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$(\varphi, \psi)-$ Biprojective Banach Algebras

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Abstract. The article studies the concept of a $(\varphi, \psi)-$ biprojective and $(\varphi, \psi)-$pseudo amenable Banach algebra A, where φ is a continuous homomorphism on A and $\psi \in \Phi_A$. We show if A is $(\varphi, \psi)-$ contractible, then A is $(\varphi, \psi)-$ biprojective. The converse holds, whenever A is either unital or commutative and there exists $a_o \in A$ such that $\varphi(a_0) = a_0$.

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1. Introduction

Amenable Banach algebra was introduced by Johnson in [6]. He showed that A is amenable Banach algebra if and only if A has a approximate diagonal that is, a bounded net (m_α) in $(A \hat{\otimes} A)$ such that $m_\alpha a - a m_\alpha \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$ for every $a \in A$. The notion of a biflat and biprojective Banach algebra was introduced by Helemskii [4, 5]. Indeed, A is called biprojective if there is a bounded A-bimodule map $\theta : A \longrightarrow A \hat{\otimes} A$ such that $\pi \circ \theta = id_A$.

He considered a Banach algebra A is amenable if A biflat and has a bounded approximate identity [3, 5]. In fact, A is called biflat if there exists a bounded A-bimodule map $\theta : (A \hat{\otimes} A)^* \longrightarrow A^*$ such that $\theta \circ \pi^* = id_{A^*}$.

Given a continuous homomorphism φ from A into A, authors in [9, 10] are defined and studied φ-derivations and φ-amenability.

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Recall that a character on A is a non-zero homomorphism from A into the scalar field. The set of all characters on A is called the character space of A and is denoted by Φ_A.

This article studies $(\varphi, \psi)-$ contractible Banach algebras, where φ is a continuous homomorphism on A and $\psi \in \Phi_A$. We show that if A is $(\varphi, \psi)-$ contractible, then A is $(\varphi, \psi)-$ biprojective. The converse holds, whenever A is either unital or commutative and there exists $a_o \in A$ such that $\varphi(a_0) = a_0$.

2. $(\varphi, \psi)-$ Biprojective Banach Algebras

Suppose that A is a Banach algebra. Let $Hom(A)$ denotes the set of all continuous homomorphisms from A into itself.

Definition 2.1. *Let A be a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. We say that A is $(\varphi, \psi)-$ biprojective if there exists a bounded A-bimodule map $\theta : A \longrightarrow (A \hat{\otimes} A)$, where $\psi \circ \pi \circ \theta \circ \varphi = \psi \circ \varphi$.*

If A is a biprojective Banach algebra, then A is a $(\varphi, \psi)-$ biprojective Banach algebra for every $\varphi \in Hom(A)$ and $\psi \in \Phi_A$.

Theorem 2.2. *Suppose that A is a $(\varphi, \psi)-$ biprojective Banach algebra. If I is a closed deal of A with one sided bounded approximate identity and $\varphi(I) \subset I$. Then I is $(\varphi|_I, \psi|_I)-$ biprojective.*

Proof. Assume that $\theta : A \longrightarrow (A \hat{\otimes} A)$ is a continuous A-bimodule map such that $\psi \circ \pi \circ \theta \circ \varphi(a) = \psi \circ \varphi(a)$ $(a \in A)$. Let $\iota : I \hookrightarrow A$ be the inclusion map. Then $\theta|_I = \theta \circ \iota : I \longrightarrow (A \hat{\otimes} A)$ is I-bimodule homomorphism. If I^3 denotes span $\{abc : \ a, b, c \in I\}^-$, then $I^3 = I$, because I has a one sided bounded approximate identity and

$$\begin{aligned} \theta|_I &= \theta(I) \\ &= \theta_(I^3) \\ &\subseteq span\{a \cdot \theta(b) \cdot c\}^- \\ &\subseteq span\{a \cdot m \cdot c : a, c \in I, m \in A \hat{\otimes} A\}^- \subseteq I \hat{\otimes} I. \end{aligned}$$

So for every $a \in I$,

$$\begin{aligned} \psi \circ \pi \circ \theta|_I \circ \varphi(a) &= \psi \circ \pi(\theta(\varphi(a))) \\ &= \psi \circ \varphi(a). \quad \square \end{aligned}$$

Proposition 2.3. *Let A be a $(\varphi_A, \psi_A)-$ biprojective Banach algebra, and let B be a $(\varphi_B, \psi_B)-$ biprojective Banach algebra with $\varphi_A \in Hom(A)$,*

$\psi_A \in \Phi_A$, $\varphi_B \in Hom(B)$ and $\psi_B \in \Phi_B$. Then $A \hat{\otimes} B$ is $(\varphi_A \otimes \varphi_B, \psi_A \otimes \psi_B)$ -biprojective.

Proof. There exists an A-bimodule map $\theta_1 : A \longrightarrow (A \hat{\otimes} A)$ with $\psi_A \circ \pi_A \circ \theta_1 \circ \varphi_A = \psi_A \circ \varphi_A$ and B-bimodule map $\theta_2 : B \longrightarrow (B \hat{\otimes} B)$ with $\psi_B \circ \pi_B \circ \theta_2 \circ \varphi_B = \psi_B \circ \varphi_B$. Let $\theta_0 : (A \hat{\otimes} A)\hat{\otimes}(B \hat{\otimes} B) \longrightarrow (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B)$ be the isometric isomorphism given by $(a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)$ $(a_1, a_2 \in A,\ b_1, b_2 \in B)$. We let $\theta = \theta_0 \circ (\theta_1 \otimes \theta_2) : A \hat{\otimes} A \longrightarrow (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B)$. Then for $a \otimes b \in A \hat{\otimes} B$ we have

$$\begin{aligned}
\pi_{A\hat{\otimes}B} \circ \theta \circ (\varphi_A(a) \otimes \varphi_B(b)) &= \pi_{A\hat{\otimes}B} \circ \theta_0 \circ (\theta_1 \otimes \theta_2) \circ (\varphi_A(a) \otimes \varphi_B(b)) \\
&= \pi_A \otimes \pi_B \circ (\theta_1 \otimes \theta_2)(\varphi_A(a) \otimes \varphi_B(b)) \\
&= \pi_A \circ \theta_1 \circ \varphi_A(a) \otimes \pi_B \circ \theta_2 \circ \varphi_B(b).
\end{aligned}$$

Thus $(\psi_A \otimes \psi_B) \circ \pi_{A\hat{\otimes}B} \circ \theta \circ (\varphi_A(a) \otimes \varphi_B(b)) = \psi_A \circ \pi_A \circ \theta_1 \circ \varphi_A(a) \otimes \psi_B \circ \pi_B \circ \theta_2 \circ \varphi_B(b) = (\psi_A \circ \varphi_A) \otimes (\psi_B \circ \varphi_B)(a \otimes b) = (\psi_A \circ \varphi_A)(a)(\psi_B \circ \varphi_B)(b) = (\psi_A \otimes \psi_B) \circ (\varphi_A \otimes \varphi_B)(a \otimes b)$.
Therefore, $A \hat{\otimes} B$ is $(\varphi_A \otimes \varphi_B, \psi_A \otimes \psi_B)-$ biprojective. □

The proof of the following result is similar to that of Proposition 2.3.

Proposition 2.4. *Let A be a (φ_A, ψ_A)- biprojective Banach algebra, and let B be a $(\varphi_B, \psi_B)-$ biprojective Banach algebra with $\varphi_A \in Hom(A)$, $\psi_A \in \Phi_A$, $\varphi_B \in Hom(B)$ and $\psi_B \in \Phi_B$. Then $A \oplus B$ is a $(\varphi_A \oplus \varphi_B, \psi_A \oplus \psi_B)-$ biprojective.*

Proposition 2.5. *Let A be a unital Banach algebra, and B be a Banach algebra containing a non-zero idempotent b_0. If $A \hat{\otimes} B$ is $(\varphi_A \otimes \varphi_B, \psi_A \otimes \psi_B)-$ biprojective, then A is $(\varphi_A, \psi_A)-$biprojective.*

Proof. There is an $A \hat{\otimes} B$-bimodule $\theta : A \hat{\otimes} B \longrightarrow (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B)$ with $(\psi_A \otimes \psi_B) \circ \pi_{A\hat{\otimes}B} \circ \theta \circ (\varphi_A \otimes \varphi_B) = (\psi_A \otimes \psi_B) \circ (\varphi_A \otimes \varphi_B)$. We regard $A \hat{\otimes} B$ as an $A-$bimodule with the actions given by

$$a_1 \cdot (a_2 \otimes b) = a_1a_2 \otimes b, \ \textit{and} \ (a_2 \otimes b) \cdot a_1 = a_2a_1 \otimes b \quad (a_1, a_2 \in A,\ b \in B)$$

Then for $a_1, a_2 \in A$ we have

$$\begin{aligned}
\theta(a_1a_2 \otimes b_0) &= \theta((a_1 \otimes b_0)(a_2 \otimes b_0)) \\
&= (a_1 \otimes b_0) \cdot \theta((a_2 \otimes b_0)) \\
&= a_1 \cdot (e_A \otimes b_0) \cdot \theta((a_2 \otimes b_0)) \\
&= a_1 \cdot \theta((e_A \otimes b_0) \cdot (a_2 \otimes b_0)) \\
&= a_1 \cdot \theta(e_A \cdot a_2 \otimes b_0^2) \\
&= a_1 \cdot \theta(a_2 \otimes b_0).
\end{aligned}$$

Similarly, we can show a right-module version of this equation. Hence we get

$$\theta(a_1a_2 \otimes b_0) = a_1 \cdot \theta(a_2 \otimes b_0) = \theta(a_1 \otimes b_0) \cdot a_2 \quad (a_1, a_2 \in A).$$

Let $\psi_B(b_0) = 1$ and we define

$$\rho : (A \hat{\otimes} B)\hat{\otimes}(A \hat{\otimes} B) \longrightarrow (A \hat{\otimes} A), \ \ (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \mapsto \psi_B(b_1b_2)a_1 \otimes a_2,$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Clearly ρ is a bounded linear operator.
We now define $\tilde{\theta} : A \longrightarrow (A \hat{\otimes} A)$ by

$$\tilde{\theta}(a) = \rho \circ \theta(a \otimes \varphi_B(b0)) \quad (a \in A).$$

Then $\tilde{\theta}$ is an A-bimodule morphism. It follows from the identity

$$\pi_A \circ \rho = (id_A \otimes \psi_B) \circ \pi_{A\hat{\otimes}B}.$$

So

$$\begin{aligned} \psi_A \circ \pi_A \circ \tilde{\theta} \circ \varphi_A(a) &= \psi_A \circ \pi_A \circ \rho \circ \theta(\varphi_A(a) \otimes \varphi_B(b0) \\ &= \psi_A \circ (id_A \otimes \psi_B) \circ \pi_{A\hat{\otimes}B} \circ \theta(\varphi_A(a) \otimes \phi_B(b0) \\ &= (\psi_A \otimes \psi_B) \circ \pi_{A\hat{\otimes}B} \circ \theta(\varphi_A(a) \otimes \varphi_B(b0) \\ &= (\psi_A \otimes \psi_B) \circ (\varphi_A \otimes \varphi_B)(a \otimes b0) \\ &= (\psi_A \circ \varphi_A)(a)(\psi_B \circ \varphi_B)(b_0) \\ &= (\psi_A \circ \varphi_A)(a). \end{aligned}$$

That is, A is $(\varphi_A, \psi_A)-$biprojective. □

Definition 2.6. *Let A be a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. We say that A is $(\varphi, \psi)-$ contractible if it has a central $(\varphi, \psi)-$ diagonal, i.e., a (φ, ψ)- diagonal $m \in A \hat{\otimes} A$ satisfying $\varphi(a) \cdot m = m \cdot \varphi(a)$ for all $a \in A$ and also $\psi \circ \pi(m) = 1$.*

Proposition 2.7. *Let A be a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. If A is $(\varphi, \psi)-$ contractible, then A is $(\varphi, \psi)-$ biprojective. The converse holds, whenever A is either unital or commutative and there is $a_o \in A$ such that $\varphi(a_0) = a_0$.*

Proof. Suppose that $m \in A \hat{\otimes} A$ is a central (φ, ψ)-diagonal for A. We define $\theta : A \longrightarrow A \hat{\otimes} A$ by $\theta(a) := a \cdot m$. Then for every $a \in A$ we have

$$\begin{aligned} \psi \circ \pi \circ \theta \circ \varphi(a) &= \psi \circ \pi(\varphi(a) \cdot m) \\ &= \psi \circ (\varphi(a))\psi \circ \pi(m) = \psi \circ (\varphi(a)). \end{aligned}$$

Thus, A is $(\varphi, \psi)-$ biprojective.

Conversely, since A is $(\varphi, \psi)-$ biprojective, there is a bounded A- module morphism $\theta : A \longrightarrow A \hat{\otimes} A$ such that $\psi \circ \pi \circ \theta \circ \varphi(a) = \psi \circ (\varphi(a))$ $(a \in A)$. Let e_A be an identity for A and let $m = \theta(e_A)$. Then m is a central $(\varphi, \psi)-$diagonal for A.

In the commutative case, let $a_o \in A$ be such that $\varphi(a_0) = a_0$. Suppose that $\psi(a_0) = 1$ and define $m = \theta(a_0)$, then m is a central $(\varphi, \psi)-$diagonal for A. Therefore, A is $(\varphi, \psi)-$ contractible Banach algebra. $\square$

Example 2.8. Consider the semigroup $\mathbb{N}_\wedge$ with the operation semigroup $m \wedge n = \min\{m, n\}$, $m, n \in \mathbb{N}$. $\Phi_{l^1(\mathbb{N}_\wedge)} = \{\psi_n : l^1(\mathbb{N}_\wedge) \to \mathbb{C} | \psi_n(\Sigma_{i=1}^{\infty} c_i \delta_i) = \Sigma_{i=1}^{\infty} c_i,\ n \in \mathbb{N}\}$. Then $l^1(\mathbb{N}_\wedge)$ is not biprojective [11]. But if we choose $\psi_1 \in \Phi_{l^1(\mathbb{N}_\wedge)}$, $\varphi \in Hom(l^1(\mathbb{N}_\wedge))$ and define $m = \delta_1 \otimes \delta_1$, then $\varphi(a) \cdot m = m \cdot \varphi(a)$ for all $a \in l^1(\mathbb{N}_\wedge)$ and also $\psi_1 \circ \pi(m) = 1$. Terefore $l^1(\mathbb{N}_\wedge)$ is a $(\varphi, \psi_1)-$ contractible. By Proposition (2.7), $l^1(\mathbb{N}_\wedge)$ is a $(\varphi, \psi_1)-$biprojective.

Definition 2.9. *Let A be a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. A is called $(\varphi, \psi)-$ biflat if there exists a bounded A-bimodule map $\theta : A \longrightarrow (A \hat{\otimes} A)^{**}$, where $\tilde{\psi} \circ \pi^{**} \circ \theta \circ \varphi = \psi \circ \varphi$.*

i) Let A be a biflat Banach algebra. Then A is $(\varphi, \psi)-$ biflat Banach algebra for every $\varphi \in Hom(A)$ and $\psi \in \Phi_A$.

ii) Let A be a $(\varphi, \psi)-$biprojective Banach algebra. Then A is $(\varphi, \psi)-$ biflat Banach algebra for every $\varphi \in Hom(A)$ and $\psi \in \Phi_A$.

The following result can be found in [9].

Lemma 2.10. *Let A be a Banach algebra. Then there exists an A-bimodule homomorphism $\gamma : (A \hat{\otimes} A)^* \longrightarrow (A^{**} \hat{\otimes} A^{**})^*$ such that for any functional $f \in (A \hat{\otimes} A)^*$, elements $\varphi, \psi \in A^{**}$ and nets $(a_\alpha), (b_\beta)$ in A with $w^* - lim_\alpha a_\alpha = \varphi$ and $w^* - lim_\beta b_\beta = \psi$ we have*

$$\gamma(f)(\varphi \otimes \psi) = lim_\alpha lim_\beta f(a_\alpha \otimes b_\beta).$$

If $\psi \in \Phi_A$, then ψ has a unique extension on A^{**} which it by $\tilde{\psi}$ and defined by $\tilde{\psi}(F) = F(\psi)$ for every $F \in A^{**}$.

Theorem 2.11. *Suppose that A is a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. If A^{**} is $(\varphi^{**}, \tilde{\psi})-$biprojective, then A is $(\varphi, \psi)-$biflat.*

Proof. Let $\kappa : A \longrightarrow A^{**}$, $\kappa_1 : A^* \longrightarrow A^{***}$ and $\kappa_* : A^{**} \longrightarrow A^{****}$ denote the natural inclusions, π ($^{**}\pi$, respectively) the product maps on A (A^{**}, respectively) and let γ be defined as in Lemma 2.10. Then the following diagram commutes:

$$\begin{array}{ccc} A^* & \xrightarrow{\ \pi^*\ } & (A \hat{\otimes} A)^* \\ \Big\downarrow \kappa_1 & & \Big\downarrow \gamma \\ A^{***} & \xrightarrow[\ {}^{**}\pi^*\ ]{} & (A^{**} \hat{\otimes} A^{**})^* \end{array}$$

for each $a^* \in A^*$, elements $a_1^{**}, a_2^{**} \in A^{**}$ and nets $(a_\alpha), (b_\beta) \subset A$ with $w^* - \lim_\alpha a_\alpha = a_1^{**}, w^* - \lim_\beta b_\beta = a_2^{**}$, we have

$$\begin{aligned} (\gamma(\pi^*(a^*)))(a_1^{**} \otimes a_2^{**}) &= \lim_\alpha \lim_\beta \pi^*(a^*)(a_\alpha \otimes b_\beta) \\ &= \lim_\alpha \lim_\beta a^*(a_\alpha b_\beta) \\ &= w^* - \lim_\alpha w^* - \lim_\beta \kappa(a_\alpha b_\beta)(a^*) \\ &= \kappa_1(a^*)(a_1^{**} a_2^{**}) \\ &= \kappa_1(a^*)({}^{**}\pi(a_1^{**} \otimes a_2^{**})) \\ &= ({}^{**}\pi^*(\kappa_1(a^*)))(a_1^{**} \otimes a_2^{**}). \end{aligned}$$

Thus $\gamma \circ \pi^* =^{**} \pi^* \circ \kappa_1$. Hence $\pi^{**} \circ \gamma^* = \kappa_1^* \circ^{**} \pi^{**}$. Since A^{**} is $(\varphi^{**}, \tilde{\psi})-$biprojective, there is an A-bimodule map $\theta_0 : A^{**} \longrightarrow (A^{**} \hat{\otimes} A^{**})$, such that $\tilde{\psi} \circ \pi \circ \theta_0 \circ \varphi^{**} = \tilde{\psi} \circ \varphi^{**}$. Putting $\theta := \gamma^* \circ \theta_0 \circ \kappa$, then for each $a \in A$ we have

$$\begin{aligned} \tilde{\psi} \circ \pi^{**} \circ \theta \circ \varphi(a) &= \tilde{\psi} \circ \pi^{**} \circ \gamma^* \circ \theta_0 \circ \kappa \circ \varphi(a) \\ &= \tilde{\psi} \circ \kappa_1^* \circ \pi^{**} \circ \theta_0 \circ \kappa \circ \varphi(a) \\ &= \tilde{\psi} \circ \kappa_1^* \circ \pi^{**} \circ \theta_0 \circ \varphi^{**}(a) \\ &= \tilde{\psi} \circ \pi^{**} \circ \theta_0 \circ \varphi^{**}(a) = \psi \circ \varphi(a). \end{aligned}$$

That is, A is $(\varphi, \psi)-$biflat. $\square$

3. (φ, ψ)-Spseudo Amenable Banach Algebras

Suppose that A is a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. Let X be a Banach A-bimodule. A linear operator $D : A \longrightarrow X$ is a $(\varphi, \psi)-$derivation if it satisfies $D(ab) = D(a) \cdot \psi(b) + \varphi(a) \cdot D(b)$ for all $a, b \in A$. A $(\varphi, \psi)-$derivation D is (φ, ψ)-inner derivation if there is $x \in X$ such that $D(a) = \varphi(a) \cdot x - x \cdot \psi(a)$ for $a \in A$. Let $\mathcal{Z}^1_{(\varphi,\psi)}(A, X)$ be the set of all continuous (φ, ψ)-derivations and $\mathcal{N}^1_\varphi(A, X)$ be the set of all (φ, ψ)-inner derivations from A

into X. The first cohomology group $\mathcal{H}^1_{(\varphi,\psi)}(A,X)$ is defined the quotient space $\mathcal{Z}^1_{(\varphi,\psi)}(A,X)/\mathcal{N}^1_{(\varphi,\psi)}(A,X)$.

A Banach algebra A is called (φ,ψ)-amenable if $\mathcal{H}^1_{(\varphi,\psi)}(A,X^*)=\{0\}$, for all A-bimodules X.

Let A be a Banach algebra and X, Y be Banach A-bimodules. Then A-bimodule morphism from X to Y is a morphism $\varphi : X \longrightarrow Y$ such that

$$\varphi(a\cdot x)=a\cdot\varphi(x),\quad \varphi(x\cdot a)=\varphi(x)\cdot a \ \ (a\in A,\ x\in X)$$

Theorem 3.1. *Assume that A is a Banach algebra with a bounded approximate identity and $a\cdot b=\psi(a)\cdot b$, $\psi\circ\varphi(a)=1$ for every $a,b\in A$. If A is $(\varphi,\psi)-$amenable, then A is a $(\varphi,\psi)-$ biflat.*

Proof. Let (e_α) be a bounded approximate identity for A and E be a w^*-cluster point of $(\varphi(e_\alpha)\otimes\varphi(e_\alpha))$ in $(A\hat{\otimes}A)^{**}$. We define a $(\varphi,\psi)-$ derivation $D: A\longrightarrow (A\hat{\otimes}A)^{**}$ by $D(a)=\varphi(a)\cdot E-E\cdot\psi(a)$. Then

$$\begin{aligned}
\pi^{**}(D(a)) &= w^*-\lim_\alpha \pi[(\varphi(a)(\varphi(e_\alpha)\otimes\varphi(e_\alpha))-(\varphi(e_\alpha)\otimes\varphi(e_\alpha))\psi(a)]\\
&= \lim_\alpha \varphi(a)\varphi(e_\alpha^2)-\varphi(e_\alpha^2)\psi(a)\\
&= \lim_\alpha \varphi(ae_\alpha^2)-\varphi(e_\alpha^2)\psi(a)\\
&= \lim_\alpha \psi(a)\varphi(e_\alpha^2)-\varphi(e_\alpha^2)\psi(a)=0.
\end{aligned}$$

Therefore, $D(A)\subseteq ker(\pi^{**})=(ker\pi)^{**}$. So there exists $N\in(ker\pi)^{**}$ such that $Da(a)=\varphi(a).N-N.\psi(a)$. Put $M=E-N$. Then

$$\begin{aligned}
\tilde{\psi}\circ\pi^{**}(M) &= \tilde{\psi}\circ\pi^{**}(E-N)=\tilde{\psi}\circ\pi^{**}(E)\\
&= w^*-\lim_\alpha \tilde{\psi}\circ\pi(\varphi(e_\alpha)\otimes\varphi(e_\alpha))\\
&= \lim_\alpha \psi\circ\varphi(e_\alpha^2)=1.
\end{aligned}$$

We now define $\theta:\ A\longrightarrow(A\hat{\otimes}A)^{**}$ by $a\mapsto\psi(a)\cdot M$ $(a\in A)$. Hence, for every $a\in A$,

$$\begin{aligned}
\tilde{\psi}\circ\pi^{**}\circ\theta\circ\varphi(a) &= \tilde{\psi}\circ\pi^{**}(\psi(\varphi(a))\cdot M)\\
&= \psi(\varphi(a)). \quad \square
\end{aligned}$$

Definition 3.2. *Let A be a Banach algebra, $\varphi\in Hom(A)$ and $\psi\in\Phi_A$. We say that A is $(\varphi,\psi)-$approximate biprojective if there is a net $\theta_\alpha : A\longrightarrow (A\hat{\otimes}A)(\alpha\in I)$ of continuous A-bimodule homomorphisms such that $\psi\circ\pi\circ\theta_\alpha\circ\varphi(a)\rightarrow\psi\circ\varphi(a)$.*

Let A be a biprojective Banach algebra. Then A is $(\varphi,\psi)-$approximate biprojective Banach algebra for every $\varphi \in Hom(A)$ and $\psi \in \Phi_A$.

Theorem 3.3. *Let A be a $(\varphi,\psi)-$approximate biprojective Banach algebra. If I is a closed ideal of A and $\varphi(I) \subset I$, then I is $(\varphi|_I, \psi|_I)-$ approximate biprojective.*

Proof. Suppose that $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)(\alpha \in I)$ satisfies $\psi \circ \pi_A \circ \theta_\alpha \circ \varphi(a) \mapsto \psi \circ \varphi(a)$ $(a \in A)$ and $i_0 \in I$ such that $\psi(i_0) = 1$. Let $T : A\hat{\otimes}A \longrightarrow I\hat{\otimes}I$ be defined by $a \otimes b \mapsto ai_0 \otimes i_0 b$ (since I is ideal, for every $a, b \in A$, then $ai_0, i_0 b \in I$). We define $\rho_\alpha = T \circ \theta_\alpha|_I$. Therefore

$$\begin{aligned}\psi \circ \pi_I \circ \rho_\alpha \circ \varphi(i) &= \psi \circ \pi_I \circ T \circ \theta_\alpha \circ \varphi(i)\\ &= \psi \circ \pi_A \circ \theta_\alpha \circ \varphi(i)\\ &\mapsto \psi \circ \varphi(i) \ \ (i \in I).\end{aligned}$$

Hence the proof is completes. $\square$

Theorem 3.4. *Suppose that A is a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. If A is $(\varphi,\psi)-$biflat, then A is $(\varphi,\psi)-$approximate biprojective.*

Proof. Assume that $\theta : A \longrightarrow (A \hat{\otimes} A)^{**}$ is a continuous A-bimodule map such that $\tilde{\psi} \circ \pi_A^{**} \circ \theta \circ \varphi(a) = \psi \circ \varphi(a)$ $(a \in A)$. By Goldstine's Theorem, there exists $(\theta_\alpha) \subset B(A, A \hat{\otimes} A)$ such that $\theta = w^* - \lim_\alpha \theta_\alpha$. For every $a \in A$ we have

$$w^* - \lim_\alpha \pi_A \circ \theta_\alpha \circ \varphi(a) = w^* - \lim_\alpha \pi_A^{**} \circ \theta_\alpha \circ \varphi(a) = \pi_A^{**} \circ \theta \circ \varphi(a).$$

So

$$\psi \circ \pi_A \circ \theta_\alpha \circ \varphi(a) \to \tilde{\psi} \circ \pi_A^{**} \circ \theta \circ \varphi(a).$$

Given $\varepsilon > 0$ and take $F = \{a_1, a_2, ..., a_r\} \subset A$. We put $M = \{\psi \circ \pi_A \circ T \circ \varphi(a_i) - \psi \circ \varphi(a_i) | T \in B(A, A \hat{\otimes} A)\}_{i=1,...,r}$. Applying Mazur's Theorem, we obtain a net $(\theta_{(F,\epsilon)}) \subset B(A, A \hat{\otimes} A)$ such that

$$\psi \circ \pi_A \circ \theta_{(F,\epsilon)} \circ \varphi(a) \longrightarrow \psi \circ \varphi(a)$$

So, A is $(\varphi,\psi)-$approximate biprojective. $\square$

Definition 3.5. *Let A be a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. We say that A is $(\varphi,\psi)-$ pseudo amenable if A admit a $(\varphi,\psi)-$ approximate diagonal, i.e., there is a net $(m_\alpha) \subset A \hat{\otimes} A$ (not necessary bounded) such that $m_\alpha \cdot \varphi(a) - \varphi(a) \cdot m_\alpha \longrightarrow 0$ and $\psi \circ \pi(m_\alpha) \longrightarrow 1$ $(a \in A)$.*

Theorem 3.6. *Suppose that A is a Banach algebra, $\varphi \in Hom(A)$ and $\psi \in \Phi_A$. If A^{**} is $(\varphi^{**}, \tilde{\psi})-$ pseudo amenable, then A is $(\varphi,\psi)-$ pseudo amenable.*

Proof. Let $(\tilde{m}_\alpha)$ be a $(\varphi, \psi)-$approximate diagonal for A^{**}. Then for every $a \in A$ we have $(\tilde{m}_\alpha \cdot \varphi(a) - \varphi(a) \cdot \tilde{m}_\alpha \to 0$ and $\psi \circ \pi(\tilde{m}_\alpha) \to 1$. By Goldstine's Theorem there is a net $(\tilde{m}_\alpha)$ in $(A \hat{\otimes} A)$, and we can replace weak* convergence in the above two limit by weak convergence. This implies, by Mazur's Theorem, that A is $(\varphi, \psi)-$ pseudo amenable. □

Theorem 3.7. *Suppose that A is a Banach algebra with an approximate identity. Then A is $(\varphi, \psi)-$ pseudo amenable if and only if A is $\varphi-$ approximate biprojective.*

Proof. Let $(e_\beta)_{\beta \in I}$ be an approximate identity for A and suppose that $\theta_\alpha : A \longrightarrow (A \hat{\otimes} A)$ $(\alpha \in \triangle)$ satisfies $\psi \circ \pi \circ \theta_\alpha \circ \varphi(a) \to \psi \circ \varphi(a)$ $(a \in A)$. Let $E = I \times \triangle^I$ be directed by the product ordering and for each $\lambda = (\beta, \alpha) \in E$, define $m_\lambda = \theta_\alpha(\varphi(e_\beta))$. Using the iterated limit theorem [7, Theorem 2.4], we get

$$\lim_\lambda (m_\lambda \cdot \varphi(a) - \varphi(a) \cdot m_\lambda) = 0 \quad (a \in A),$$

and also

$$\begin{aligned} \lim_\lambda \psi \circ \pi(m_\lambda) &= \lim_\lambda \psi \circ \pi(\theta_\alpha(\varphi(e_\beta))) \\ &= \lim_\lambda \psi \circ \varphi(e_\beta) = 1. \end{aligned}$$

That is, A is $(\varphi, \psi)-$ pseudo amenable. Conversely, let (m_β) be a $(\varphi, \psi)-$approximate diagonal for A and define $\theta_\beta : A \longrightarrow (A \hat{\otimes} A)$ by $a \mapsto a \cdot m_\beta$. Then for every $a \in A$ we have

$$\begin{aligned} \psi \circ \pi \circ \theta_\beta \circ \varphi(a) &= \psi \circ \pi \circ (\varphi(a) \cdot m_\beta) \\ &= \psi \circ \varphi(a) \psi \circ \pi(m_\beta) \\ &\to \psi \circ \varphi(a). \quad \square \end{aligned}$$

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