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Evaluation and Ranking of Rail Freight and Passenger Transportation in Same Asian Countries with New Method in Data Envelopment Analysis

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Abstract. The evaluation of organizational performance can have a significant impact on the activities of the organization. One of the most useful applications for performance assessment is the use of Data Envelopment Analysis (DEA). DEA is a mathematical programming method that measures the relative efficiency of organizational units having different inputs and outputs. The inability to rank efficient units is one of the major weaknesses of traditional DEA methods. Different methods for ranking efficient units have been proposed by researchers. In this paper, we present a method for ranking all DMUs based on sound supporting hyperplanes. Strong Hyperplanes of production possibility set (PPS) have always been the discerning focus of researchers and managers of organizations as well as calculating the replacement rates of inputs and outputs of hyperplane applications. Moreover, having an explicit form of a production possibility set is beneficial for the managers for decision-making. The proposed method practically, does not undergo common ranking problems, such as, model permeability, small data instability, inability to rank the non-extreme units and with no false rankings. Since rail transport plays an important role in the economic

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development of a country, many researchers have focused their attention on measuring efficiency and ranking in the rail transport industry. As an applied project, we shall assess rail and passenger transport in some Asian countries. The evaluation is based on data from the International Rail Union (UIC) in 2016.

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1. Introduction

Data envelopment analysis (DEA) is a methodology for assessing the performances of a group of decision-making units (DMUs) that utilize multiple inputs to produce multiple outputs. DEA, which was originally presented by Charnes et al. [1], is a well-known mathematical programming tool for evaluating the relative efficiency of a set of comparable processing decision-making units (DMUs). DEA successfully divides the units into two categories: efficient DMUs and inefficient DMUs. Unlike the inefficient DMUs, the efficient ones cannot be ranked based on their efficiencies, because they all secure the efficiency score equating to one.

However, it is not reasonable to claim that the efficient DMUs have the same performance in actual practice. Now, the question arises as to how to rank the efficient DMUs? To address this question, different methods have been developed to achieve complete ranking of the said. So, one of the interesting subjects in research is to discriminate between the efficient DMUs. Hence, the researchers proposed some methods to distinguish the efficient units. This concept is known as ranking efficient units in DEA. There are many ranking methods and each of them has some advantages and drawbacks in this arena. To review the ranking methods refer to Adler et al. [2] and Hosseinzadeh et al. [3].

In the following, we summarized some of renowned methods for ranking DMUs.

Charnes et al. [4], tallied the number of times that an efficient DMU is accounted as a benchmark unit for other DMUs, and used it to rank the units. As a reference set for a DMU is not found easily, thereby, a model in this relative is not an applicable method. Charnes et al. [5], proposed another method to find the benchmark DMUs. They modified the outputs of units and then they evaluated as to the manner that the efficiency score of DMUs altered. However, they did not perceive as to how it was to be achieved. A super-efficient approach is another method pioneered by Anderson and Peterson [6] (AP model). In their method, the corresponding column to the DMU under evaluation is omitted from the technological matrix. Later, Mehrabian et al. (MAJ) [7] has modified the AP model. In some circumstances, the mentioned models may be infeasible and in particular, the AP model may be unstable because the extreme sensitivity could occur by small variations in the data, when certain units have trivial values in some inputs. Saati et al. [8] have modified the MAJ model and solved its infeasibility.

Subsequently, Jahanshahloo et al. [9] changed the type of data normalization in order to attain a much better result. So as to eliminate the snags of the AP and MAJ models, some authors have used specific norms. For instance, Jahanshahloo et al [10] has applied the norm for ranking the efficient units. Amirteimoori et al. [11] have employed the norm to seek the gap between the evaluated efficient units and the new PPS. Jahanshahloo et al [12] have used the gradient line and ellipsoid norms, in order to rank the efficient units. Tone [13] and [14] has utilized the SBM model in this approach. Sexton et al. [15] proposed the cross-efficiency method. In cross-efficiency assessments, each DMU is self and peer evaluated. Each unit specifically determines a set of weights in the traditional DEA model, resulting in n sets of weights. Then, each DMU is evaluated by the n sets of weights obtaining n efficiency scores. The cross-efficiency of each unit is the average of the n efficiency scores.

Although, cross-efficiency evaluation has been extensively applied in various cases, but there is a factor which probably reduces the efficacy of the cross-efficiency evaluation method. This aspect is that the cross-efficiency scores may not be exclusive due to the presence of alternative optimal weights. As a result, it is suggested that secondary goals are introduced in cross-efficiency evaluation. For more studies in relative secondary goal models, see Doyle and Green [16], Liang et al. [17], Wang and Chin [18], Dotoli et al. [19], Wu et al. [20], Ma et al. [21], Jahanshahloo et al. [22] and Wu et al. [23].

Although secondary goal models were suggested to solve the problem of the cross-efficiency evaluation, the existing secondary goal models have some shortcomings in the literature. Note that, none of the secondary goal models in the literature guarantees that the optimal weights are unique. Hence, the problem of the alternative optimal solutions present is not solved entirely. This is the focal drawback of secondary goal models. Though, most of the existing secondary goal models in literature solve $n(n\text{-}1)$ model are to obtain the rank of units, Thereby, if n is a large integer, then the number of models that must be solved is extremely copious, thence, the computational complexity is very high and this is yet another encumbrance of secondary goal models. All ranking methods evaluate units from a particular perspective and each of them has its advantages and hindrances in comparison to others. Therefore, none of the methods has superiority over the others. In this paper, we present a method for ranking all DMUs based on strong supporting hyperplanes. As rail, transport plays an important role in the economic development of a country; as a case study we will evaluate and rank the rail freight and passenger transportation in Asian countries. The comparison is based on data from the International Union of Railways members in 2016. Section 2, consists of two sub-sections, where first the basic definitions and the required theorems are introduced and proved, because the proposed model is basically designed on the super efficiency approach. Thus, first model of super efficiency (AP) as the basic model in this field is discussed and presented in the second section. In Section 3, the proposed method is given. In the next section, with help of the proved theorems in the proposed model and along with a summary of the proposed method observed in Section 4, a numerical example is set. The conclusion is drawn in Section 6.

2. Preliminaries and Basic Ranking Methods

In this section, we describe some of the basic DEA methods and the main ranking method with their advantages and drawbacks.

Let a set consist of ?? homogeneous decision-making units to be evaluated. Assume that each of these units uses m inputs $x_{ij}(i = 1, ..., m))$ to produceproduce s outputs $y_{rj}(r = 1, ..., s)$. Moreover, $X_j \in R^m$ and $Y_j \in R^s$ are considered as non-negative vectors. We define the set of production possibility as ?? = {(??,Y) | ?? can produce ??}. One of the most demonstrative DEA models for evaluating the relative efficiency of a set of DMUs is the BCC model, proposed by Banker et al. [24]. The production possibility set (PPS) of BCC model can be defined as follows:

$$T = \left\{ (x, y) \middle| \sum_{j\in J} \lambda_j x_j \leqslant x, \sum_{j\in J} \lambda_j y_j \geqslant y, \sum_{j\in J} \lambda_j = 1 \quad \lambda_j \geqslant 0 \ \text{ j} \in J \right\}$$

In which x_j and y_j are vectors of input and output of DMU_j , respectively where J= $\{j \mid j = 1, \ldots, n\}$. To rank DMU_P $p \in \{1, ..., n\}$ first it is omitted from the observation there for
T_v and T'_v is defined as

$$T_v{}' = \left\{ (x, y) \middle| \sum_{j\neq p} \lambda_j x_j \leqslant x \,, \sum_{j\neq p} \lambda_j y_j \geqslant y, \ \sum_{j\neq p} \lambda_j = 1, \lambda_j \geqslant 0, j = 1, ..., n \right\}$$

always $T'_v \subseteq T_v$.

The input-oriented BCC model, corresponds to DMU_p , $p \in J$ is given by

$$
\begin{array}{ll}
\bar{\min}\ \theta - \varepsilon\,(1s^- + 1s^+) & \\
subject\,to: & \\
\sum_{j=1}^{n} \lambda_j x_{ij} + s_i{}^- = \theta x_{ip} & i = 1, ..., m \\
\sum_{j=1}^{n} \lambda_j y_{rj} - s_r{}^+ = y_{rp} & r = 1, ..., \mathrm{s} \\
\sum_{j=1}^{n} \lambda_j = 1 & \\
\lambda_j \geqslant 0 & j \in J \\
s_i{}^- \geqslant 0 & i = 1, ..., m \\
s_r{}^+ \geqslant 0 & \mathrm{r} = 1, ..., \mathrm{s}
\end{array}
\tag{1}
$$

Where ε is non-Archimedean small and positive number in model (1) $s_i{}^-, s_r{}^+, i = 1, ..., m,\ r = 1, ..., s$ are called slack variables belong to R^+ . Note that $s_i{}^-$ represents the input waste and $s_r{}^+$ represents the output shortfalls. θ and $\lambda_j\ j \in J$, are also real numbers, including $\lambda_j \in R^+$. The model (1) is called the envelopment form. The dual of model (1) is as follows:

$$
\begin{array}{ll}
\max\ \sum_{r=1}^{s} u_r y_{rp} + u_o & \\
subject\,to: & \\
\sum_{r=1}^{s} \mathrm{u_r y}_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leqslant 0 & \mathrm{j} = 1, ..., \mathrm{n} \\
\sum_{i=1}^{m} \mathrm{v_i x}_{ip} = 1 & \\
\mathrm{u_r} \geqslant \varepsilon & \mathrm{r} = 1, ..., \mathrm{s} \\
\mathrm{v_i} \geqslant \varepsilon & \mathrm{i} = 1, ..., \mathrm{m} \\
\mathrm{u}_o \text{ free} &
\end{array}
\tag{2}
$$

If $(u*, v*, u_o*)$ is an optimal solution of model (2) then $u * y - v * x + u_o* = 0$ is equation of supporting hyperplane of the PPS.

It is clear that, the evaluated DMU_p is efficient if and only if $\theta^* = 1$ and all slack variables in every optimal solution are zero in model (1). Equivalently DMU_p is efficient if and only if an optimal solution for model (2) such that $(u^*, v^*) > 0$ and $\sum_{r=1}^{s} u_r y_{rp} - u_o = 1$ exists. Then $H = \{(x, y) \mid u^* y - v^* x - u_o = 0\}$ hyperplane is said to be a strong efficient hyperplane when an optimal solution (u^*, v^*, u_o^*) of model (2) which $u^* > 0$ and $v^* > 0$ and $u^* y - u_o^* = 1$ is present. For DMU_p, its reference set E_p, is defined by

$$E_p = \left\{ j \mid \lambda_j^* > 0 \ in\ some\ optimal\ solution\ of\ model\ (1) \right\} \subset \{1, \ldots, n\}$$

References of DMU_p, are efficient as DMU_s, whereas, if, DMU_p is inefficient then, a combination of them dominate DMU_p.

Definition 2.1. $\mathrm{DMU_p}$ *is inefficient; if and only if* $p \notin E_P$.

Definition 2.2. $\mathrm{DMU_p}$ *is extreme efficient if* $E_p = \{p\}$.

Definition 2.3. $\mathrm{DMU_p}$ *is non-extreme efficient if* $p \in E_p$ *and* $|E_p| \geqslant 2$.

In the course of improving various abilities of data envelopment analysis (DEA) models, many investigations have been carried out for ranking decision-making units (DMUs). A variety of papers exist which apply to different ranking methods in relative to a real data set. We describe some of the main ranking methods and their advantages and drawbacks. Definition 2.1 $\mathrm{DMU_p}$ is inefficient; if and only if $p \notin E_P$

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In the course of improving various abilities of data envelopment analysis (DEA) models, many investigations have been carried out for ranking decision-making units (DMUs). A variety of papers exist which apply to different ranking methods in relative to a real data set. We describe some of the main ranking methods and their advantages and drawbacks.

Theorem 2.6. *suppose that the unit*(x_p, y_p) *is efficient, the unit*(x_p, y_p) *is non-extreme efficient, if and only if, there is an optimal solution in model* (1) *such as* (λ^*, θ^*)*in which*$\lambda_p^* = 0$.

Proof. in appendix [1]. □

Theorem 2.7. *The unit*(x_p, y_p) *is extreme efficient if and only if model* (4) *has an optimal solution as* (θ'^*, λ^*)*with* $\theta'^* > 1$ *or model* (4) *is infeasible.*

Proof. in appendix [2]. □

Theorem 2.8. *The unit* (x_p, y_p)*is extreme efficient if and only if*$(x_p, y_p) \notin {T_v}'$.

Proof. in appendix [3]. □

2.1 AP model

Super efficiency models introduced in DEA techniques are based upon the idea of skip one and assess this unit through the remaining units. In this subsection, we are going to review the AP ranking model in DEA. Andersen and Petersen [6] developed a new procedure for ranking efficient units. The methodology enables an extreme efficient DMUp to achieve an efficiency score greater than or equal to one by removing the p-th constraint in the primal formulation. They omitted the efficient DMU from the PPS, and solved CCR model [1] for other units to rank them. The mathematical formulation of model (3) is as follows:

$$
\begin{aligned}
&\max \quad z = \textstyle\sum_{r=1}^{s} u_r y_{ro} \\
&subject\,to: \\
&\textstyle\sum_{i=1}^{m} v_i x_{io} = 1, \\
&\textstyle\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \geqslant 0 \qquad for \;\; j = 1, ..., n \;,\; j \neq o, \\
&u_r \geqslant 0 \qquad\qquad for \quad r = 1, ..., s, \\
&v_i \geqslant 0 \qquad\qquad for \quad i = 1, ..., m.
\end{aligned}
\tag{3}
$$

The dual formulation of the super-efficient model, as seen in model (4), computes the distance between the Pareto frontier, evaluated without DMUp, and the unit itself i.e. for$\{j = 1, \ldots, n, j \neq p\}$.

$$
\begin{aligned}
&\min \quad \theta \\
&subject\,to: \\
&\textstyle\sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j x_{ij} \leqslant \theta x_{ip} \quad for \;\; i = 1, ..., m, \\
&\textstyle\sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j y_{rj} \geqslant y_{rp} \quad for \;\; r = 1, ..., s, \\
&\lambda_j \geqslant 0 \qquad\qquad for \;\; j = 1, ..., n.
\end{aligned}
\tag{4}
$$

However, The AP method has the following problems:

Primarily, Andersen and Petersen refer to the DEA objective function value as a rank score for all units, despite the fact that each unit is evaluated according to different weights. This value in fact explains the proportion of the maximum efficiency score that each unit P attained with its chosen weights in relation to a virtual unit closest to it on the frontier. Furthermore, if we assume that the weights reflect prices, then each unit has different prices for the same set of inputs and outputs within the same organization.

Secondly, the super-efficient methodology can give “specialized” DMUs an excessively high ranking. To avoid this problem, Sueyoshi [25] suggested a method to avoid this problem.

The third problem lies with an infeasibility issue, which if occurs, means that the super-efficient technique cannot provide a complete ranking of all DMUs. Mehrabian et al. [7] suggested a modification to the dual formulation in order to ensure feasibility; we will refer to it later. Note that, the AP model is feasible when we use this model in output oriented form.

Fourthly, in some cases, small changes in the data may change the θ^*immensely, though, of course, this problem does not occur in output oriented form.

In the fifth instance, the AP model does not have any suggestion for ranking non-extreme efficient units. In fact, the super efficiency method cannot rank the non-extreme efficient DMUs.

Researchers have provided various methods for ranking efficient units. In this paper, a method is suggested for ranking all the DMUs based on strong and weak supporting hyperplanes. Strong hyperplanes of production possibility set (PPS) have always been within the focus of researchers and managers of organizations, including the calculations of the replacement rates of inputs and outputs of hyperplane applications. Moreover, having an explicit form of the production possibility set is beneficial for the managers for decision-making purposes. The distance of the under-evaluated DMU and the new efficient frontier indicates the extent to which inputs or outputs can get worse, so that the DMU remains efficient. This is referred to as the stability radius in most studies; the larger the stability radius, the greater the area, where the under-evaluated DMU remains stable. Therefore, a better ranking is expected for it. Hence, the maximum distance of the under-evaluated DMU from the new PPS (after eliminating the DMU), could be a better criterion for ranking the DMUs. The units ranked on the assumption that, the ranking of the extreme efficient units is higher than those of non-extreme efficient units, whilst, the ranking of non-extreme units is superior to that of the weak efficient units and the ranking of the weak efficient units is greater than the ones of inefficient units. The proposed method virtually does not endure the common problems in ranking, including infeasibility of the model, instability for small data, inability to rank the non-extreme units and with no false ranking.

3. The Proposed Method

To rank DMU_P $p \in \{1, ..., n\}$ first it is omitted from the observation there may be two cases as follow: $(x_p, y_p) \notin T'_v$: After elimination of DMU_P in figure 1, the frontier of PPS changes which is showed with a hyphened line.

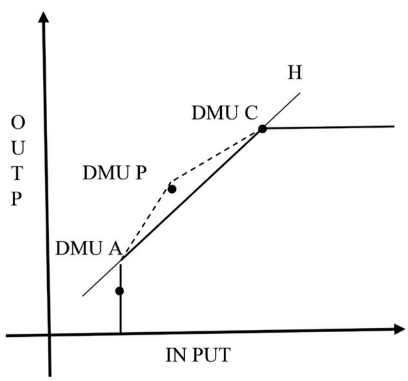

Figure 1. $(x_p, y_p) \notin T'_v$

We want to find a strong supporting hyperplane on a PPS such as $H = \{(x, y) \,|\, u^*y - v^*x + u_o^* = 0\}$, which has the greatest distance from the unit under evaluation. The hyperplane H divides the space R^{m+s} into two half spaces:

$$H^- = \{(x, y) \,|\, u^*y - v^*x + u_o^* \leqslant 0\}$$

$$H^+ = \{(x, y) \,|\, u^*y - v^*x + u_o^* \geqslant 0\}$$

According to the definition of H^- :

$$T_v' \subseteq H^-$$

So the supporting hyperplane H is chosen so that$(x_p, y_p) \notin H^-$, then, $u^*y_P - v^*x_P + u_o^* > 0 \exists\; z > 0 :\; u^*y_P - v^*x_P + u_o^* - z = 0$, Note that if $DMU_P \in H$ then z= 0 , The greater distance from the $DMU_P$to the H, the higher the value of Z.

$(x_p, y_p) \in {T_v}'$: After eliminating DMU_P in figure 2, the frontier of PPS does not alter.

$$\begin{array}{ll}
\max z & \\
subject\,to: & \\
uy_j - vx_j + u_o \leqslant 0 & j = 1, ..., n, \quad j \neq p \\
uy_p + u_o - z = 1 & \\
vx_p = 1 & \\
z \leqslant M.u_r & forall\ r \\
z \leqslant M.v_i & forall\ i \\
u_r \geqslant 0 & forall\ r \\
v_i \geqslant 0 & forall\ i \\
u_o, z\ free & \\
\text{M is positive number} &
\end{array} \tag{5}$$

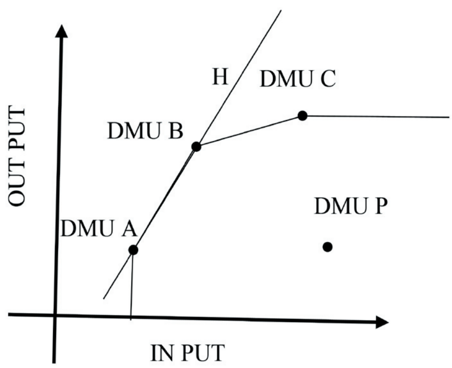

Figure 2. $(x_p, y_p) \in T_v'$

As $T_v' \subseteq H^-$, then $(x_p, y_p) \in H^-$ and it can be written as $u^* y_P - v^* x_P + u_o^* \leqslant 0$ in this case $\exists\, z \leqslant 0; \quad u^* y_P - v^* x_P + u_o^* - z = 0$

In both cases, we recommend the following linear model to find a strong supporting hyperplane which has the greatest distance to the unit under evaluation.

In model (5)$v x_P = 1$, then, $1 +$ z $= u y_P + u_o$. Therefore, maximizing the value Z means maximizing$u y_P + u_o$. Since Z is maximized, thus the large positive number M is added to the problem to achieve positive values U and V if the constraints in the problems are as z $\leqslant M.u_r$, $\forall$r H z $\leqslant M.v_i$ $\forall$i . This makes the achieved hyperplane a strong hyperplane.

Theorem 3.1. model (5) is always feasible.

Proof. in appendix [4])

Theorem 3.2. Assume that (u^*, v^*, u^*, z^*) is the optimal solution for model (5), then $H = \{(x, y)\,|\,u^* y - v^* x + u_o^* = 0\}$ supports hyperplane on ${T_v}'$.

Proof. in appendix [5]. □

Theorem 3.3. Assume that (u^*, v^*, u^*, z^*) is the feasible solution for model (5), then $z^* > 0$ if and only if the unit (x_p, y_p) is extreme efficient unit.

Proof. in appendix [6]. □

Theorem 3.4. Assume that $(u^*, v^*, u_o^*, z^*)$is the optimal solution for model (5); if $z^*{=}\,0$ and$(u^*, v^*) > 0$, then the unit (x_P, y_P) is non-extreme efficient.

Proof. in appendix [7]. □

Remark 3.5. If the unit (x_P, y_P) is efficient, then$z^* \geqslant 0$.

Consider a case in which (u^*, v^*, u^*, z^*) is the optimal solution for model (6) and$z^*{<}\,0$. In this case, the unit (x_P, y_P) is either inefficient or weak efficient. In case$z^*{<}\,0$, to distinguish if the unit (x_P, y_P) is either inefficient or weak efficient, model (6) is solved as follows:

$$\begin{aligned}
&\max z_1\\
&\textit{subject to}:\\
&uy_j - vx_j + u_o \leqslant 0 \qquad j = 1, ..., n\ , j \neq p\\
&vx_p - u_o + z_1 = 1\\
&uy_p = 1\\
&z_1 \leqslant M.u_r \qquad r = 1, ..., s\\
&z_1 \leqslant M.v_i \qquad i = 1, ..., m\\
&u_r \geqslant 0 \qquad \forall r\\
&v_i \geqslant 0 \qquad \forall i\\
&u_o, z_1\ free\\
&\text{M is positive number}
\end{aligned} \tag{6}$$

Theorem 3.6. Assume that the unit (x_P, y_P) is weak efficient; if $(u^*, v^*, u^*, z^*)$is the optimal solution for model 5 and $(\hat{u}, \hat{v}, \hat{u}_o, \hat{z}_1)$is the optimal solution for model (6), then z^*= 0 or $\hat{z}_1$= 0 .

Proof. in appendix [8]. □

Theorem 3.7. Assume that $(u^*, v^*, {u_o}^*, z^*)$ is the optimal solution for model (5) and $(\hat{u}, \hat{v}, \hat{u}_o, \hat{z}_1)$is the optimal solution for model (6).

Proof. in appendix [9]. □

1) If$\hat{z}_1 < 0$, $z^* < 0$, then the unit (x_P, y_P) is inefficient.

2) If $(z^* = 0, \hat{z}_1 < 0)$ or$(z^* < 0, \hat{z}_1 = 0)$, then the unit (x_P, y_P) is weak efficient.

4. Summary of the Proposed Method for Ranking All Units

Step 1: model (5) is solved for all the units. The set w_o is defined as follows:

$$w_o = \{j \,|z_j{}^* > 0\}\ ,\ E_o =; \{j \,|z_j{}^* = 0, (u^*, v^*) > 0\}\ ,$$

$$s_1 = \left\{ j \,\middle|\, z_j{}^* = 0, (u^*, v^*)\ \geqslant\ 0 \& \exists i; v_i^* = 0\ or \exists r; u_r^* = 0 \right\}$$

$$s_{2=} \{j \,|z_j{}^* < 0\}$$

Where, w_o is the set of all the extreme efficient units having the ranking $w_o$and E_o is the set of all the non-extreme efficient units. Ifcard $(\mathrm{E}_o) \leqslant 1$, so will be the ranking of the only member of the set E_o in case that it has just one member. Therefore, the ranking of all the extreme and non-extreme units is achieved and ifcard $(\mathrm{E}_o) \geqslant 2$, the second step is performed as follows:

Step 2: Place $k_o = \{1, ..., n\} - w_o$, so the new PPS for the members of the set $k_o$is formed as follows:

$$\begin{array}{ll}
\max \mathrm{z} & \\
subject\,to: & \\
\mathrm{uy}_j - vx_j + u_o \leqslant 0 \quad \mathrm{j} \in k_o\ \&\mathrm{j} \neq \mathrm{q} & \\
\mathrm{vy}_q - z + u_o = 1 & \\
\mathrm{vx}_q = 1 & \\
\mathrm{z} \leqslant M.u_r,\ \forall \mathrm{r} & (7) \\
\mathrm{z} \leqslant M.v_i\ \ \forall \mathrm{i} & \\
u_r \geqslant 0 & \\
\mathrm{v}_i \geqslant 0 & \\
u_o, z_1\ free & \\
\text{M is positive number} &
\end{array}$$

By considering$w_1 = \{j \in E_0\,|z_j{}^* > 0\}$, $\mathrm{E}_1 = \{j \in E_0\,|z_j{}^* = 0, (u^*, v^*) > 0\}$ In this case, $z_j{}^*$will be criteria for ranking of the units which they are members of. It is obvious that the sectional members ofw_1 are non-extreme efficient units which are ranked like this. Now if card $(\mathrm{E}_o) \leqslant 1$, then the ranking of all the efficient units including extreme and non-extreme is gained and if card $(\mathrm{E}_o) \geqslant 2$, the set k_1 is defined as follows:

$$k_1 = \{1, ..., n\} - w_1$$

And the new PPS for the members of $k_1$is formed as follows

$$T_1 = \left\{ \left(\begin{array}{c} x \\ y \end{array} \right) \middle| \sum_{j \in k_1} \lambda_j x_j \leqslant x, \sum_{j \in k_1} \lambda_j y_j \geqslant y, \sum_{j \in k_1} \lambda_j = 1 \quad \lambda_j \geqslant 0\ \ j \in k_1 \right\}$$

Continuing this process, model (5) is solved for members of E_1until we gain the ranking of all the non-extreme efficient units

The l^{th} step of the method: The sets w_l , E_l, k_l are defined as follows:

$$w_l = \{j \in E_{l-1} \,|z_j{}^* > 0\} \quad \mathrm{l} = 1, ..., \mathrm{t}$$
$$\mathrm{E}_l = \{j \in E_{l-1} \,|z_j{}^* = 0, \;\; (u^*, v^*) > 0\} \;\mathrm{l} = 1, ..., \mathrm{t}$$
$$\mathrm{k}_l = k_{l-1} - w_l \quad \mathrm{l} = 1, ..., \mathrm{t}$$

In this case, $z_j{}^* > 0$will be the ranking of the units, which are in the setw_l. T_l is formed on the members of k_l as follows

$$T_l = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \Big| \sum_{j \in k_l} \lambda_j x_j \leqslant x, \sum_{j \in k_l} \lambda_j y_j \geqslant y_P, \sum_{j \in k_l} \lambda_j = 1 \quad \lambda_j \geqslant 0 \;\; j \in k_l \right\}$$

Model (8) is solved for the members of E_l; therefore, if$d \in E_l$, we have

$$\begin{aligned}
&\max \mathrm{z} \\
&subject\, to: \\
&\mathrm{uy}_j - v x_j + u_o \leqslant 0 \quad\;\; \mathrm{j} \in k_l \\
&\mathrm{vy}_l - z + u_o = 1 \\
&\mathrm{vx}_l = 1 \\
&\mathrm{z} \leqslant M.u_r, \;\forall \mathrm{r} \\
&\mathrm{z} \leqslant M.v_i \;\; \forall \mathrm{i} \\
&u_r \geqslant 0 \\
&\mathrm{v}_i \geqslant 0 \\
&\;\; u_o, z_1 \; free \\
&\mathrm{M\ is\ positive\ number}
\end{aligned} \tag{8}$$

Noting the definition of E_l , it can easily be understood that $\mathrm{E}_t \subset \mathrm{E}_{t-1} \subset ... \subset E_1 \subset E_0$ Assume that in the repetition we havecard $(\mathrm{E}_o) \leqslant 1$, then the ranking of the entire extreme and non-extreme units are gained.

$rank\,(w_0) > rank\,(w_1) > ... > rank\,(w_t)$ The set S_1, S_2 are weak efficient or inefficient units. Assume that means $z_p^* = 0, (u^*, v^*) \ngtr 0$ and in this case DMU_P is on the weak frontier and, therefore, it is weak efficient; and if $p \in s_2$ means$z_P{}^* < 0$, then model (a) –which is mentioned in Appendix- is solved for the unit(x_P, y_P), and if$z'_P{}^* < 0$, the unit (x_P, y_P) is inefficient and if$z'_P{}^* = 0$, the unit (x_P, y_P) is weak efficient. The ranking criterion for the weak efficient units is in this manner that if $z_P{}^* = 0$ then $z'_P{}^* < 0$ is the ranking criterion and if $z'_P{}^* = 0$ then $z_P{}^* < 0$ will be a criterion for ranking. In addition, we consider this issue that$rank\,(S_1) > rank\,(S_2)$.

Given that the modeling is designed in T_v which has at least one extreme efficient unit in the poorest case; and since we know that, at least one strong supporting hyperplane passes through each extreme efficient unit, then, it can be concluded that T_v has one strong supporting hyperplane in the minimum. But it is probable that the production possibility set does not have a strong defining hyperplane. Any extreme efficient DMU, "n" linear programing problems must be solved to rank the "n" decision making units, if it is not present in the proposed model. But since, the modeling is formed in T_v, "n-1" extreme efficient DMUs or inefficient DMUs may exist in the worst case and the model must solved "2n-1" times.

4.1 Numerical example

To illustrate the algorithm process in this subsection, we describe a small numerical example and then apply the algorithm to real data. Consider 13 DMUs with one input and one output. Table 1 shows 13 decision-making units with one input and one output.

Table 1: Data of 13 DMUs

DMU	A	B	C	D	E	F	G	H	I	J	K	M	N
In put	1	1	2	3	4	5	6	8	9	12	13	7	9
Out put	1	2	4	6	8	9	10	12	13	13	14	3	6

The PPS for the numerical example is shown in figure 4.
The optimal solution of model (5) for the DMUs is show in Table 2.

Table 2: The optimal solution of model (5)

	A	B	C	D	E	F	G	H	I	J	K	M	N
Z^*	0	0.33	0	0	0.08	0	0	0	0.11	-0.1	-0.3	-0.7	-0.1
V^*	1	1	0.5	0.33	0.25	0.2	0.167	0.143	0.11	0.1	0.07	0.143	0.111
U^*	0	0.33	0.25	0.167	0.167	0.2	0.167	0.143	0.167	0.1	0.07	0.071	0.056
$U_O{}^*$	1	0.66	0	0	-0.25	-0.8	-0.66	-0.57	-1.1	-0.4	-0.3	0	0

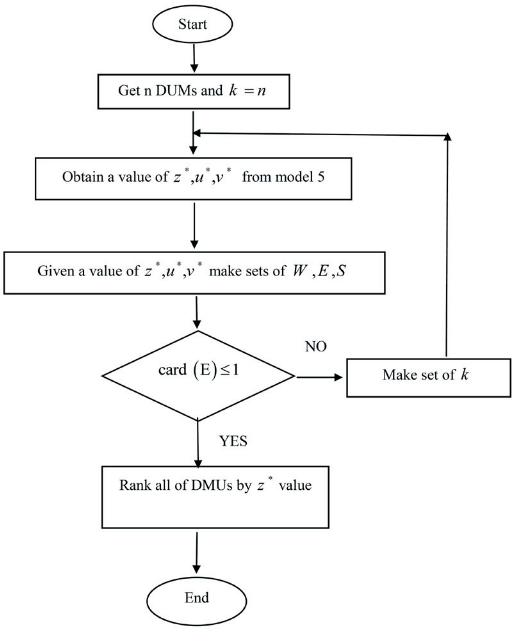

Figure 3. PPS of numerical example

The set w_o , E_o,S_1,S_2 are defined as follows:

$w_o = \{j\,|z_j{}^* > 0\} = \{B, E, I\}$

$\mathrm{E}_o =; \{j\,|z_j{}^* = 0, (u^*, v^*) > 0\} = \{C, D, F, G, H\}$

$$s_1 = \left\{ j \,\middle|\, z_j{}^* = 0, (u^*, v^*) \;\geqslant\; 0\&\exists i; v_i^* = 0 \; or \exists r; u_r^* = 0 \right\} = \{A\}$$

$s_{2=} \{j\,|z_j{}^* < 0\} = \{J, K, M, N\}$

The ranking of the units B, E, I are specified to be $z_B{}^* = 0.33$ $z_E{}^* = 0.08$ $z_I{}^* = 0.11$ respectively. Therefore, the units B, E, I are extreme efficient DMUs. Since the implementation of the proposed approach is the same for all units conceptually, we illustrate the proposed process for DMU_B in the Fig. 5. It is important to note that for the other DMUs, the ranking process can also be described. After the elimination of DMU_B, we have illustrated the new frontier in Fig. 5. The PPS supporting hyperplane from which DMU_B has the maximum distance, is displayed with a hyphened line.

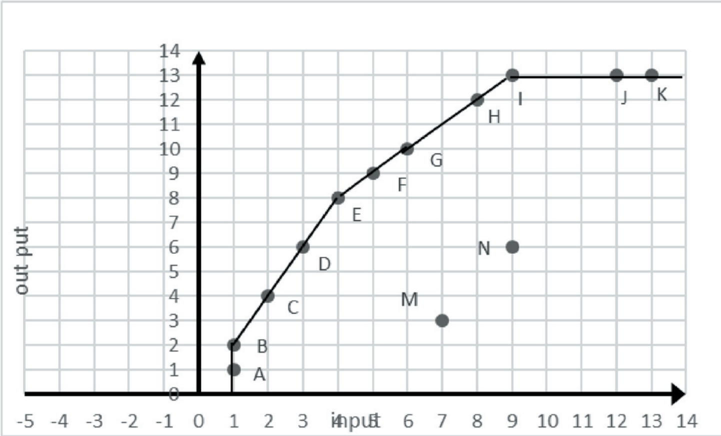

Figure 4. PPS of numerical example

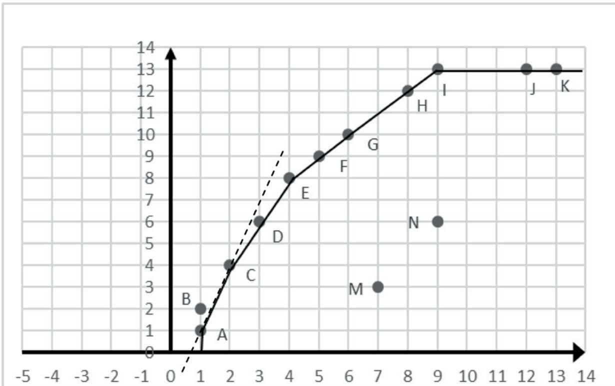

Figure 5. PPS of numerical example after omitted DMUB

The units C, D, F, G and H are non-extreme efficient DMUs. The set k_o is formed as follows:

$$k_o = \{A, B, C, D, E, F, G, H, I, J, K, M, N\} - W_0 = \{A, C, D, G, H, J, K, M, N\}$$

As $card\,(E_0) = 5$, so PPS is formed for the units in the set k_o as follows. Then, model (5) is solved again for the members of E_0 in the PPS. The new PPS for members of k_o and the optimal solution of model (5) for the members of E_0in the new PPS is explained as follows:

Table 3: The optimal solution of model (5)

	C	D	F	G	H
Z^*	0.1	0.066	0.05	0	0.047
V^*	0.5	0.33	0.2	0.167	0.143
U^*	0.2	0.2	0.15	0.167	0.19
$U_O{}^*$	0.3	-0.13	-0.3	-0.66	-1

$$w_1 = \{j \in E\,|z^* > 0\} = \{C, D, F, H\}$$
$$\mathrm{E}_1 = \{j \in E_0\,|z^* = 0, (U^*, V^*) > 0\} = \{G\}$$

The optimal value of $Z^*$for the members of $w_1$set is as follows: $Z^*_C = 0.1 \quad Z^*_D = 0.066 \quad Z^*_F = 0.05 \quad Z^*_H = 0.047$ So ranking of the members of w_1 set is as follows:

$$R_C > R_D > R_F > R_H$$

In addition, as$card\,(\{E_1\}) = 1$, then the ranking of the unit G is also gained$Z^*_G = 0$. As a result, the ranking of the extreme and non-extreme units is gained as follows:

$$R(B) > R(I) > R(E) > R(C) > R(D) > R(F) > R(H) > R(G)$$

Ranking the weak efficient and inefficient units: The optimal value of the objective function for members of $S_1 \cup S_2$ in the set is as follows:

$$Z^*_A = 0$$
$$Z^*_J = -0.1 < 0$$
$$Z^*_K = -0.3 < 0$$
$$Z^*_M = -0.7 < 0$$
$$Z^*_N = -0.66 < 0$$

Since$Z^*_A = 0$, then it is located on a weak hyperplane and it is weak efficient and incapable of reducing inputs. Therefore, it has a higher ranking among the members of the set$S_1 \cup S_2$. Now, to determine the ranking of the units J, K, M, and N; model 6 is solved for the members of$S_1 \cup S_2$. The optimal solution of model 6 for the members of $S_1 \cup S_2$ is as follows:

$$Z^{*\prime}_A = -1 \quad Z^{*\prime}_J = 0 \quad Z^{*\prime}_K = 0 \quad Z^{*\prime}_M = -2.67 \quad Z^{*\prime}_N = -1.17 < 0$$

This shows that the units J and K are weak efficient and the units M and N are inefficient. Since priority is given to reducing the inputs, ranking of the units in the set$S_1 \cup S_2$ is as follows:

$$R(A) > R(J) > R(K) > R(N) > R(M)$$

Therefore, ranking all the units will be as follows:

$$R_B > R_I > R_E > R_C > R_D > R_F > R_H > R_G > R_A > R_J > R_K > R_N > R_M$$

5. Analysis of Indicators of Productivity Evaluation

In this section, we shall evaluate and rank rail freight and passenger transportation in Asian countries. The statistical community of recent studies includes all railways in the world. Information and statistics of 60 UIC (International Railway Statistics, Union International des Chemins defer) member countries are gathered and used. The data and information have been extracted from the statistical yearbook of the International Union of railways till 2016. The database includes information on population and area of the country, the length of rail lines, number of freight and passenger wagons, the number of locomotives and the amount of freight and passenger transported in the countries. As an example, information related to the Asian countries in 2016, are shown in Table 4. [http://www.uic.org].

Information on the entire countries in Asia is rendered in the Table 4. In this article, we evaluate the performance and ranking of the populous Asian countries. It must also be noted that, information in relative to some countries was not available in the abovementioned Table and they are signified with (na). For accurate evaluation, these countries have not been investigated. The inputs and outputs in this study are presented in Table 5.

The countries evaluated in this paper, along with their inputs and outputs, are presented in Table 6:

Table 4: Information about the countries in Asia.

Country	Area (1000 km2)	Population (million)	The Total length of lines (km)	Two-lane lines (km)	The number of locomotives	Passengers tansported (million)	Passenger-km (million)	Tonnage of goods tansported (million)	Ton-km (million)	Average staff strength (thousands)
China	9,563	1,370.84	67,092	34,777	19,988	1,544.36	723,006	2,294.10	1,980,061	2004
India	3,287	1,311.05	66,030	na	10,730	8,224	1,147,190	1,095.26	681,696	1326
Indonesia	1,911	257.56	5,279	486	357	198	18,510	20.44	5,452	27
Pakistan	796	188.93	9,255	2,866	452	52.95	20,288	3.60	3,301	32.98
Bangladesh	148	161.00	2,835	na	286	65.60	7,305	2.71	710	27.97
Japan	378	126.82	19,204	7,593	194	9,090.74	260,192	31.00	20,255	128.89
Vietnam	331	91.71	2,480	0	296	na	4,233	na	na	28.36
Iran	1,745	79.11	8,576	1,900	915	24.45	14,938	35.65	25,014	9.02
Thailand	513	67.96	5,327	346	265	44	7,504	10.86	2,455	26.32
South Korea	100	50.63	3,944	2,342	492	134.44	23,071	37.38	9,564	27.85
Iraq	435	35.87	2,138	na	na	na	99	1.00	249	8.8
Saudi Arabia	2,150	31.54	1,412	0	na	0.99	297	4.03	1,852	1.59
Uzbekistan	447	31.19	4,192	na	na	17.12	3,437	82.39	22,686	20.94
Malaysia	331	30.33	2,250	350	92	40.20	3,293	11.83	3,071	5.45
China-Taiwan	36	23.31	1,410	1,069	281	276.75	19,757	10.91	634	13.18
Syria	185	22.35	2,139	0	na	3.59	1,857	8.51	2,206	12.57
Kazakhstan	2,725	17.51	14,758	3,759	1,892	21	16,595	280.00	223,583	78.81
Azerbaijan	87	9.65	2,068	803	326	1.89	494	17.09	6,210	16.15
Tajikistan	143	8.48	621	na	na	0.55	24	8.41	554	3.06
Israel	22	8.35	1,340	na	na	52.81	2,608	7.50	1,155	3.3
Jordan	89	6.74	509	na	na	0.04	503	2.13	344	6.5
Kyrgyzstan	200	5.93	417	na	na	0.55	75	6.91	922	2.94
Turkmenistan	488	5.37	3,115	na	na	6.47	1,811	26.84	11,992	14.39
Georgia	70	3.72	1,491	na	50	2.73	549	16.68	4,987	9.068
Armenia	30	3.02	703	8	61	0.84	50	1.64	345	3.43
Mongolia	1,564	2.96	1,810	na	na	3.31	1,194	19.15	11,463.00	14.68

Table 5: The inputs and outputs

ID	Variables	Type of Variables
1	The Total length of lines (km)	input
2	The number of locomotives	input
3	Passenger - km (million)	output
4	Ton - km (million)	output

Table 6: The inputs and outputs for 7 countries

Country	The Total length of lines (km)	The number of locomotives	Passenger - km (million)	Ton - km (million)
China	67,092	19,988	723,006	1,980,061
Indonesia	5,279	357	18,510	5,452
Pakistan	9,255	452	20,288	3,301
Japan	19,204	194	260,192	20,255
Iran	8,576	915	14,938	25,014
Thailand	5,327	265	7,504	2,455
South Korea	3,944	492	23,071	9,564

The optimal sum of the model 5 with the rank of each country is shown in Table 7.

Table 7: The optimal sum of the model 6 with rankings

Country	z^*	rank
China	0.00114	2
Indonesia	-0.0003	5
Pakistan	-0.372	6
Japan	0.002	1
Iran	-0.38	7
Thailand	0.00058	4
South Korea	0.00091	3

The assessment results show that East Asian countries have a particular interest in rail transport and the optimal use of this industry. Other countries, especially developing countries in Asia, can develop the rail industry by simulating East Asian countries and transferring advanced technologies in the coming years. The development of the rail industry can play an important role in improving environmental conditions and reducing pollution. Given the safety of rail travel, the use of this industry can play an imperative role in reducing mortality in all countries. An indicator is a variable, which is used to measure the status and efficiency of the system and should be comparable. In this study, several indicators are defined, whereby; the status and efficiency of rail freight transport could be compared between various countries. Most of the variables of the database cannot be considered as an indicator, but significant and comparable indices could be created with their no dimensionality and application of algebraic relationships between several variables. The definition of seven indicators made by the database variables and the manner of their calculation are shown in Table 8. In this Table, the indicators are divided into four general parts including the development of railways, navigation development, exploitation of rail lines, and the utilization of rail fleet.

Table 8: Sample of information about countries in Asia

Types of Indicators	**Indicators**	**Symbols and Formulas**
'Railroads development indicators	Density of the length of lines in the area	$I_1 = \frac{The\,total\,lenght\,of\,rail\,road(km)}{Area(1000Km^2)}$
	The length of rail road per capita	$I_2 = \frac{The\,total\,lenght\,of\,rail\,road(km)}{Pupulation(million)}$
Locomotive number	the number of locomotive to the length of rail lines	$I_3 = \frac{The\,number\,of\,locomotive}{The\,total\,lenght\,of\,lines(Km)}$
Efficiency of lines	Annual ton-Km per a rail road-Km	$I_4 = \frac{Ton-Km(million)}{The\,total\,lenght\,of\,lines(Km)}$
	Annual Passenger-Km per a rail road-Km	$I_5 = \frac{Passenger-Km(million)}{The\,total\,lenght\,of\,lines(Km)}$
Efficiency of locomotive	Annual Passenger-Km per The number of locomotives	$I_6 = \frac{Passenger-Km(million)}{The\ number\ of\ locomotives}$
	Annual ton-Km per The number of locomotives	$I_7 = \frac{ton-Km(million)}{The\ number\ of\ locomotives}$

The important indicators commonly used to measure rail performance of different countries are shown in Table 8 and the following diagrams (Figure 6) present the performance of Asian countries per indicator, which are useful for comparing the ranking of the proposed model with them.

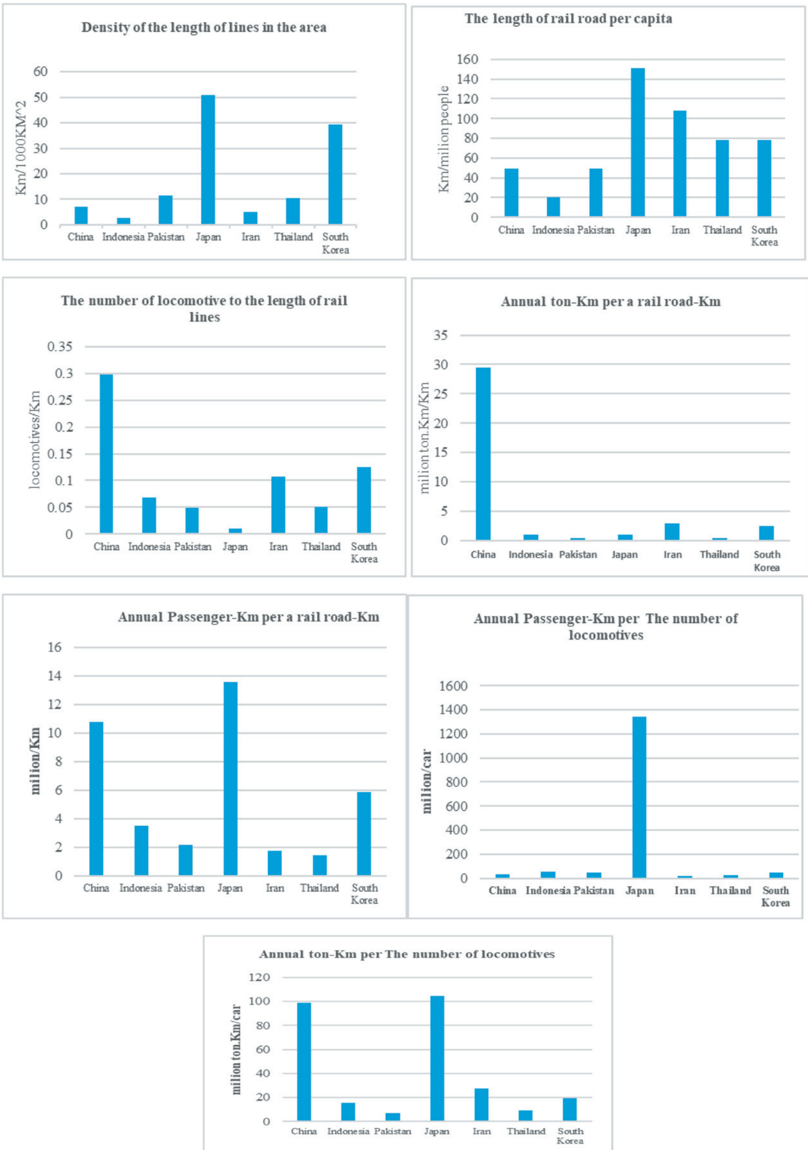

Figure 6. The status countries of Asia by comparing indicators of development and fleet

In general, China and Japan have the highest values based on the indexes. In Japan, in particular, rail infrastructure has a high density and spread. In addition, in China, the fleet used has a significant difference with other countries. So two countries rank first and second in the annual passenger and freight turnover, followed by Korea, which ranked third, with an average fleet.

6. Conclusion

Inability in ranking the efficient units is one of the major weaknesses of traditional methods of Data Envelopment Analysis (DEA). Researchers have provided various methods for ranking efficient units. In this paper, we suggested a method for ranking all the DMUs, which are based on strong and weak supporting hyperplanes. In addition, the decision-making units, which are ranked on the assumption that, the ranking of the extreme efficient units is higher than those of the non-extreme efficient units, the ranking of non-extreme units is higher than the one of weak efficient units and that the ranking of the weak efficient units is higher than the ones of the inefficient units. The proposed method virtually does not endure the common problems of ranking, including the infeasibility of the model, instability for small data, inability to rank the non-extreme units and false ranking. Applying the proposed method to the common set of weight (CSW), as well as developing the proposed method for conditions where strong defining hyperplanes can be suggested for future research. As a case study, we implemented the presented model on railway transportation data for some Asian countries. By presenting performance indicators, we conclude that, the efficiency of rail transportation in the East Asian States supersedes the other countries. As we are aware, in the mentioned countries, which attain higher rankings such as, Japan, China and South Korea railroad is a major means of transport, signifying the fact that, their rail networks are among the busiest in the world, which have expanded with the new high speed and conventional and commuter lines during last decades. Furthermore, there are long-term plans to expand the networks in the future. At times, the privatized networks are highly efficient, requiring few subsidies and run extremely punctually. Hence, the efficiency of rail transportation in East Asia is more substantial.

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Appendix

Proof:

[**1**] Proof Theorem 1:

Suppose that it is non-extreme efficient, thus $\bar{\lambda} = e_p, \bar{\theta} = 1$ is the optimal solution for model (1) and since(x_p, y_p) is non-extreme efficient it has another optimal solution such as $(\lambda^*, \theta^*)$where $\lambda^* \neq \bar{\lambda}$. If $\lambda_p^* = 0$ is the proved assertion, or otherwise $0 < \lambda_p^* < 1$ and so

$$\sum_{j\neq P} \lambda_j^* x_j + \lambda_p^* \mathrm{x}_p = \mathrm{x}_p \quad , \quad \sum_{j\neq P} \lambda_j^* y_j + \lambda_p^* y_p = \mathrm{y}_p$$

$$\sum_{j\neq P} \lambda_j^* = 1 - \lambda_p^* \quad \Rightarrow \begin{cases} \sum_{j\neq P} \lambda_j^* x_j = (1-\lambda_p^*)\mathrm{x}_p \Rightarrow \sum_{j\neq P} \frac{\lambda_j^*}{1-\lambda_p^*} x_j = \mathrm{x}_p \\ \sum_{j\neq P} \lambda_j^* y_p = (1-\lambda_p^*) y_p \Rightarrow \sum_{j\neq P} \frac{\lambda_j^*}{1-\lambda_p^*} y_j = y_p \end{cases}$$

We place$\hat{\lambda}_j = \sum_{j\neq p} \frac{\lambda_j^*}{1-\lambda_p^*} \geqslant 0$, j $\neq p$ so we have

$$\sum_{j\neq P} \hat{\lambda}_j x_j = \mathrm{x}_p \quad \sum_{j\neq P} \hat{\lambda}_j y_j = \mathrm{y}_p \ , \quad \sum_{j\neq p} \hat{\lambda}_j = \sum_{j\neq P} \frac{\lambda_j^*}{1-\lambda_p^*} = \frac{1-\lambda_P^*}{1-\lambda_P^*} = 1$$

Now by choosing $\tilde{\theta} = 1$ and $\tilde{\lambda} = \left(\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_{p-1}, 0, \hat{\lambda}_{p+1}, .., \hat{\lambda}_n\right)$ and $s^- = 0, s^+ = 0$ it is concluded that

$\left(\tilde{\theta}, \tilde{\lambda}, s^-, s^+\right)$is the optimal solution for model(1) and $\tilde{\lambda}_p = 0$.

Proof by contraposition: Let us assume that model (1) has an optimal solution like (λ^*, θ^*) in which $\lambda_p^* = 0$. Based on the assumption that, as(x_p, y_p) is efficient, then $\theta^* = 1$; this shows that the unit(x_p, y_p) of the reference set is$R_1 = \left\{DMU_j \left| \lambda_j^* > 0 \right.\right\}$. As $\lambda_p{}^* = 0$, then$DMU_p \notin R_1$. We place$\bar{\lambda} = e_p, \bar{\theta} = 1$, this shows that $\left(\bar{\theta}, \bar{\lambda}\right)$ is another optimal solution for model (1).Thus, the unit (x_p, y_p)has another reference set as $R_2 = \{DMU_P\}$. Therefore, according to the definition 3, the unit $(x_p, y_p)$is non-extreme efficient.

[**2**] Proof Theorem 2:

Proof: In assuming that the unit (x_p, y_p) is extreme efficient. Proof by contradiction: Assume that model (4) is feasible and its optimal solution is (θ'^*, λ^*) and$\theta'^* \leqslant 1$. One of the following cases would be seen

i) $\theta'^* < 1$ So we place$\bar{\lambda} = (\lambda^*{}_1, \lambda^*{}_2, ..., \lambda^*{}_{P-1}, 0, \lambda^*{}_{P+1}, ..., \lambda^*{}_n)and\bar{\theta} = \theta'^*$, then$\left(\bar{\theta}, \bar{\lambda}\right)$

the solution is feasible for model 1 and as $\bar{\theta} < 1$, it is understood that the unit $(x_p, y_p)$is inefficient which is in contradiction to the proposition.

ii) $\theta'^* = 1$, so we put $\bar{\lambda} = (\lambda^*{}_1, \lambda^*{}_2, ..., \lambda^*{}_{P-1}, 0, \lambda^*{}_{P+1}, ..., \lambda^*{}_n)$ and $\bar{\theta} = \theta'^*$, in this case $\left(\bar{\theta}, \bar{\lambda}\right)$is the optimal solution for model 1 and$\bar{\lambda}_p = 0$. Based on the theorem 1, the unit (x_p, y_p) is non-extreme efficient, which is in contradiction to the proposition.

Proof by contraposition: Now, assume that$(\theta'^*, \lambda^*)$is the optimal solution for model (4) and $\theta'^* > 1$or model (4) is infeasible. We show that the unit (x_p, y_p) is extreme efficient. Proof by contradiction: Assume that the unit $(x_p, y_p)$is not extreme efficient, then there would be two cases; either the unit is inefficient or it is weak efficient or it is non-extreme efficient. Based on model (1), it has an optimal solution like (θ^*, λ^*) where $\lambda_p{}^* = 0$ and$\theta^* \leqslant 1$.
We put $\bar{\lambda} = (\lambda^*{}_1, \lambda^*{}_2, ..., \lambda^*{}_{P-1}, 0, \lambda^*{}_{P+1}, ..., \lambda^*{}_n)$, then $\left(\theta^*, \bar{\lambda}\right)$ is the feasible solution for model (5). Thus, it is in contradiction to the proposition denoting model (4) is infeasible or it is in contradiction to the optimality of θ'^* for model (5). Therefore, the contradictory assumption is nullified and the assertion is proved.

[**3**] Proof Theorem 3:

Proof. assume that the unit $(x_p, y_p)$is extreme efficient. Proof by contradiction: $(x_p, y_p) \in {T_v}'$assume that(θ^*, λ^*) is the optimal solution of model 1 and $\left(\theta'^*, \widehat{\lambda}\right)$is the optimal solution of model 5. Since$(x_p, y_p) \in {T_v}'$it is understood that$T_v \subseteq {T_v}'$ and we have $T_v = {T_v}'$Therefore $\theta^* = \theta'^* = 1$ which is in contradiction to the theorem 2.

Proof by contraposition: Assume that$(x_p, y_p) \notin {T_v}'$we show that $(x_p, y_p)$is extreme-efficient.

Proof by contradiction: if the unit is not extreme efficient, then it is either inefficient or weak efficient or non-extreme efficient, then model 1 has an optimal solution $(\theta^*, \lambda^*)$where $\lambda^*_P = 0$ and $\theta^* \leqslant 1$.
Let $\bar{\lambda} = ({\lambda^*}_1, {\lambda^*}_2, ..., {\lambda^*}_{P-1}, 0, {\lambda^*}_{P+1}, ..., {\lambda^*}_n)$ and$\bar{\theta}' = \theta^*$, then $\left(\bar{\theta}', \bar{\lambda}\right)$is the feasible solution for model (5) and $(x_p, y_p) \in T_v'$ which is in contradiction to the proposition of the theorem.

[**4**] Proof Theorem 4:

Let $\bar{z} = -1$, $\bar{u}_o = 0$, $\bar{u} = 0$ and $\bar{v}_i = \frac{1}{kx_{ip}}$ for $x_{ip} > 0$ and $\bar{v}_i = 0$ for otherwise. where $k = card\left(\{i \,|x_{ip} > 0\}\right)$, then $(\bar{u}, \bar{v}, , \bar{u}_o, \bar{z})$ is the feasible solution for model (5).

[**5**] Proof Theorem 5:
Assume that $(\bar{x}, \bar{y}) \in T_v'$, we show that $u^*\bar{y} - v^*\bar{x} + u_o^* \leqslant 0 (\bar{x}, \bar{y}) \in T_v' \Rightarrow \exists\bar{\lambda} \geqslant 0 \;\; ; \sum_{j\neq P} \bar{\lambda}_j x_j \leqslant \bar{x} \;, \sum_{j\neq P} \bar{\lambda}_j y_j \geqslant \bar{y}, \;\; \sum_{j\neq P} \bar{\lambda}_j = 1$

Then we have $u^*\bar{y} \leqslant u^* \sum_{j\neq p} \bar{\lambda}_j y_j \;\; , v^*\bar{x} \geqslant v^* \sum_{j\neq p} \bar{\lambda}_j x_{ij} \Rightarrow$
$u^*\bar{y} - v^*\bar{x} + u_o^* \leqslant \sum_{j\neq p} \bar{\lambda}_j \left(u^* y_j - v^* x_{ij} + u_o^*\right) \leqslant 0$
Now, we show that$H \bigcap T_v' \neq \emptyset$, $\exists q \;\; q \neq p; \quad u^* y_q - v^* x_q + u_o^* = 0$ Proof by contradiction, Assume that for each index such as j $\forall j \quad u^* y_j - v^* x_j + u_o^* < 0 \quad j \neq p$. $\bar{u}$ is defined as$\bar{u} = (u_1^* + \Delta, u_2^* + \Delta, ..., u_s^* + \Delta)$ and the value of $\Delta$is found in a way that $\bar{u}$ is feasible. To reach this goal:

$$
\begin{aligned}
&\bar{u} y_j - v^* x_j + u_o^* \leqslant 0 \quad j = 1, .., n \;, j \neq p \\
\Rightarrow \;& \bar{u} y_j + \Delta \textstyle\sum_{r=1}^{s} y_{rj} - v^* x_j + u_o^* \leqslant 0 \quad j = 1, .., n \;, j \neq p \\
\Rightarrow \;& \Delta \textstyle\sum_{r=1}^{s} y_{rj} \leqslant -\left(u^* y_j - v^* x_j + u_o^*\right) \quad j = 1, .., n \;, j \neq p \\
\Rightarrow \;& \Delta \leqslant -\frac{1}{\sum_{r=1}^{s} y_{rj}} \left(u^* y_j - v^* x_j + u_o^*\right) \quad j = 1, .., n \;, j \neq p
\end{aligned}
$$

$$\Delta' = \min\left\{ \frac{-1}{\sum_{r=1}^{s} y_{rj}} \left(\mathrm{u}^* y_j - v^* x_j + u'_o\right) \mid \ \mathrm{j} = 1, ..., \mathrm{n} \ , \ \mathrm{j} \neq \mathrm{p} \ \right\}$$

As $\mathrm{u}^* y_j - v^* x_j + u_o^* < 0 \quad \mathrm{j} \neq \mathrm{p}$

so $\Delta' > 0$ However, it can be $\mathrm{z}^* \leqslant \ M.u_r^* \leqslant M.\bar{u}_r \quad \forall \mathrm{r}\ , \ \mathrm{z}^* \leqslant M.v_i^* \quad \forall \mathrm{i}$ Putting$\bar{z} = \bar{u}y_p + u_o^* - 1$, as M is a very large positive number, so$\bar{\mathrm{z}} \leqslant M.\bar{u}_r$ $\forall$r , $\bar{\mathrm{z}} \leqslant M.v_i^*$ $\forall$i. This shows that $(\bar{u}, v^*, u^*, \bar{z})$ is a feasible solution for model (5). However, $\bar{z} = \bar{u}y_p + u_o^* - 1 = u^*\mathrm{y}_p + \Delta' \sum_{r=1}^{s} y_{rp} + u_o^* - 1 = z^* + \Delta' \sum_{r=1}^{s} y_{rp}$ And as$\Delta' \sum_{r=1}^{s} y_{rp} > 0$, so it can be understood that $\bar{z} > z^*$ which in contradiction to the optimality of $z^*$for model (5). Therefore, the contradiction proposition is nullified and the assertion is proved.

[**6**] Proof Theorem 6:

Assume that $z^* > 0$ as (u^*, v^*, u^*, z^*) is the feasible solution for model (5).

$u^* y_P + u_o^* - z^* = 1 \ \& \ v^* x_P = 1 \quad \Rightarrow \ u^* y_P - v^* x_p + u_o^* = z^* > 0$ On one hand it shows that, $(x_p, y_p) \notin H^-$ and on the other hand, $T'_v \subseteq H^-$, thus$(x_p, y_p) \notin T'_v$. Based on the theorem 3 it is implied that the unit (x_p, y_p) is extreme efficient. It is implied that the unit (x_p, y_p) is an extreme efficient unit in a sufficient condition. Now, we assume that the unit (x_p, y_p) is extreme efficient we show that $z^* > 0$. The duality of model 5 is as follows:

$$\begin{array}{ll}
z^* = \min \ \theta - \phi & \\
\mathrm{s.t} & \\
\sum_{j \neq P} \lambda_j x_j \leqslant \theta x_j - M s_i^- \ \ \mathrm{i} = 1, ...\mathrm{m} & \\
\sum_{j \neq P} \lambda_j y_{rj} \leqslant \phi y_{rp} + M s_r^+ \ \ \mathrm{r} = 1, ..., \mathrm{s} & \\
\sum_{j \neq P} \lambda_j = \phi & \\
\phi + \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ = 1 & \\
\lambda_j \geqslant 0 \qquad \mathrm{j} = 1, ..., \mathrm{n} \ \mathrm{j} \neq \mathrm{p} & \\
\mathrm{s}_i^- \geqslant 0 \qquad \mathrm{i} = 1, ..., \mathrm{m} & \\
\mathrm{s}_r^+ \geqslant 0 \qquad \mathrm{r} = 1, ..., \mathrm{s} &
\end{array} \tag{9}$$

Assume that $(\theta^*, \phi^*, \lambda^*, s^{-*}, s^{+*})$ is the optimal solution for model (a). First, we show that$\theta^* > 0$. Assume that$\theta^* \leqslant 0$, then we have $\sum_{j \neq P} \lambda_j^* x_j + M s_i^+ \leqslant \theta^* x_{ip} \leqslant 0 \quad \Rightarrow \lambda^* = 0 \ \ , s^{-*} = 0$ by the equation$\sum_{j \neq P} \lambda_j^* =$

ϕ^*, it is understood that$\phi^* = 0$; by the in equation $\sum_{j \neq P} \lambda_j y_{rj} \leqslant \phi y_{rp} + M s_r^{+*}$it is also understood thats^{-*}= 0 . This is in contradiction to the equation$\phi^*+\sum_{i=1}^{m} s_i^{-*} + \sum_{r=1}^{s} s_r^{+*} = 1$. Therefore, the contradiction proposition is nullified and the assertion is proved now, we show that$z^* > 0$. Assume that $z^* = \theta^* - \phi^* \leqslant 0$ because$\theta^* > 0$, then$\phi^* > 0$. However, we have:

$$\begin{array}{l} \sum_{j=P} \lambda_j^* x_j \leqslant \sum_{j \neq P} \lambda_j^* x_j + M s_i^{+} \leqslant \theta^* x_{ip} \leqslant \phi^* \mathrm{x}_P \\ \sum_{j \neq P} \lambda_j^* y_{rj} \geqslant \sum_{j \neq P} \lambda_j^* y_{rj} - M s_r^{+*} \geqslant \phi^* y_P \end{array}$$

As$\phi^* > 0$, it can be expressed that:

$$\sum_{j=P} \frac{\lambda_j^* x_j}{\phi^*} \leqslant \mathrm{x}_P \quad \& \sum_{j \neq P} \frac{\lambda_j^* y_{rj}}{\phi^*} \geqslant y_P$$

put$\frac{\lambda_j^*}{\phi^*} = \bar{\lambda}_j$, we have:

$$\sum_{j \neq p} \bar{\lambda}_j = \sum_{j \neq P} \frac{\lambda_j^*}{\phi^*} = \frac{\phi^*}{\phi^*} = 1 \quad , \sum_{j=P} \bar{\lambda}_j x_j \leqslant \mathrm{x}_P \quad \& \sum_{j \neq P} \bar{\lambda}_j y_{rj} \geqslant y_P$$

In addition, this shows that $(x_P, y_P) \in T_v'$ and this is contradiction to the unit (x_P, y_P) being extreme efficient. Therefore, the contradiction proposition is nullified and it is understood that $z^* = \theta^* - \phi^* > 0$

[**7**] Proof Theorem 7:

Let z^*= 0 , $u^* y_P + u_o^* - z^* = 1$ & $v^* x_P = 1 \Rightarrow u^* y_P - v^* x_p + u_o^* = 0$ This shows that the unit (x_P, y_P) is on the supporting hyperplane $H = \{(x, y) \,|\, u^* y - v^* x + u_o^* = 0\}$ and based on the proposition, which is$(u^*, v^*) > 0$, then the unit (x_P, y_P) is efficient. However, the unit (x_P, y_P) is non-extreme efficient because if we consider this unit as extreme efficient, based on the theorem 3 $z^* > 0$ which is in contradiction to the propositionz^*= 0 .

[**8**] Proof Theorem 8:

Based on the proposition of the problem, the unit $(x_P, y_P)$is weak efficient. Therefore, it is not possible to improve all its elements. Thus, we have the two following cases:

1) It is not possible to improve all the elements on the vectorx_P. In this case, $(\theta^*, \lambda^*, s^{-*}, s^{+*})$is the optimal solution for model (1) and (u'^*, v'^*, u'_o) is the optimal solution for model (2), therefore

$$\theta^* = 1 \,, (s^{-*}, s^{+*}) \neq (0,0)$$
$$\theta^* = u^* y_P + u'_o = 1 \,, v' x_P = 1$$
$$u' y_j - v' x_j + u'_o \leqslant 0 \;\; \forall \text{j}$$

Choosing $z' = 0$ and regarding $z' \leqslant M u'_r \; \forall \text{r}$, $z' \leqslant M v'_i \; \forall \text{i}$, it is understood that $(u', v', u'_o, z')$is the feasible solution for model (5). On one hand, as z^*, is the optimal value for model (5), then$z^* \geqslant z' = 0$. On the other hand$z^* = 0$; otherwise, if $z^* > 0$, based on the theorem 8 it is understood that the unit (x_P, y_P) is extreme efficient which is in contradiction to the proposition of the theorem

2) It is not possible to improve all the elements on the vectory_P. (Proof like the part (1))

[**9**] Proof Theorem 9:
If$\hat{z}_1 < 0$, $z^* < 0$, then the unit (x_P, y_P) is inefficient.

Proof. By contradiction, Assume the unit (x_P, y_P) is not inefficient; subsequently, one of the following cases occurs:

1) The unit (x_P, y_P) is efficient, subsequently it is concluded that $z^* \geqslant 0$since (based on the result 1) and this is in contradiction to the problem proposition.

2) The unit (x_P, y_P) is weak efficient, subsequently based on the theorem 10 $z^*= 0$ or $\hat{z}_1= 0$ which is in contradiction to the problem proposition. Thus, the contradiction proposition is nullified and the result is achieved.

If $(z^* = 0, \hat{z}_1 < 0)$ or$(z^* < 0, \hat{z}_1 = 0)$, then the unit (x_P, y_P) is weak efficient.

Proof.

1) If the case $(z^* = 0, \hat{z}_1 < 0)$ occurs $u^* y_P + u^*_o - z^* = 1$, $v^* x_P = 1 \Rightarrow u^* y_P - v^* x_P + u^*_o = 0$ This shows that the unit (x_P, y_P) is on the supporting hyper plane $H = \{(x, y) \,|\, u^* y - v^* x + u^*_o = 0\}$ and as, based on the proposition, $\hat{z}_1 < 0$; then the unit $(x_P, y_P)$is not efficient and this unit is weak efficient.

2) If the case ($z^* < 0$,$\hat{z}_1 = 0$) occurs, $\hat{\mathrm{v}}\mathrm{x}_p - \hat{u}_o + \hat{z}_1 = 1$, $\hat{\mathrm{u}}\mathrm{y}_P = 1 \Rightarrow \hat{\mathrm{u}}\mathrm{y}_p - \hat{v}x_p + \hat{u}_o = 0$. This shows that the unit (x_P, y_P) is on the supporting hyperplane $H = \{(x, y) \,|\, u^*y - v^*x + u^*_o = 0\}$ and as, based on the proposition, $z^* < 0$; then the unit $(x_P, y_P)$is not efficient. As the frontier T_v includes efficient or weak efficient points, then, it is understood that the unit (x_P, y_P) is weak efficient.

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