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Some Relations on Noetherian and Boolean Artinian BL-algebras

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Abstract. In this paper, we derive some new results on Noetherian and Boolean Artinian BL-algebras. We further obtain some relations between local and semilocal BL-algebras and Boolean Artinian BL-algebras.

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1. Introduction

BL-algebras have been invented by Hájek [2] in order to provide an algebraic proof of the completeness theorem of "basic logic" (BL, for short) arising from the continuous triangular norms, familiar in the fuzzy logic frame-work. BL-algebras are the algebraic structures for Hájek [2] basic logic in order to investigate many-valued logic by algebraic

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means. He provided an algebraic counterpart of a propositional logic, (BL), which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic is proposed as the most general many-valued logic with truth values in interval [0, 1] and BL-algebras are the corresponding LindenbaumTarski algebras. The language of propositional Hájek basic logic [2] contains the binary connectives o, $\Rightarrow$ and the constant $\overline{0}$. Axioms of BL are given as:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow w) \Rightarrow (\varphi \Rightarrow w))$;
(A2) $(\varphi o \psi) \Rightarrow \varphi$;
(A3) $(\varphi o \psi) \Rightarrow (\psi o \varphi)$;
(A4) $(\varphi o (\varphi \Rightarrow \psi)) \Rightarrow (\psi o (\psi \Rightarrow \varphi))$;
(A5a) $(\varphi \Rightarrow (\psi \Rightarrow w)) \Rightarrow ((\varphi o \psi) \Rightarrow w)$;
(A5b) $((\varphi o \psi) \Rightarrow w) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow w))$;
(A6) $((\varphi \Rightarrow \psi) \Rightarrow w) \Rightarrow ((\psi \Rightarrow \varphi) \Rightarrow w) \Rightarrow w)$;
(A7) $\overline{0} \Rightarrow w$.

Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view. There has been some recent interest in applying rings theory notions to non-mainstream algebras. The situation is analogous to that of rings theory. However, the details are slightly different although some of the ideas are similar.

Hájek [2] introduced the concepts of filters and prime filters in BL-algebras. From logical point of view, filters correspond to sets of provable formula. E. Turunen studied some properties of filters theory, which plays important role in studying logical algebras. He showed how BL-algebras can be studied by deductive systems. Deductive systems correspond to subsets closed with respect to Modus Ponens and they are called filters, too. Also he introduce the notion of Boolean Artinian BL-algebras [12]. Motamed and Moghaderi [5], introduced the notions of Noetherian and Artinian on BL-algebras. They obtained some equivalent definitions of Noetherian and Artinian BL-algebras. Meng B. L and Xin X. L [6], introduced the co-Noehterian BL-algebras.

In this paper, we obtain some new relations between the above notions.

The structure of the paper is as follows:

In Section 2, we recall some definitions and results about BL-algebras that we use in the sequel. In Section 3, we recall the notion of Noetherian and Boolean Artinian BL-algebras and we derive some results about the relations between them.

2. Preliminaries

In this section, we give some definitions and theorems which are needed in the rest of the paper.

An algebra$(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of the type (2, 2, 2, 2, 0, 0) is called a BL-algebra if satisfies the following axioms [2]:

$(BL1)$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

$(BL2)$ $(A, \odot, 1)$ is a commutative monoid;

$(BL3)$ $\odot$ and $\rightarrow$ form an adjoint pair, i.e., $c \leqslant a \rightarrow b$ if and only if $a \odot c \leqslant b$;

$(BL4)$ $a \wedge b = a \odot (a \rightarrow b)$;

$(BL5)$ $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

For all $a, b, c \in A$. We denote $\overline{x} = x \rightarrow 0$ and $x^{--} = (\overline{x})^{-}$, for all $x \in A$. Most familiar example of a BL-algebra is the unit interval $[0, 1]$ endowed with the structure induced by a continuous t-norm. In the rest of this paper A, denotes the universe of a BL-algebra [2].

A BL-algebra is nontrivial if $0 \neq 1$. For any BL-algebra A, the deduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. We denote the set of all natural numbers by $\mathbb{N}$ and define $a^0 = 1$ and $a^n = a^{n-1} \odot a$, for $n \in \mathbb{N} \setminus \{0\}$. Hájek [2] defined a filter of a BL-algebra A to be a non-empty subset F of A such that (i) if a, b$\in F$ implies $a \odot b \in F$, (ii) if $a \in F, a \leqslant b$ then $b \in F$. In each BL-algebra A, for every $x, y \in A$, $x \odot y \leqslant x, y$ [8]. E. Turunen [7] defined a deductive system of a BL-algebra A to be a non-empty subset D of A such that (i) $1 \in D$, (ii) $x \in D$ and $x \rightarrow y \in D$ imply $y \in D$. The set of all deductive systems of a BL-algebra A is denoted by $\mathtt{D}(A)$. A subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A [7]. A deductive system D of $BL-$algebra A is Boolean if, for all $x \in A$, $x \vee \overline{x} \in D$ [9].

A filter F of a BL-algebra A is proper if $F \neq A$. A proper filter P of A is called a prime filter of A if for all $x, y \in$A$, x \vee y \in P$ implies $x \in P$ or $y \in P$. A proper filter P of A is Prime if and only if P can not be expressed as an intersection of two filters properly containing P or equivalently, for all $x, y \in$A, either $x \to y \in P$ or $y \to x \in P$ [8].

If F, G and P are filters of A, then P is a prime filter of A if and only if $F \cap G \subseteq P$ then $F \subseteq P$ or $G \subseteq P$.

A proper filter M of A is a maximal filter if and only if for any $x \notin M$ there exists $n \in \mathbb{N}$ such that $(x^n)^- \in M$ [7]. Every maximal filter of A is a prime filter [8].

The set of all filters, prime filters and maximal filters of a BL-algebra A are denoted by $\mathcal{F}(A)$, $Spec(A)$ and $Max(A)$, respectively. For every subset $X \subseteq A$, the smallest filter of A which contains X, i.e., the intersection of all filters $F \in \mathcal{F}(A)$ such that $F \supseteq X$, is said to be the filter generated by X [1]. The filter of A generated by X will be denoted by $\langle X \rangle$, where $X \subseteq A$, in which $\langle \emptyset \rangle$=$\{1\}$ and $\langle X \rangle = \{a \in A : x_1 \odot x_2 \odot \cdots \odot x_n \leqslant a$, for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X\}$ [8]. A filter $F \in \mathcal{F}(A)$ is called finitely generated, if $F = \langle x_1, x_2, \ldots, x_n \rangle$, for some $x_1, \ldots, x_n \in$A and $n \in \mathbb{N}$. For $F \in \mathcal{F}(A)$ and $x \in A \backslash F$, $F\langle x \rangle = \langle F \bigcup \{x\} \rangle$ and so $F\langle x \rangle = \{a \in A : a \geqslant f \odot x^n$, for some $f \in F$, and $n \geqslant 1\}$ [8].

Definition 2.1. ([5]) *Let A be a BL-algebra and F be a filter of A. A prime filter P of A, is called a minimal prime filter of F if $F \subseteq P$ and if further, there exists $Q \in Spec(A)$ such that $F \subseteq Q \subseteq P$, then $P = Q$. The set of all minimal prime filter of $\{1\}$, is denoted by $Min(A)$.*

Definition 2.2. ([5]) *A BL-algebra A is called Noetherian (Artinian), if for every increasing (decreasing) chain of its filters $F_1 \subseteq F_2 \subseteq \ldots$ $(F_1 \supseteq F_2 \supseteq \ldots)$, there exists $n \in \mathbb{N}$ such that $F_i = F_n$, for all $i \geqslant n$.*

Theorem 2.3. ([5]) *Let A be a BL-algebra. Then A is Noetherian BL-algebra if and only if every filter of A is finitely generated.*

Definition 2.4. ([2]) *Let A be a BL-algebra. A non-empty subset $I \subseteq A$ is called an ideal of A, if the following conditions are satisfied:*

(*i*) $0 \in I$,

(*ii*) *If* $x, (x^- \to y^-)^- \in I$ *then* $y \in I$.

Definition 2.5. ([6]) *A BL-algebra A is said to be co-Noetherian with respect to ideals if every ideal of A is finitely generated. A BL-algebra A is satisfies the ascending chain condition with respect to its ideals if for every ascending chain sequence $I_1 \subseteq I_2 \subseteq ...$ of ideals of A, there exists $n \in \mathbb{N}$ such that $I_i = I_n$, for all $i \geqslant n$.*

Remark 2.6. ([8]) *Let F and G be two filters of A such that $F \subseteq G$. It is evident that $\dfrac{G}{F}$ is a filter of $\dfrac{A}{F}$. Since G is a filter, then it can be easily shown that $\dfrac{a}{F} \in \dfrac{G}{F}$ if and only if $a \in G$. Moreover, $\mathcal{F}(\dfrac{A}{F}) = \{\dfrac{H}{F} : H \in \mathcal{F}(A), F \subseteq H\}$.*

Definition 2.7. ([7]) *A BL-algebra A is called locally finite if all non-unit elements are of finite order, i.e., for any non-unit element x of A, $x^n = 0$ for some $n \geqslant 1$. Obviously, the only proper deductive system of a locally finite BL-algebra is $\{1\}$, thus, $M(A) = \{1\}$, where $M(A) = \bigcap\{M : M$ is a maximal deductive system of $A\}$.*

From [7], a BL-algebra A is semisimple if $M(A) = \{1\}$. In [7], it is also proved that locally finite BL-algebras and locally finite MV-algebras are coincide. Moreover, for any BL-algebras A, M is a maximal deductive system of A if and only if $\dfrac{A}{M}$ is a locally finite BL-algebra if and only if for any $x \notin M$, $(x^n)^- \in M$ for some $n \geqslant 1$. E. Turunen [9], defined a BL-algebra A is local if it has a unique maximal deductive system. He proved that a BL-algebra A is local if and only if the unique maximal deductive system of A is equal to $\{x \in A : x^n > 0$ for all $n \geqslant 1\}$.

From [11], Given two locally finite BL-algebras A and B, a product BL-algebra $A \times B$ contains two disjoint descending chains of deductive systems, namely $A \times B \supseteq A \times \{1\} \supseteq \{(1, 1)\}$ and $A \times B \supseteq \{1\} \times B \supseteq \{(1, 1)\}$. Clearly, $A \times B$ is a semisimple BL-algebra and both maximal deductive system $A \times \{1\}$ and $\{1\} \times B$ are disjoint. More generally, given

n locally finite BL-algebras $A_1, A_2, A_3, \dots A_n$, a product BL-algebra $\prod_{k=1}^{n} A_k$ is semisimple, contains $2^n - 1$ proper deductive system and n disjoint maximal deductive system $M_k = A_1 \times \cdots \times \{1\} \times \cdots \times A_n$, $k = 1, 2, \dots, n$. Also, any properly descending chain of deductive systems is finite.

Proposition 2.8. ([11]) *Let A be a BL-algebra and $M_1, M_2, M_3, \dots, M_n$ be n maximal deductive systems, (not necessarily disjoint), of A. Then the product BL-algebra $\mathcal{A} = \prod_{k=1}^{n} A/M_k$ is semisimple, contains $2^n - 1$ proper deductive systems, and n disjoint maximal deductive systems further any properly descending chain of deductive system of $\mathcal{A}$ is finite.*

Definition 2.9. ([11]) *A BL-algebra A is semilocal if it contains only finite many disjoint maximal deductive systems.*

Definition 2.10. ([11]) *A BL-algebra A is Boolean Artinian if, any properly descending chain of Boolean deductive systems is finite.*

Definition 2.11. ([6]) *Let X be a subset of a BL-algebra A. The least ideal containing X in A is called the ideal generated by X and denoted by $(X]$. If $X = \{a_1, a_2, \dots, a_n\}$ then $(X]$ is denoted by $(a_1, a_2, \dots, a_n]$ instead of $(\{a_1, a_2, \dots, a_n\}]$. An ideal I of A is said to be finitely generated if there exist $a_1, a_2, \dots, a_n \in A$ such that $I = (a_1, a_2, \dots, a_n]$.*

Proposition 2.12. ([6]) *Let A be a BL-algebra. Then for any $x_1, \dots, x_n \in A$ and $n \in \mathbb{N}$, $((x_1] \cup (x_2] \cup \cdots \cup (x_n]] = ((\bar{x}_1 \odot \bar{x}_2 \odot \cdots \odot \bar{x}_n)^-]$.*

Definition 2.13. ([12]) *Let A be a BL-algebra. If I is an ideals of A, $I = (x]$ where $x \in A$, then I is called a principal ideal of A.*

Definition 2.14. ([8]) *Let A and B be two BL-algebras. A map $f : A \longrightarrow B$ defined on A, is called a BL-homomorphism if, for all $x, y \in A$, $f(x \to y) = f(x) \to f(y)$, $f(x \odot y) = f(x) \odot f(y)$ and $f(0_A) = 0_B$. Also, we define $ker(f) = \{a \in A : f(a) = 1\}$ and $Im(f) = \{f(a) : a \in A\}$.*

Theorem 2.15. ([4]) *Let X be a subset of A. Then $\langle X \rangle = \{a \in A : (x_1 \odot x_2 \odot \cdots \odot x_n) \to a = 1$, for some $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n \in X\}$, where $\langle X \rangle$ is the filter of A generated by X.*

Theorem 2.16. ([8]) *A subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A.*

Theorem 2.17. ([3]) *Let A be a nontrivial BL-algebra, $X \subseteq A$ and $D(X) = \{P \in Spec(A) : X \nsubseteq P\}$. Then the following hold:*
(i) $X \subseteq Y \subseteq A$ implies $D(X) \subseteq D(Y) \subseteq Spec(A)$;
(ii) $D(\{0\}) = Spec(A)$ and $D(\emptyset) = E(\{1\}) = \emptyset$;
(iii) $D(X) = Spec(A)$ if and only if $A = \langle X \rangle$;
(iv) $D(X) = \emptyset$ if and only if $X = \emptyset$ or $X = \{1\}$;
(v) If $\{X_i\}_{i\in I}$ is any family of subsets of A, then $D(\bigcup_{i\in I} X_i) = \bigcap_{i\in I} D(X_i)$;
(vi) $D(X) = D\langle X \rangle$;
(vii) $D(X) \bigcup D(Y) = D(\langle X \rangle \bigcup \langle Y \rangle)$;
(viii) If $X, Y \subseteq A$, then $\langle X \rangle = \langle Y \rangle$ if and only if $D(X) = D(Y)$;
(ix) If F, H are filters of A, then $F = H$ if and only if $D(F) = D(H)$.

Theorem 2.18. ([11]) *Let $M_1, M_2, M_3, \ldots, M_n$ be n maximal deductive systems of a BL-algebra A and $\mathcal{A} = \prod_{i=1}^{n} A/M_i$ be the product BL-algebra. Then a map $h : A \longrightarrow \mathcal{A}$ defined, for all $a \in A$, by $h(a) = (a/M_1, a/M_2, \ldots, a/M_n)$ is an onto BL-homomorphism such that $h(a) = 1_{\mathcal{A}}$ if and only if $a \equiv 1 \mod M_i$, for all $i = 1, 2, \ldots, n$ where, for $x, y \in A$, $x \equiv y \mod D$ iff $(x \to y) \odot (y \to x) \in D$. Thus $A/\bigcap_{i=1}^{n} M_i$ is isomorphic to $\mathcal{A} = \prod_{i=1}^{n} A/M_i$.*

3. Some Results on Noetherian and Boolean Artinian BL-Algebras

In this section, we derive some new results on Noetherian, semilocal, local and Boolean Artinian BL-algebras.

Lemma 3.1. *Let A be a BL-algebra and $F \in \mathcal{F}(A)$. If F is generated by a finite set of generators, then F is generated by an element.*

Proof. If F is generated by $\{x_1, x_2, \ldots, x_n\}$, for some $x_1, x_2, \ldots, x_n \in A$, then F is the smallest filter containing, $x_1, x_2, \ldots, x_n$. We claim that F is generated by $x_1 \odot x_2 \odot \cdots \odot x_n$. It is enough to show that F is the smallest filter of A, which contains $x_1 \odot x_2 \odot \cdots \odot x_n$. Assume on the contrary that F is not the smallest filter of A with the above condition. Then, there exists a filter G of A which contains $x_1 \odot x_2 \odot \cdots \odot x_n$ and $G \subset F$, so $x_1 \odot (x_2 \odot \cdots \odot x_n) \in G$. Since $x_1 \odot (x_2 \odot \cdots \odot x_n) \leqslant x_1$, and $x_1 \odot (x_2 \odot \cdots \odot x_n) \leqslant (x_2 \odot \cdots \odot x_n)$ and G is a filter, so $x_1 \in G$ and $(x_2 \odot x_3 \odot \cdots \odot x_n) = x_2 \odot (x_3 \odot \cdots \odot x_n) \in G$. Therefore, $x_1 \in G$, $x_2 \in G$ and $x_3 \odot \cdots \odot x_n \in G$. Continuing this procedure, we conclude $x_1, x_2, \ldots, x_n \in G$, which is a contradiction. Thus, F is the smallest filter of A such that containing all x_i for all $1 \leqslant i \leqslant n$. □

Theorem 3.2. *Let A be a BL-algebra and $A \neq \emptyset$. Then $Spec(A)$ has minimal element with respect to inclusion.*

Proof. Since $A \neq \emptyset$, so $Spec(A) \neq \emptyset$. We consider $(Spec(A), \supseteq)$ and suppose $\{P_i\}$ be a chain in $(Spec(A), \supseteq)$. Put $P = \bigcap_i P_i$. Since, $1 \in P$, so $P \neq \emptyset$ and we conclude that P is a filter of A. We prove P is prime. Let $a \vee b \in P$ and $a \notin P, b \notin P$. Then there exist $i, j \in \mathbb{N}, a \notin P_i$, $b \notin P_j$. Since $\{P_i\}$ is a chain, so $P_i \supseteq P_j$, and $a \notin P_j$. Therefore, $a \vee b \notin P_j$ and $a \vee b \notin P = \bigcap_i P_i$, which is a contradiction, so P is prime and $Spec(A)$ has a minimal element with respect to inclusion. □

Corollary 3.3. *Let A be a BL-algebra. If A satisfy the ascending chain condition on finitely generated filters, then A is a Noetherian BL-algebra.*

Proof. Put $F = \{F_i \subseteq A : F_i$ is finitely generated filter of $A\}$. We prove that F has a maximal element. Clearly $\langle 1 \rangle \in F$, then $F \neq \emptyset$. We show that F has a maximal element. Let $F_1 \in F$, if F_1 is maximal of F, it is obvious, otherwise, there exists $F_2 \in F$, $F_1 \neq F_2$ such that $F_1 \subset F_2$. Now if F_2 is maximal element of F, we are done. But if F_2 is not maximal element of F, then there exist $F_3 \in F$, $F_3 \neq F_2$ and $F_1 \subset F_2 \subset F_3$. Continuing this procedure, we have $F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_n \subset \ldots$. Since $F_i \in F$, for all $i \in \mathbb{N}$, so every F_i is finitely generated filter therefore, there exist s $n \in \mathbb{N}$ such that $F_i = F_n$, for all $i \geqslant n$. Hence F_n is a maximal element of F. To prove that A is Noetherian, it is sufficient to show that every filter of A, is finitely generated. Let G be an arbitrary filter of A, then we prove that G is finitely generated. Let $H = \{F_i \subseteq G : F_i$ be a finitely generated filter of $A\}$. For every $x \in G$, we have, $\langle x \rangle \subset G$, so $H \neq \emptyset$. According to the previous argument, H has a maximal element like F_1. We prove $F_1 = G$ otherwise, if $F_1 \neq G$, since $F_1 \in H$, so $F_1 \subset G$ and there exists $x \in G - F_1$. Since F_1 is finitely generated, then $F_1 = \langle x_1, x_2, \ldots, x_n \rangle$, for some $x_1, x_2, \ldots, x_n \in A$. Thus $F_1 \subset F_2 = \langle x_1, x_2, \ldots, x_n, x \rangle \subset G$, so $F_2 \in H$ and $F_1 \subset F_2$, which is a contradiction. Hence G is a finitely generated filter of A and by Theorem 2.3, A is a Noetherian BL-algebra. □

Theorem 3.4. *Let A be a BL-algebra. Then A is a semilocal if and only if $A/M(A)$ is a Boolean Artinian BL-algebra.*

Proof. Let A be a semilocal BL-algebra. We show that by Definition 2.10, any properly descending chain of Boolean deductive system in $A/M(A)$ is finite. Suppose $M_1, M_2, M_3, \ldots, M_n$ be the n disjoint maximal deductive systems of A. Then by Definition 2.7, $M(A) = \cap\{M : M$ is a maximal deductive system of $A\}$, i.e., $M(A) = \bigcap_{k=1}^{n} M_k$. By Proposition 2.18, $A/M(A)$ is isomorphic to $\mathcal{A} = \prod_{k=1}^{n} A/M_k$. Also by Proposition 2.8, every properly descending chain of deductive system of $\mathcal{A}$ is finite. Thus $A/M(A)$ is Boolean Artinian. □

Conversely, Let $A/M(A)$ be Boolean Artinian, then any properly descending chain of deductive system in $A/M(A)$ is finite. So any properly descending chain of deductive system of A is finite, thus, A is a semilocal BL-algebra. If A contains infinite many maximal deductive system like $M_1, M_2, M_3, \ldots$, then $M_1 \supseteq (M_1 \cap M_2) \supseteq (M_1 \cap M_2 \cap M_3) \supseteq \ldots$, is an infinite properly descending chain of deductive system generating an infinite properly descending chain of deductive system $M_1/M(A) \supseteq (M_1 \cap M_2)/M(A) \supseteq (M_1 \cap M_2 \cap M_3)/M(A) \supseteq \ldots$, of deductive system into $A/M(A)$, this is a contradiction. Since $A/M(A)$ is Boolean Artinian, so A contains only finite many disjoint maximal deductive system, i.e., A is a semilocal BL-algebra.

By Theorem 3.4, the following conditions are equivalent:

Corollary 3.5. *Let A be a BL-algebra. Then the following conditions are equivalent:*

(i) A is a semilocal and A has the n disjoint maximal deductive systems;

(ii) $A/M(A)$ is Boolean Artinian;

(iii) $A/M(A)$ is a semilocal BL-algebra.

Proof. $(i) \Rightarrow (ii)$ Let A be a semilocal and A has the n disjoint maximal deductive systems. Then, by Theorem 3.4, $A/M(A)$ is Boolean Artinian.

$(ii) \Rightarrow (i)$ Let $A/M(A)$ be Boolean Artinian. Then, by Theorem 3.4, A is a semilocal BL-algebra and by Definition 2.9, A has the n disjoint maximal deductive systems.

$(i) \Rightarrow (iii)$ Let A be a semilocal and A has the n disjoint maximal deductive systems. Suppose $M_1, M_2, M_3, \ldots, M_n$ be the n disjoint maximal deductive systems of A. Then by Definition 2.7, $M(A) = \cap\{M : M$ is a maximal deductive system of $A\}$, i.e., $M(A) = \bigcap_{k=1}^{n} M_k$. By Proposition 2.18, $A/M(A)$ is isomorphic to $\mathcal{A} = \prod_{k=1}^{n} A/M_k$. Also by Proposition 2.8, every properly descending chain of deductive systems of $\mathcal{A}$ is finite. Thus $A/M(A)$ is a semilocal BL-algebra.
$(iii) \Rightarrow (ii)$ Let $A/M(A)$ be a semilocal BL-algebra. Suppose $M_1/M(A) \supseteq M_2/M(A) \supseteq M_3/M(A) \supseteq \ldots,$ be properly descending chain of deductive system of a semilocal BL-algebra $A/M(A)$ in which $M_1, M_2, M_3, \ldots,$ are maximal deductive systems of A. Since $A/M(A)$ is a semilocal, so it contains only finite many disjoint maximal deductive systems. Hence for some $n \geqslant 1$, $M_n/M(A) = M_{n+1}/M(A)$. Thus $A/M(A)$ is a Boolean Artinian BL-algebra. □

Theorem 3.6. *Let A be a BL-algebra and $A \neq \emptyset$. Then the set of prime ideals of A has minimal element with respect to inclusion.*

Proof. It is similar to proof of Theorem 3.2. □

Theorem 3.7. *Let A be a BL-algebra. Then every finitely generated ideal in A is principal.*

Proof. Let F be a finitely generated ideal of A, then $F = (x_1, \ldots, x_n]$ for some $x_1, \ldots, x_n \in A$. By induction on n, we complete the proof. If $n = 1$, the claim is true by hypothesis. Now assume it is true for $n = k$ and set $(x_1, \ldots, x_n] = (x]$. Let $n = k+1$ then $(x_1, \ldots, x_k, x_{k+1}] \subseteq ((x] \cup (x_{k+1}]]$, by Proposition 2.12, we have $((x] \cup (x_{k+1}]] = ((\bar{x} \odot \bar{x}_{k+1})^{-}]$. □

Conversely, since $(x], (x_{k+1}] \subseteq (x_1, , x_k, x_{k+1}]$, then $((\bar{x} \odot \bar{x}_{k+1})^{-}] = (x] \cup (x_{k+1}]] \subseteq (x_1, \ldots, x_k, x_{k+1}]$, so $(x_1, \ldots, x_k, x_{k+1}] = ((\bar{x} \odot \bar{x}_{k+1}^{-})^{-}]$ is a principal ideal of A, and the claim holds for all $n \in \mathbb{N}$.

Theorem 3.8. *If Every ideal of BL-algebra A is principal, then A is a co-Noetherian BL-algebra.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$, be an increasing chain of ideals of A. Put $I = I_1 \cup I_2 \cup \ldots$, it is clear that I is an ideal of A. By hypothesis $I = (x]$ for some $x \in A$. So there exists $n \in \mathbb{N}$ such that $x \in I_n$. Therefore, $I_i = I_n$, for all $i \geqslant n$. Hence A is a co-Noetherian BL-algebra. □

Proposition 3.9. *If A is a Noetherian BL-algebra. Then $(Spec(A), T_A)$ is a Noetherian topological space, where $D(F) = \{P \in Spec(A) : F \nsubseteq P\}$ and $T_A = \{D(F) : F \in \mathcal{F}(A)\}$.*

Proof. From [3], we deduce, $T_A = \{D(F) : F \in \mathcal{F}(A)\}$ is a topological space. Let $F_1 \subseteq F_2 \subseteq \ldots$, be a chain of filters of A. By Theorem 2.17 (i) clearly, $D(F_1) \subseteq D(F_2) \subseteq \ldots$, is a chain of T_A. Since A is Noetherian, so there exists $i \in \mathbb{N}$, such that for all $n \geqslant i$ $F_i = F_n$. Then by Theorem 2.17 (ix), $D(F_i) = D(F_n)$, for all $n \geqslant i$ and hence $(Spec(A),\ T_A)$ is Noetherian topological space. □

Proposition 3.10. *Let A be a Boolean Artinian BL-algebra, B be a BL-algebra and $f : A \longrightarrow B$ be an onto BL-homomorphism. Then $f(A) = B$ is a Boolean Artinian BL-algebra.*

Proof. It is easy to see that for any deductive system D of B, $f^{-1}(D)$ is a deductive system of A. Let $D_1 \supsetneqq D_2 \supsetneqq \cdots \supsetneqq D_n \supsetneqq \ldots$, be a properly descending chain of Boolean deductive systems of B, then $f^{-1}(D_i)$, $i \in \mathbb{N}$ are Boolean deductive systems of A. Since f is onto, $f^{-1}(D_1) \supsetneqq f^{-1}(D_2) \supsetneqq \cdots \supsetneqq f^{-1}(D_n) \supsetneqq \ldots$, is a properly descending chain of Boolean deductive system of A. By hypothesis, A is Boolean Artinian, then there exists a $n \in \mathbb{N}$ such that, $f^{-1}(D_i) = f^{-1}(D_n)$, for all $i \geqslant n$. Since f is an onto BL-homomorphism, so we get $f(f^{-1}(D_i)) = f(f^{-1}(D_n))$ and $D_i = D_n$, for all $i \geqslant n$. Thus B is a Boolean Artinian BL-algebra. □

Proposition 3.11. *Let A be a Boolean Artinian BL-algebra and $f : A \longrightarrow A$ be a one to one BL-homomorphism. Then f is an onto BL-homomorphism.*

Proof. Suppose f is not an onto BL-homomorphism of BL-algebras, i.e., $A \supsetneqq f(A)$. Since f is one to one, so $f(A) \supsetneqq f^2(A)$ and hence $f^{n-1}(A) \supsetneqq f^n(A)$ for all $n \geqslant 2$. This means that $A \supsetneqq f(A) \supsetneqq f^2(A) \supsetneqq \cdots \supsetneqq f^{n-1}(A) \supsetneqq f^n(A) \supsetneqq \dots$, is a properly descending chain of Boolean deductive systems of A. This chain is not stationary, because if there exists $k \in \mathbb{N}$ such that $f^{k+1}(A) = f^k(A)$, then by the injectivity of f, there exists a map $g : A \longrightarrow A$, $g(f(A)) = I_A$, thus $g(f^{k+1}(A)) = g(f^k(A))$, i.e., $f^k(A) = f^{k-1}(A)$, by continuing this procedure, we get, $A = f(A)$. This is a contradiction, hence $A = f(A)$ and f is an onto BL-homomorphism. $\square$

Proposition 3.12. *Let A be a Boolean Artinian BL-algebra and D be a deductive system of A. Then $\frac{A}{D}$ is a Boolean Artinian BL-algebra.*

Proof. Let $\frac{D_1}{D} \supsetneqq \frac{D_2}{D} \supsetneqq \cdots \supsetneqq \frac{D_n}{D} \supsetneqq \dots$, be a properly descending chain of Boolean deductive system of $\frac{A}{D}$. Then $D_1 \supsetneqq D_2 \supsetneqq \cdots \supsetneqq D_n \supsetneqq \dots$, is a properly descending chain of Boolean deductive system of A. Since A is a Boolean Artinian BL-algebra, there exists a $n \in \mathbb{N}$ such that for all $i \geqslant n$, $D_i = D_n$, so for all $i \geqslant n$, $\frac{D_i}{D} = \frac{D_n}{D}$. Thus $\frac{A}{D}$ is Boolean Artinian. $\square$

Proposition 3.13. *Let A be a BL-algebra. Then A is a Boolean Artinian BL-algebra if and only if every non-empty set of deductive systems of A has a minimal element.*

Proof. Let A be a Boolean Artinian BL-algebra and G be a non-empty set of deductive systems of A that does not have a minimal element. There exists a $D_1 \in G$, since G is a non-empty set. Now as G does not have a minimal element, there exists $D_2 \in G$ such that $D_1 \supsetneqq D_2$. Continuing this method, we have $D_1 \supsetneqq D_2 \supsetneqq D_3 \supsetneqq \dots$, which is a properly descending chain of Boolean deductive systems of A. This chain is not stationary, so which is a contradiction. Thus G has a minimal element. $\square$

Conversely, suppose $D_1 \supsetneqq D_2 \supsetneqq D_3 \supsetneqq \dots$, is a properly descending chain of Boolean deductive systems of A. Put $G = \{D_i : i \in \mathbb{N}\}$. Since

G is a non-empty set, G has a minimal element, like D_n. Hence for all $i \geqslant n$, $D_i = D_n$ and A is a Boolean Artinian BL-algebra.

Proposition 3.14. *Let A be a Boolean Artinian BL-algebra. Then the set of all maximal deductive systems of A is finite.*

Proof. Put $G = \{D \in \mathtt{D}(A) : D$ is the intersection of finitely many maximal deductive systems of $A\}$. If $Max(A)$ is a non-empty set, then G is also a non-empty set. Thus by Proposition 3.13, G has a minimal element D_1. So there exist $M_1, M_2, \ldots, M_n$ of the set of all maximal deductive systems of A such that $D_1 = M_1 \cap M_2 \cap \cdots \cap M_n$. Suppose M is an element of the set of all maximal deductive systems of A. Then $M \cap D_1 \subseteq D_1$, $M \cap D_1 = M \cap M_1 \cap M_2 \cap \cdots \cap M_n \in G$ and D_1 is a minimal element of G, so $M \cap D_1 = D_1$, Hence $D_1 = M_1 \cap M_2 \cap \cdots \cap M_n \subseteq M$. Since any deductive system is a filter and every maximal filters is prime, so M is in $Spec(A)$, thus there exists $i \in \mathbb{N}$, such that $M_i \subseteq M$. Now as M, M_i are elements of the set of all maximal deductive systems of A, we get $M_i = M$. Hence the set of all maximal deductive systems of A is finite. $\square$

Proposition 3.15. *Let A and B be two local BL-algebras and $f : A \longrightarrow B$ be a BL-homomorphism. Then $f(A)$ is a Boolean Artinian BL-algebra.*

Proof. First we show that A is a Boolean Artinian BL-algebra. Suppose $D_1 \supsetneq D_2 \supsetneq \cdots \supsetneq D_n \supsetneq \ldots$, be a properly descending chain of Boolean deductive system of a local BL-algebra A. We claim that, there exists a $n \in \mathbb{N}$ such that for all $i \geqslant n$, $D_i = D_n$. It is clear that, $D_1 \supsetneq D_2 \supsetneq \cdots \supsetneq D_n \supsetneq \ldots$, generates a properly descending chain $\langle D_1 \cup M(A)\rangle / M(A) \supsetneq \langle D_2 \cup M(A)\rangle / M(A) \supsetneq \ldots$, of deductive system in $A/M(A)$, where $M(A) = \bigcap\{M : M$ is a maximal deductive system of $A\}$. Since A is a local BL-algebra, so $A/M(A)$ is semilocal and for some $i \geqslant n$, we have $\langle D_i \cup M(A)\rangle / M(A) = \langle D_n \cup M(A)\rangle / M(A)$. We show that $D_i = D_n$. Suppose $a \in D_i$, then $a \in D_i \cup M(A) \subseteq \langle D_i \cup M(A)\rangle$ and $a \in \langle D_i \cup M(A)\rangle = \langle D_n \cup M(A)\rangle$, $a \in \langle D_n \cup M(A)\rangle$, so for some $b \in D_n$ and $c \in M(A)$, we have $b \odot c \leqslant a$. Thus $b \leqslant c \to a$ and since $b \in D_n$, D_n is a deductive system, hence $c \to a \in D_n$. Since

$M(A)$ is a properly deductive system, $c \in M(A)$, and by Definition 2.7, $M(A) = \{x \in A : x^n > 0 \ for \ all \ n \geqslant 1\}$, so $\overline{c} \notin M(A)$, $(\overline{c})^k = 0$, for some $k \geqslant 1$. Since D_n is Boolean, then $c \vee \overline{c} \in D_n$, thus $(\mathrm{c} \vee \overline{c})^k \in D_n$ and $c \vee 0 = c$, $(\mathrm{c} \vee 0)^k = c^k$, so $c^k = (\mathrm{c} \vee \overline{c})^k$, i.e., $c^k \in D_n$. We know that $c^k \leqslant c$ implies $c \rightarrow a \leqslant c^k \rightarrow a$, therefore, from $c \rightarrow a \in D_n$, We conclude that $a \in D_n$, i.e., $D_i = D_n$, for all $i \geqslant n$ and A is Boolean Artinian. Since B is a local BL-algebra, so B is Boolean Artinian, and by Proposition 3.10, since A and B are Boolean Artinian, so $f(A)$ is a Boolean Artinian BL-algebra. $\square$

Corollary 3.16. *If A is a local BL-algebra and $f : A \longrightarrow A$ be a one to one BL-homomorphism. Then f is an onto BL-homomorphism.*

Proof. Since A is local BL-algebra, so by Proposition 3.15, A is also Boolean Artinian. Thus by Proposition 3.11, f is an onto BL-homomorphism. $\square$

Corollary 3.17. *Let A and B be two BL-algebra and $f : A \longrightarrow B$ be an onto BL-homomorphism. If A is a local BL-algebra. Then B is a Boolean Artinian BL-algebra.*

Proof. Since A is a local BL-algebra, so by Proposition 3.15, A is also Boolean Artinian, and by Proposition 3.10, B is a Boolean Artinian BL-algebra. $\square$

Corollary 3.18. *Let A be a local BL-algebra and D be a deductive system of A. Then $\frac{A}{D}$ is a Boolean Artinian BL-algebra.*

Proof. Since A is a local BL-algebra, so by Proposition 3.15, A is Boolean Artinian and by Proposition 3.12, $\frac{A}{D}$ is a Boolean Artinian BL-algebra. $\square$

Corollary 3.19. *Let A be a local BL-algebra. Then every non-empty set of deductive systems of A has a minimal element.*

Proof. Since A is local BL-algebra, so by Proposition 3.15, A is Boolean Artinian, and by Proposition 3.13, A is a Boolean Artinian BL-algebra if and only if every non-empty set of deductive systems of A has a minimal element. $\square$

Corollary 3.20. *Let A be a local BL-algebra. Then the set of all maximal deductive systems of A is finite.*

Proof. Since A is local BL-algebra, so by Proposition 3.15, A is Boolean Artinian and by Proposition 3.14, the set of all maximal deductive systems of A is finite. □

Proposition 3.21. *Let A and B be two local BL-algebras and $h : A \longrightarrow B$ be an one to one BL-homomorphism. If $\langle h(D_1)\rangle = \langle h(D_2)\rangle$, then $D_1 = D_2$, for all deductive systems D_1, D_2 of A.*

Proof. Let D_1 and D_2 be two deductive systems of A and $\langle h(D_1)\rangle = \langle h(D_2)\rangle$. Then for any $x \in D_1$, $h(x) \in \langle h(D_1)\rangle = \langle h(D_2)\rangle$, by Theorem 2.15, there exists $n \in \mathbb{N}$ such that $(h(x_1) \odot h(x_2) \odot \cdots \odot h(x_n)) \to h(x) = 1$, since h is a BL-homomorphism, then $h(x_1 \odot x_2 \odot \cdots \odot x_n) = (h(x_1) \odot h(x_2) \odot \cdots \odot h(x_n)) \to h(x) = 1$, i.e., $h((x_1 \odot x_2 \odot \cdots \odot x_n) \to x) = 1$, $h(1) = 1$ and by assumption, $(x_1 \odot x_2 \odot \cdots \odot x_n) \to x = 1$. Since x_1, $x_2, \ldots, x_n \in D_1$ and by Theorem 2.15, $x \in \langle D_2\rangle = D_2$, so $D_1 \subseteq D_2$.
Suppose $x \in D_2$, then $h(x) \in \langle h(D_1)\rangle = \langle h(D_2)\rangle$, by Theorem 2.15, there exists $n \in \mathbb{N}$ such that $(h(x_1) \odot h(x_2) \odot \cdots \odot h(x_n)) \to h(x) = 1$, since h is a BL-homomorphism, then $h(x_1 \odot x_2 \odot \cdots \odot x_n) = (h(x_1) \odot h(x_2) \odot \cdots \odot h(x_n)) \to h(x) = 1$, i.e., $h((x_1 \odot x_2 \odot \cdots \odot x_n) \to x) = 1$, $h(1) = 1$ and by assumption, $(x_1 \odot x_2 \odot \cdots \odot x_n) \to x = 1$. Since $x_1, x_2, \ldots, x_n \in D_2$ and by Theorem 2.15, $x \in \langle D_1\rangle = D_1$, so $D_2 \subseteq D_1$. Hence $D_1 = D_2$, for all $D_1, D_2 \in D(A)$. □

Corollary 3.22. *Let A be a BL-algebra such that every ideals of A is finitely generated. If B is sub BL-algebra of A, and B is a co-Noetherian BL-algebra, then every sub BL-algebra S between B and A ($B \subseteq S \subseteq A$) is also a co-Noetherian BL-algebra.*

Proof. Let S be every sub BL-algebra between B and A, i.e., $B \subseteq S \subseteq A$. By Definition 2.5, since every ideal of A is finitely generated, so A is a co-Noetherian BL-algebra and every ascending chain in S, stops in A, thus S is co-Noetherian. □

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