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Original Research Paper

The Canonical Mapping T_H for Weighted L^p-Spaces the General Case

F. Esmaeelzadeh
Bojnourd Branch, Islamic Azad University

Abstract. Let G be a locally compact group and H be a closed subgroup of G. It is well-known that G/H as a homogeneous space admits a strongly quasi invariant measure and the linear mapping T_H of $L^1(G)$ into $L^1(G/H)$ is bounded and surjective. In this note it is shown that by means of complex interpolation theorem, that under restrictions on weight function ω, the mapping T_H of weighted spaces $L^p(G,\omega)$ into $L^p(G/H,\varpi)$ is well-defined, bounded linear and surjective, for $1 \leqslant p \leqslant \infty$.

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1. Introduction and Preliminaries

For a locally compact group G and closed subgroup H of G, Reiter in [7] has studied the bounded linear mapping T_H of $L^1(G)$ onto $L^1(G/H)$ defined by

$$T_H f(\pi(x)) = \int_H \frac{f(xh)}{\rho(xh)} dh, \tag{1}$$

where ρ is a strictly positive continuous function of G and π is a canonical mapping of G onto G/H given by $x \to xH$. Feichtinger in [3] has

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shown that the linear canonical mapping T_H maps a weighted L^p-space, $L^p(G,\omega)$ onto another weighted L^p-space, $L^p(G/H,\varpi)$, for $1 \leqslant p \leqslant \infty$, when H is a closed normal subgroup of G. It is well-known that if H is a closed normal subgroup of G, then G/H admits a Haar measure as a quotient group. In this note, we consider H as a closed subgroup of G and G/H as a quotient space. We take on this task in the general case considering a relatively invariant measure μ on G/H, when H is a closed subgroup of G.

Throughout this note H is a closed subgroup of the locally compact group G. The quotient space G/H on which G acts by left is called a homogeneous space. For a pair (G,H) there is a continuous function $\rho : G \to (0,\infty)$ satisfying

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x) \qquad (x \in G, h \in H), \tag{2}$$

where Δ_G, Δ_H are the modular functions on G and H, respectively. Moreover, it is worthwhile to recall that the Radon measure μ on G/H is called strongly quasi invariant if there is a positive continuous function λ of $G \times G/H$ such that $d\mu_x(yH) = d\mu(xyH) = \lambda(x,yH)d\mu(yH)$, for $x,y \in G$. If the function $\lambda(x,.)$ reduce to constants, the measure μ is called relatively invariant measure. For each rho-function ρ for the pair (G,H) there is a strongly quasi invariant measure μ on G/H such that

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}, \tag{3}$$

for $x,y \in G$. Every strongly quasi invariant measure on G/H arises from a rho function and all such measures are strongly equivalent. It is known that the existence of a homomorphism rho-function for the pair (G,H) is a necessary and sufficient condition for the existence of a relatively invariant measure on G/H. Assume that μ is a relatively invariant measure on G/H which arises from a rho-function ρ. It has been shown that

$$\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)}, \tag{4}$$

and

$$\frac{d\mu(xy)}{d\mu(y)} = \frac{\rho(x)}{\rho(e)}.$$

It is well known that the mapping $T_H : L^1(G) \to L^1(G/H)$ is a surjective map such that for any $\varphi \in L^1(G/H)$,

$$\|\varphi\| = inf\{\|f\|; f \in L^1(G),\ T_H(f) = \varphi\}.$$

Moreover, for $f \in L^1(G)$ satisfy the following weil's formula.

$$\int_G f(x)dx = \int_{G/H} T_H(f)(xH)d\mu(xH).$$

(for more information about homogeneous spaces see [4, 7]).

For a convenience's readers we recall that a real-valued function ω on locally compact group G is said to be a submultiplicative weight function if it is a locally bounded and positive measurable function on G such that

$$\omega(xy) \leqslant \omega(x)\omega(y) \qquad (x, y \in G).$$

It is immediately concluded $\omega(e) \geqslant 1$ in which e is a neutral element of group G. A weight function m on G is ω-moderate if

$$m(xy) \leqslant C\omega(x)m(y),$$

for $x, y \in G$. In [7, Theorem 3.7.5] has been shown that for every weight function ω_1 on a locally compact group G, there is a continuous weight function ω such that ω, ω_1 define the same subalgebra of $L^1(G)$ with equivalent norms. Moreover, a function f belongs to $L^p(G, \omega)$ if $f \cdot \omega$ is in $L^p(G)$. For $1 \leqslant p \leqslant \infty$, L^p-space $L^p(G, \omega)$ is a Banach space with the norm $\|f\|_{p,\omega} := \|f \cdot \omega\|_p$. (See more details about weight functions in [2, 5, 6]).

In this note it is proven by complex interpolation, that under restrictions on weight function ω, the canonical mapping T_H maps weight L^p-space $L^p(G, \omega)$ to another weight L^p-space $L^p(G/H, \varpi)$, where ϖ depende on p. It should be emphasized that H has not considered normal necessarily, so the homogeneous space G/H does not admit a group structure; consequently, we do not expect a Haar measure on G/H!

2. Main Result

In this section the image under the mapping T_H of weight spaces $L^1(G,\omega)$ and $L^\infty(G,\omega)$ is characterized and then it is investigated for weight L^p-space $L^p(G,\omega)$, $\;1<p<\infty$, by the interpolation theorem.

For a closed subgroup H of G consider a relatively invariant measure μ on homogeneous space G/H which arises from a rho-function ρ. The following condition on ω is to be imposed occasionally:

$$\rho^{-\frac{1}{q}}\omega^{-1}|_H \in L^q(H) \tag{5}$$

in which $\frac{1}{q}+\frac{1}{p}=1$. Or equivalently,

$$\left(\frac{\Delta_G}{\Delta_H}\right)^{\frac{1}{q}}\omega^{-1}|_H \in L^q(H).$$

Throughout this section we assume that the condition (5) holds and the following lemma shows that this condition is G-invariant.

Lemma 2.1. *If $\rho^{-\frac{1}{q}}\omega^{-1}|_H \in L^q(H)$, then $L_x\rho^{-\frac{1}{q}}\omega^{-1}|_H \in L^q(H)$, for each $x\in G$, where L_x is the left translation operator on G.*

Proof. By the submultiplicativity of ω and (5) we get,

$$\omega(h) \leqslant \omega(xh)\omega(x^{-1})$$

$$\frac{1}{\rho^{\frac{1}{q}}(xh)\omega(xh)} \leqslant \frac{\omega(x^{-1})}{\rho^{\frac{1}{q}}(xh)\omega(h)}$$

$$\frac{1}{\rho^{\frac{1}{q}}(xh)\omega(xh)} \leqslant \frac{\omega(x^{-1})}{(\frac{\Delta_H(h)}{\Delta_G(h)})^{\frac{1}{q}}\rho^{\frac{1}{q}}(x)\omega(h)}$$

$$\frac{1}{\rho^{\frac{1}{q}}(xh)\omega(xh)} \leqslant \frac{\omega(x^{-1})\Delta_G^{\frac{1}{q}}(h)}{\rho^{\frac{1}{q}}(x)\Delta_H^{\frac{1}{q}}(h)\omega(h)}.$$

By (5) we get,

$$\begin{aligned}\int_H |L_x\rho^{-1/q}\omega^{-1}(h)|^q dh &= \int_H |\frac{1}{\rho^{1/q}(xh)\omega(xh)}|^q dh \\ &\leqslant \frac{\omega(x^{-1})^q}{\rho(x)}\int_H |\frac{1}{\rho^{1/q}(h)\omega(h)}|^q dh.\end{aligned}$$

Thus the proof is complete. □

Now we define a moderate weight function on homogeneous space G/H.

Definition 2.2. *A weight function ϖ on homogeneous space G/H is defined as a locally bounded, positive measurable function on G/H such that*

$$\varpi(\pi(x)) = inf_{h\in H}\omega(xh).$$

Moreover, a weight function on homogeneous space G/H is moderete if satisfying

$$\varpi(\pi(xy)) \leqslant \omega(x)\varpi(\pi(y)), \tag{6}$$

in which ω is a submultiplicative weight function on G and π is the canonical mapping of G onto G/H.

At first we show that the linear mapping T_H of $L^1(G,\omega)$ onto $L^1(G/H,\varpi_1)$ exists in which ω is a weight on G such that $\omega^{-1}|_H \in L^\infty(H)$.

Proposition 2.3. *The mapping $T_H : L^1(G,\omega) \to L^1(G/H,\varpi_1)$ defined in (1) is well-defined and surjective linear map. Moreover,*

$$\|T_H(f)\|_{1,\varpi_1} \leqslant \|f\|_{1,\omega},$$

where ω satisfy in (5) and $\varpi_1(\pi(x)) = \operatorname{essinf}\{\omega(xh), h \in H\}$.

Proof. It is straightforward to see ϖ_1 is a weight function on G/H. For $f \in L^1(G,\omega)$ we have $f\cdot\omega \in L^1(G)$. So $L_x(\frac{f\cdot\omega}{\rho}) \in L^1(H)$. Now we get,

$$\begin{aligned}\int_H |\frac{f(xh)}{\rho(xh)}|dh &= \int_H |\frac{f(xh)\omega(xh)}{\rho(xh)\omega(xh)}|dh \\ &= \int_H |\frac{f\cdot\omega(xh)}{\rho(xh)}|.\frac{1}{\omega(xh)}dh \\ &\leqslant \int_H |\frac{f\cdot\omega(xh)}{\rho(xh)}|dh.\operatorname{esssup}_{h\in H}\omega^{-1}(xh).\end{aligned}$$

Hence for $f \in L^1(G,\omega)$, $T_H(f)$ is well defined. Therefore by [7, Theorem 3.4.6] it can be concluded that the function $xH \mapsto T_H(f)\cdot\varpi_1$ defined almost every where on G/H is integrable in $L^1(G/H)$, in which $\varpi_1(xH) = \operatorname{essinf}_{h\in H}\omega(xh) = (\operatorname{esssup}\omega^{-1}(xh))^{-1}$. Then $T_H(f) \in L^1(G/H,\varpi_1)$. Moreover, since

$$T_H(f)\cdot(\operatorname{esssup}\omega^{-1}(xh))^{-1} \leqslant T_H(f\cdot\omega),$$

we get

$$\|T_H(f)\|_{1,\varpi_1} = \|T_H(f).\varpi_1\|_1 \leqslant \|T_H(f\cdot\omega)\|_1 \leqslant \|f\cdot\omega\|_1 = \|f\|_{1,\omega}.$$

Now let $\varphi \in C_c(G/H)$. Put $L = \operatorname{supp}\varphi$. So there exists a compact set K in G such that $\pi(K) = L$. Take a $k \in C_c^+(G)$ is strictly positive on K, then $T_Hk(xH) > 0$, for $xH \in L$. Now define for $x \in G$, R_H as a right inverse T_H given by

$$R_H\varphi(x) = \begin{cases} \dfrac{\varphi\circ\pi(x)k(x)}{(T_Hk)\circ\pi(x)} & x \in K \\ 0 & otherwise \end{cases}$$

Then $R_H\varphi$ is continuous and $\operatorname{supp} R_H\varphi$ is compact. So $R_H\varphi \in C_c(G)$ and

$$T_HR_H\varphi(xH) = \int_H \frac{\varphi(xH)k(xh)}{\rho(xh)T_Hk(xH)}dh = \varphi(xH).$$

Thus by density $C_c(G)$ in $L^1(G,\omega)$ with the norm $\|.\|_{1,\omega}$, the mapping T_H is surjective. □

It is worthwhile that if X, Y be two dense subspaces of Banach spaces X_1 and Y_1, respectively. If $T : X \to Y$ is a linear bounded mapping such that for $x \in X$

$$\|T(x)\| = inf\{\|z\|; z \in X, T(z) = T(x)\},$$

then

(i) The quotient mapping $\bar{T} : X/kerT \to Y$ defined as $\bar{T}(x + kerT) = T(x)$, is isometry. Also, If T is surjective, then $\bar{T}$ is isometric isomorphism.

(ii) For the linear mapping T there exist a unique bounded linear extension $\bar{T}: X_1 \to Y_1$ such that $\overline{kerT} = ker\bar{T}$ and for $x \in X_1$,

$$\|\bar{T}(x)\| = inf\{\|z\|; z \in X_1, \bar{T}(z) = \bar{T}(x)\}.$$

Also, $\bar{T}$ is surjective if T is surjective.

Secondly, we determine image under mapping T_H of $L^\infty(G,\omega)$ in the following proposition. In this case G is a σ-compact group.

Proposition 2.4. *Let ω be a submultiplicative weight function on G such that $(\rho\omega)^{-1}|_H \in L^1(H)$. The mapping $T_H : L^\infty(G,\omega) \to L^\infty(G/H, \varpi_\infty)$ is linear and surjective. Also,*

$$\|T_H(f)\|_{\infty,\varpi_\infty} \leqslant \|f\|_{\infty,\omega},$$

in which $\varpi_\infty = (T_H\omega^{-1})^{-1}$.

Proof. Note that $\frac{\omega\rho}{\rho(e)}$ is a submultiplicative weight function on G, then ϖ_∞ is a weight function on G/H. Indeed,

$$\begin{aligned} T_H\omega^{-1}(xyH) &= \int_H \frac{\omega^{-1}(xyh)}{\rho(xyh)} dh \\ &\geqslant \int_H \frac{\omega^{-1}(x)\omega^{-1}(yh)}{\rho(xyh)} dh \\ &= \frac{\omega^{-1}(x)\rho(e)}{\rho(x)} T_H\omega^{-1}(yH). \end{aligned}$$

Therefore, $\varpi_\infty(xyH) \leqslant \dfrac{\rho(x)\omega(x)}{\rho(e)}\varpi_\infty(yH)$. Now, since $(\rho\cdot\omega)^{-1}|_H \in L^1(H)$, then by Lemma 2.1 $L_x(\rho\cdot\omega)^{-1}$ in $L^1(H)$. Thus ϖ_∞ is well-defined. Now for $f \in L^\infty(G,\omega)$

$$\begin{aligned} |T_H f(xH)| &\leqslant \int_H |\frac{f(xh)}{\rho(xh)}| dh \\ &\leqslant \int_H |\frac{f(xh)\omega(xh)}{\rho(xh)\omega(xh)}| dh \\ &\leqslant T_H(\omega^{-1}) \operatorname{esssup}_{h\in H} f\cdot\omega(xh). \end{aligned}$$

Then by [7, Proposition 3.4.10] we have,

$$\begin{aligned}
\|T_H(f)\|_{\infty,\varpi_\infty} &= \|T_H(f)\cdot\varpi_\infty\|_\infty \\
&= esssup_{xH\in G/H}|T_Hf(xH)|\varpi_\infty \\
&\leqslant \operatorname{esssup}_{xH\in G/H}\operatorname{esssup}_{h\in H}|f\cdot\omega(xh)| \\
&\leqslant \operatorname{esssup}_{x\in G}|f\cdot\omega(x)| \\
&= \|f\cdot\omega\|_\infty \\
&= \|f\|_{\infty,\omega}
\end{aligned}$$

where $\varpi_\infty = (T_H\omega^{-1})^{-1}$. We show next that T_H is surjective. Define R_H as a right inverse of T_H by

$$R_H(\varphi) := ((\varphi.\varpi_\infty)\circ\pi).\omega^{-1}, \qquad \varphi\in L^\infty(G/H,\varpi_\infty). \tag{7}$$

Then $T_HR_H\varphi=\varphi$. Indeed,

$$\begin{aligned}
T_HR_H\varphi(xH) &= \int_H \frac{(\varphi.\varpi_\infty)\circ\pi)(xh).\omega^{-1}(xh)}{\rho(xh)}dh \\
&= \varphi(xH)\varpi_\infty(xH)\int_H\frac{1}{\rho(xh)\omega(xh)}dh \\
&= \varphi(xH).
\end{aligned}$$

Also

$$\|R_H\varphi\|_{\infty,\omega} = \|R_H\varphi.\omega\|_\infty = \|\varphi.\varpi_\infty\|_\infty = \|\varphi\|_{\infty,\varpi_\infty}. \quad \square \tag{8}$$

Finally, the main theorem can be proven by the interpolation theorem. Before that the Stein-Weiss interpolation theorem, which apply for the proof of the main theorem, are explained.

Let k_1, k_2 be two non-negative measurable functions on measure space N. Also, let u_1, u_2 be two non-negative measurable functions on measure space M. The following theorem has been proven by M. Stein in [8] (one may see [8, Theorem2] and also, [1, Corollary 5.5.4].

Theorem 2.5. *Let T be a linear transformation defined on simple functions of M to measurable functions on N. Suppose $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$ and $\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}$, where $0 \leqslant t \leqslant 1$. Assume that for simple f,*

$$\|Tf.k_1\|_{q_1} \leqslant M_1 \|f.u_1\|_{p_1},$$

and

$$\|Tf.k_2\|_{q_2} \leqslant M_2 \|f.u_2\|_{p_2}.$$

Let $k = k_1^{1-t} k_2^t, u = u_1^{1-t} u_1^t$. The we may conclude that T may be uniquely extended to a linear transformation on functions f, for which $\|f.u\|_p < \infty$, so that

$$\|Tf.k\|_q \leqslant M_t \|f.u\|_p,$$

in which $M_t = M_1^{1-t} M_2^t$.

Theorem 2.6. *Let G be a locally compact group and H be a closed subgroup of G. The mapping $T_H : L^p(G, \omega) \to L^p(G/H, \varpi)$, by $f \mapsto T_H f$ is a surjective linear map and*

$$\|T_H(f)\|_{p,\varpi} \leqslant \|f\|_{p,\omega},$$

in which $\rho^{-\frac{1}{q}} \omega^{-1}|_H \in L^q(H)$.

Proof. At first given $1 < p < \infty$ and $f \in L^p(G, \omega)$ we show that $\int_H \frac{f(xh)}{\rho(xh)} dh$ is convergent. By Lemma 2.1 and Hölder's inequality we obtain that,

$$\begin{aligned} \int_H \left|\frac{f(xh)}{\rho(xh)}\right| dh &= \int_H \frac{|f(xh)| \cdot \omega(xh)}{\rho^{\frac{1}{p}+\frac{1}{q}}(xh) \cdot \omega(xh)} dh \\ &\leqslant \left(\int_H \frac{f^p(xh) \cdot \omega^p(xh) dh}{\rho(xh)} \right)^{\frac{1}{p}} \left(\int_H \frac{dh}{\rho(xh) \cdot \omega^q(xh)} \right)^{\frac{1}{q}}. \end{aligned}$$

By Propositions 2.3, 2.4 and Theorem 2.5 [1] we have a linear mapping

$$T_H : L^p(G, \omega) \to L^p(G/H, \varpi),$$

with

$$\|T_H(f)\|_{p,\varpi} \leqslant \|f\|_{p,\omega},$$

for $f \in L^p(G,\omega)$, $1 < p < \infty$ in which $\varpi = \varpi_\infty^{1-t}\varpi_1^t, 0 < t < 1$. Now, we prove that T_H is surjective. By Proposition 2.3, R_H is a right inverse for T_H of $L^1(G/H,\varpi_1)$ such that $\|R_H(\varphi)\|_{1,\omega} = \|\varphi\|_{1,\varpi}$. Infact, by the weil's formula we have $\|R_H\varphi\|_{1,\omega} = \|T_H R_H\varphi\|_{1,\varpi} = \|\varphi\|_{1,\varpi_1}$. Also, in Proposition 2.4 R_H is a right inverse for T_H of $L^\infty(G/H,\varpi_\infty)$ such that $\|R_H(\varphi)\|_{\infty,\omega} = \|\varphi\|_{\infty,\varpi_\infty}$ (see (8)). We apply again theorem 2.5 and get the right inverse R_H of $L^p(G/H,\varpi_p)$ such that $\|R_H(\varphi)\|_{p,\omega} = \|\varphi\|_{p,\varpi_p}$. $\square$

Remark 2.7. *Let G, H, ω, p, ϖ be as in Theorem 2.6. Then T_H maps $L^1(G) \cap L^p(G,\omega)$ onto $L^1(G/H) \cap L^p(G/H,\varpi)$.*

Remark 2.8. *If μ is a G-invariant measure on G/H, Then the mapping $T_H : L^p(G,\omega) \to L^p(G/H,\varpi)$, by $f \mapsto \int_H f(xh)dh$ is a linear surjective map.(for more details one may refer to [3])*

Remark 2.9 *If G is a locally compact group and H a compact subgroup of G, then the mapping $T_H : L^p(G) \to L^p(G/H)$ is well-defined and also T_H is a linear surjective map (note that since H is compact so the rho function $\rho = 1$ and the Haar measure on H is finite. Then the (5) holds and the mapping T_H is well defined.)*

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References

[1] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction.* Number 223 in Grundlehren Math. Wiss. Springer, Berlin, 1976.

[2] R. E. Edwards, The stability of weighted Lebesgue spaces. *Trans. Amer. Math. Soc.*, 93 (1959), 369-394.

[3] H. G. Feichtinger, Weighted L^p-spaces and the canonical mapping $T_H : L^1(G) \to L^1(G/H)$. *Boll. Un. Mat. Ital.*, 15 (3) (1979), 989–999.

[4] G. B. Folland, *Real Analysis. Modern Techniques and their Applications. 2nd ed.* Wiley, New York, NY, 1999.

[5] G. I. Gaudry, Multipliers of weighted Lebesgue and measure spaces. *Pro. Lond. Math.Soc.*, 3 (19) (1969), 327-340.

[6] K. Gröchenig, Weight functions in time-frequency analysis. *In L. Rodino and et al., editors, Pseudodifferential Operators: Partial Equations and Time Frequency Analysis,* 52 (2007), 343-366.

[7] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, Oxford, 2000.

[8] E. M. Stein, Interpolation of Linear Operators. *Transactions of the American Mathematical Society*, 83 (2) (1956), 482-492.

Fatemeh Esmaeelzadeh
Assistant Professor of Mathematics
Department of Mathematics
Bojnourd Branch, Islamic Azad University
Bojnourd, Iran
E-mail: esmaeelzadeh@boujnourdiau.ac.ir