not true in general. Our aim is to find conditions from $\widetilde{s}(t) \xrightarrow{st} l \ (\overline{N}, q)$ to $\widetilde{s}(t) \xrightarrow{st} l$.

Remark 3.4. (i) *The condition (11) is equivalent with*

$$st\text{-}\liminf_{t\to\infty} \frac{q(t)}{q(\lambda t)} > 1 \text{ for every } 0 < \lambda < 1$$

(cf. [16]).

(ii) *In [16], Fekete proved, if f is real or complex valued function and st-$\lim_{t\to\infty} f(t) = l$, then for each $\lambda > 0$ st-$\lim_{t\to\infty} f(\lambda t) = l$. Now, if $\widetilde{f}$ is a fuzzy number valued function and $l \in \mathbb{R}_{\mathcal{F}}$ then st-$\lim_{t\to\infty} D\left(\widetilde{f}(t), l\right) = 0$ implies st-$\lim_{t\to\infty} D\left(\widetilde{f}(\lambda t), l\right) = 0$, for each $\lambda > 0$, since $D\left(\widetilde{f}(t), l\right)$ is a real valued function.*

Lemma 3.5. *Assume that the function $q \in Q$ has the property (11) and let st-$\lim_{t\to\infty} \widetilde{\sigma}(t) = l$. Then for every $\lambda > 1$*

$$st\text{-}\lim_{t\to\infty} D\left(\frac{1}{q(\lambda t) - q(t)} \int_t^{\lambda t} \widetilde{s}(x)\, dq(x), l\right) \tag{12}$$

and for every $0 < \lambda < 1$

$$st\text{-}\lim_{t\to\infty} D\left(\frac{1}{q(t) - q(\lambda t)} \int_{\lambda t}^{t} \widetilde{s}(x)\, dq(x), l\right) = 0. \tag{13}$$

Proof. It is enough to prove the case $\lambda > 1$, the case $0 < \lambda < 1$ is similar. By Lemma 2.1 (ii)-(iii) we have

$$\begin{aligned} & D\left(\frac{1}{q(\lambda t) - q(t)} \int_t^{\lambda t} \widetilde{s}(x)\, dq(x), l\right) \\ & = D\left(\frac{1}{q(\lambda t) - q(t)} \int_t^{\lambda t} \widetilde{s}(x)\, dq(x) + \widetilde{\sigma}(t), \widetilde{\sigma}(t) + l\right) \\ & \leqslant D\left(\frac{1}{q(\lambda t) - q(t)} \int_t^{\lambda t} \widetilde{s}(x)\, dq(x), \widetilde{\sigma}(t)\right) + D(\widetilde{\sigma}(t), l) \\ & \frac{q(\lambda t)}{q(\lambda t) - q(t)} \left(D(\widetilde{\sigma}(\lambda t), l) + D(\widetilde{\sigma}(t), l)\right) + D(\widetilde{\sigma}(t), l). \end{aligned} \tag{14}$$

From (11), for every $\lambda > 1$ we have

$$\begin{aligned} st\text{-}\limsup_{t\to\infty} \frac{q(\lambda t)}{q(\lambda t) - q(t)} & = st\text{-}\limsup_{t\to\infty} \frac{1}{1 - \frac{q(t)}{q(\lambda t)}} \\ & = \left[st\text{-}\liminf_{t\to\infty} \left(1 - \frac{q(t)}{q(\lambda t)}\right)\right]^{-1} \\ & = \left[1 - \frac{1}{st\text{-}\liminf_{t\to\infty} \frac{q(\lambda t)}{q(t)}}\right]^{-1} < \infty \end{aligned} \tag{15}$$

Thus the desired result follows from (14) and (15). □

Theorem 3.6. *Assume that the function* $q \in Q$ *has the property (11) and,* $\widetilde{f} : [0, \infty) \to \mathbb{R}_{\mathcal{F}}$ *be a continuous fuzzy number valued function such that st-*$\lim\limits_{t\to\infty} \widetilde{\sigma}(t) = l$. *Then st-*$\lim\limits_{t\to\infty} \widetilde{s}(t) = l$ *if and only if for each* $\varepsilon > 0$, *we have*

$$\inf_{\lambda > 1} \limsup_{a\to\infty} \frac{1}{a} \left| \left\{ t \in [0, a] : D\left(\frac{1}{q(\lambda t) - q(t)} \int_t^{\lambda t} \widetilde{s}(x)\, dq(x), \widetilde{s}(t)\right) \geqslant \varepsilon \right\} \right| = 0 \tag{16}$$

or

$$\inf_{0<\lambda<1} \limsup_{a\to\infty} \frac{1}{a} \left| \left\{ t \in [0, a] : D\left(\frac{1}{q(t) - q(\lambda t)} \int_{\lambda t}^{t} \widetilde{s}(x)\, dq(x), \widetilde{s}(t)\right) \geqslant \varepsilon \right\} \right| = 0 \tag{17}$$

Proof. *Necessity.* Assume that $st\text{-}\lim_{t\to\infty}\widetilde{\sigma}(t)=l$ and $st\text{-}\lim_{t\to\infty}\widetilde{s}(t)=l$ hold. Then by Lemma 3.5 we have (12) and (13), respectively for every $\lambda>1$ and $0<\lambda<1$. Thus for $\lambda>1$ we have

$$\begin{aligned} D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}(x)\,dq(x),\widetilde{s}(t)\right) &\leqslant D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}(x)\,dq(x),l\right)\\ &+D\left(\widetilde{s}(t),l\right)\xrightarrow{st}0+0=0. \end{aligned}$$

Hence we get (16). Also for $0<\lambda<1$ we have

$$\begin{aligned} D\left(\frac{1}{q(t)-q(\lambda t)}\int_{\lambda t}^{t}\widetilde{s}(x)\,dq(x),\widetilde{s}(t)\right) &\leqslant D\left(\frac{1}{q(t)-q(\lambda t)}\int_{\lambda t}^{t}\widetilde{s}(x)\,dq(x),l\right)\\ &+D\left(\widetilde{s}(t),l\right)\xrightarrow{st}0+0=0 \end{aligned}$$

and so (17) holds.
Sufficiency. Let $st\text{-}\lim_{t\to\infty}\widetilde{\sigma}(t)=l$ and assume that condition (16) holds. We prove that $st\text{-}\lim_{t\to\infty}\widetilde{s}(t)=l$. For this it is enough to show that $D\left(\widetilde{\sigma}(t),\widetilde{s}(t)\right)\xrightarrow{st}0$. In the case of $\lambda>1$ we have

$$\begin{aligned} D\left(\widetilde{\sigma}(t),\widetilde{s}(t)\right) &\leqslant D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}(x)\,dq(x),\widetilde{s}(t)\right)\\ &+\frac{q(\lambda t)}{q(\lambda t)-q(t)}D\left(\widetilde{\sigma}(t),\widetilde{\sigma}(\lambda t)\right). \end{aligned}$$

So we have the inequality

$$\begin{aligned} &\left|\left\{t\in[0,a]:D\left(\widetilde{\sigma}(t),\widetilde{s}(t)\right)\geqslant\varepsilon\right\}\right|\\ &\leqslant\left|\left\{t\in[0,a]:D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}(x)\,dq(x),\widetilde{s}(t)\right)\geqslant\frac{\varepsilon}{2}\right\}\right| \qquad (18)\\ &+\left|\left\{t\in[0,a]:\frac{q(\lambda t)}{q(\lambda t)-q(t)}D\left(\widetilde{\sigma}(t),\widetilde{\sigma}(\lambda t)\right)\geqslant\frac{\varepsilon}{2}\right\}\right|. \end{aligned}$$

By the condition (16) for each $\delta>0$ there exists $\lambda>1$ such that

$$\limsup_{a\to\infty}\frac{1}{a}\left|\left\{t\in[0,a]:D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}(x)\,dq(x),\widetilde{s}(t)\right)\geqslant\frac{\varepsilon}{2}\right\}\right|\leqslant\delta. \qquad (19)$$

On the other hand by (15) and Remark 3.4 (ii) we have

$$\limsup_{a\to\infty}\frac{1}{a}\left|\left\{t\in[0,a]:\frac{q(\lambda t)}{q(\lambda t)-q(t)}D\left(\widetilde{\sigma}\left(t\right),\widetilde{\sigma}\left(\lambda t\right)\right)\geqslant\frac{\varepsilon}{2}\right\}\right|=0. \tag{20}$$

Combining (18), (19) and (20) we have

$$\limsup_{a\to\infty}\frac{1}{a}\left|\left\{t\in[0,a]:D\left(\widetilde{\sigma}\left(t\right),\widetilde{s}\left(t\right)\right)\geqslant\varepsilon\right\}\right|\leqslant\delta.$$

Since $\delta>0$ is arbitrary we have

$$\lim_{a\to\infty}\frac{1}{a}\left|\left\{t\in[0,a]:D\left(\widetilde{\sigma}\left(t\right),\widetilde{s}\left(t\right)\right)\geqslant\varepsilon\right\}\right|=0$$

for each $\varepsilon>0$. Thus we conclude that $st\text{-}\lim_{t\to\infty}\widetilde{s}\left(t\right)=l$. In the case of $0<\lambda<1$, one can easily show that $st\text{-}\lim_{t\to\infty}\widetilde{s}\left(t\right)=l$. □

Following Fekete [16], we say that the function $\widetilde{s}:[0,\infty)\to\mathbb{R}_{\mathcal{F}}$ is statistically slowly oscillating if

$$\inf_{\lambda>1}\limsup_{a\to\infty}\frac{1}{a}\left|\left\{t\in[0,a]:\max_{t\leqslant x\leqslant\lambda t}D\left(\widetilde{s}\left(x\right),\widetilde{s}\left(t\right)\right)\geqslant\varepsilon\right\}\right|=0 \tag{21}$$

or equivalently

$$\inf_{0<\lambda<1}\limsup_{a\to\infty}\frac{1}{a}\left|\left\{t\in[0,a]:\max_{\lambda t\leqslant x\leqslant t}D\left(\widetilde{s}\left(x\right),\widetilde{s}\left(t\right)\right)\geqslant\varepsilon\right\}\right|=0 \tag{22}$$

holds for each $\varepsilon>0$. By inequality (2) and Lemma 2.1 (i) we have

$$\begin{aligned}
&D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}\left(x\right)dq(x),\widetilde{s}\left(t\right)\right)\\
&=D\left(\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}\left(x\right)dq(x),\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}\widetilde{s}\left(t\right)dq(x)\right)\\
&\leqslant\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}D\left(\widetilde{s}\left(x\right),\widetilde{s}\left(t\right)\right)dq(x)\\
&\leqslant\max_{t\leqslant x\leqslant\lambda t}D\left(\widetilde{s}\left(x\right),\widetilde{s}\left(t\right)\right)\frac{1}{q(\lambda t)-q(t)}\int_t^{\lambda t}dq(x)=\max_{t\leqslant x\leqslant\lambda t}D\left(\widetilde{s}\left(x\right),\widetilde{s}\left(t\right)\right)
\end{aligned}$$

Hence the condition (21) implies (16). Similarly (22) implies (17). Thus we deduce the following result from Theorem 3.6.

Corollary 3.7. *Assume that the function $q \in Q$ has the property (11) and, $\widetilde{f} : [0,\infty) \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy number valued function such that its integral function $\widetilde{s}(t)$ is statistically slowly oscillating. If $st\text{-}\lim_{t\to\infty} \widetilde{\sigma}(t) = l$ then $st\text{-}\lim_{t\to\infty} \widetilde{s}(t) = l$.*

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