Journal of Mathematical Extension
Vol. 14, No. 4, (2020), 101-112
ISSN: 1735-8299
URL: http://www.ijmex.com
Original Research Paper

Regular Fuzzy Closed Sets and Extensions of a Double Fuzzy Topological Space

Vivek S
St. Thomas College

S. C. Mathew*
St. Thomas College

Abstract. This paper investigates how the extensions of a double fuzzy topological space affect its regular fuzzy closed sets. While some type of extensions leave all the regular fuzzy closed sets intact, there are regular fuzzy closed sets which remain so under all its extensions.

AMS Subject Classification: 54A40
Keywords and Phrases: Double fuzzy topology, extension, regular fuzzy closed set

1. Introduction

The concept of intuitionistic fuzzy sets was introduced by K. Attanassov [1] in 1983. In 2001, E. P. Lee and Y. B. Im [7] introduced the concept of mated fuzzy topological spaces as a generalization of intuitionistic fuzzy topological spaces introduced by Coker [4] and smooth topological spaces by Ramadan [14]. Also they introduced the notion of (r,s)-fuzzy open set, (r,s)-fuzzy closed set, closure operator and interior operator in mated fuzzy topological spaces. Then, as a generalization of the regular fuzzy open and regular fuzzy closed sets introduced by Azad [2] in 1981, Ramadan et.al. [15] brought the concept of (r,s)-regular fuzzy closed sets in intuitionistic fuzzy topological spaces.

Conforming to the view of J. G. Garcia and S.E. Rodabaugh [5], that Intuitionistic Fuzzy Sets by definition cannot be Intuitionistic Mathematics, scholars started to use the term “double fuzzy topological spaces” instead of “intuitionistic fuzzy topological spaces”.

Later in 2011, Ghareeb [6] introduced and studied various notions of normality using regular fuzzy closed sets in a double fuzzy topological space. Regular

Received: December 2018; Accepted: October 2019
*Corresponding author

fuzzy closed sets were further explored and generalized by several authors [3, 11, 12, 16].

In [8] Levine introduced the concept of simple extension of a topological space and studied various properties of the same. Extending this notion to fuzzy topological context, many papers came out later such as [9, 10]. Recently the authors defined extensions of a double fuzzy topological space and investigated certain properties of it in [17].

In this paper we compare the families of (r, s)-rfc sets in a double fuzzy topological space and its extensions. Though these families are different they are closely related as there is a non-empty intersection always. Certain type of extensions in which the family of (r, s)-rfc sets remain unchanged are obtained. Given a fuzzy set f in a double fuzzy topological space, extensions that make f, (r, s)-rfc are found. Investigating the structure of various families of fuzzy sets related to (r, s)-rfc sets, a complemented lattice has been identified.

2. Preliminaries

Throughout the paper, X denotes a nonempty set, $I = [0, 1]$, the closed unit interval of the real line, $I_0 = (0, 1]$, $I_1 = [0, 1)$, I^X = the set of all fuzzy subsets of X. The constant fuzzy subset taking the value α is denoted by $\underline{\alpha}$. Also, $I_0 \oplus I_1$denotes the set $\{(r, s) \in I_0 \times I_1 : r + s \leqslant 1\}$.

Definition 2.1. (see [13]) *Consider the pair (τ, τ^*) of functions from $I^X \to I$ such that*

1. $\tau(f) + \tau^*(f) \leqslant 1, \forall f \in I^X$,
2. $\tau(\underline{0}) = \tau(\underline{1}) = 1, \tau^*(\underline{0}) = \tau^*(\underline{1}) = 0$,
3. $\tau(f_1 \wedge f_2) \geqslant \tau(f_1) \wedge \tau(f_2)$ *and* $\tau^*(f_1 \wedge f_2) \leqslant \tau^*(f_1) \vee \tau^*(f_2), f_i \in I^X, i = 1, 2$,
4. $\tau\Big(\bigvee_{i\in\Delta} f_i\Big) \geqslant \bigwedge_{i\in\Delta} \tau(f_i)$ *and* $\tau^*\Big(\bigvee_{i\in\Delta} f_i\Big) \leqslant \bigvee_{i\in\Delta} \tau^*(f_i), f_i \in I^X, i \in \Delta$

The pair (τ, τ^*) is called a double fuzzy topology on X. The triplet (X, τ, τ^*) is called a double fuzzy topological space.

Notation: For a given $g \in I^X$ and for any $f \in I^X$, $R_g f$ denotes the set $\big\{(f_1, f_2) : f = f_1 \vee (f_2 \wedge g), f_1, f_2 \in I^X\big\}$.

Definition 2.2.(see [17]) *Let (X, τ, τ^*) be a double fuzzy topological space and $g \in I^X$. For $\alpha \in I_0$ and $\beta \in I_1$ with $\alpha \geqslant \tau(g)$, $\beta \leqslant \tau^*(g)$ and $\alpha + \beta \leqslant 1$ define $\mathcal{U}, \mathcal{U}^* : I^X \to I$ by*

1. $\mathcal{U}(g) = \alpha$ *and* $\mathcal{U}^*(g) = \beta$.

2. For all $f \in I^X - \{g\}$

$$\mathcal{U}(f) = \max\left\{\tau(f), \bigvee\{\tau(f_1) \wedge \tau(f_2) \wedge \alpha : (f_1, f_2) \in R_g f\ \}\right\}$$
$$\mathcal{U}^*(f) = \min\left\{\tau^*(f), \bigwedge\{\tau^*(f_1) \vee \tau^*(f_2) \vee \beta : (f_1, f_2) \in R_g f\ \}\right\}$$

Then the double fuzzy topological space $(X, \mathcal{U}, \mathcal{U}^*)$ *is said to be the* (g, α, β)*-extension of* (X, τ, τ^*).

Lee and Im [7] introduced the concept of fuzzy open set and fuzzy closed set in a double fuzzy topological space.

Definition 2.3. (see [7]) *Let* (X, τ, τ^*) *be a double fuzzy topological space. For* $(r, s) \in I_0 \oplus I_1$, *a fuzzy set* f *is called an* (r, s)*-fuzzy open (*(r, s)*-fo, for short) if* $\tau(f) \geqslant r$ *and* $\tau^*(f) \leqslant s$. *A fuzzy set* f *is called an* (r, s)*-fuzzy closed (*(r, s)*-fc, for short) set if and only if* f^c *is an* (r, s)*-fo set.*

Definition 2.4. (see [7]) *Let* (X, τ, τ^*) *be a double fuzzy topological space. For each* $(r, s) \in I_0 \oplus I_1, f \in I^X$ *the operator* $C_{\tau,\tau^*} : I^X \times\ I_0 \oplus I_1 \to I^X$ *defined by*

$$C_{\tau,\tau^*}(f, r, s) = \bigwedge\left\{g \in I^X | f \leqslant g, \tau(g^c) \geqslant r, \tau^*(g^c) \leqslant s\right\}$$

is called the double fuzzy closure operator on (X, τ, τ^*).

Definition 2.5. (see [7]) *Let* (X, τ, τ^*) *be a double fuzzy topological space. For each* $(r, s) \in I_0 \oplus I_1, f \in I^X$ *the operator* $I_{\tau,\tau^*} : I^X \times\ I_0 \oplus I_1 \to I^X$ *defined by*

$$I_{\tau,\tau^*}(f, r, s) = \bigvee\left\{g \in I^X | f \geqslant g, \tau(g) \geqslant r, \tau^*(g) \leqslant s\right\}$$

is called the double fuzzy interior operator on (X, τ, τ^*).

Theorem 2.6. (see [7]) *Let* (X, τ, τ^*) *be a double fuzzy topological space. Then for any* $(r, s) \in I_0 \oplus I_1$, $I_{\tau,\tau^*}(f^c, r, s) = \big(C_{\tau,\tau^*}(f, r, s)\big)^c$.

Later in 2005, Ramadan et.al. [15] introduced the concepts of regular fuzzy open set and regular fuzzy closed set in mated fuzzy topological spaces which when restricted to double fuzzy topological spaces give the following:

Definition 2.7. *Let* (X, τ, τ^*) *be a double fuzzy topological space,* $f \in I^X$, $(r, s) \in I_0 \oplus I_1$. *Then*

1. f *is called* (r, s)*-regular fuzzy open (or* (r, s)*-rfo) if* $f = I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(f, r, s), r, s\big)$.

2. f is called (r,s)-regular fuzzy closed (or (r,s)-rfc) if
$f = C_{\tau,\tau^*}\big(I_{\tau,\tau^*}(f,r,s),r,s\big)$.

Remark 2.8. *Let (X,τ,τ^*) be a double fuzzy topological space, $f \in I^X$ and $(r,s) \in I_0 \oplus I_1$. Then, f is (r,s)-rfc $\Leftrightarrow$ f^c is (r,s)-rfo.*

3. Regular Fuzzy Open and Regular Fuzzy Closed Sets in Extensions of a DFTS

Given an extension $(X,\mathcal{U},\mathcal{U}^*)$ of a double fuzzy topological space (X,τ,τ^*), it follows easily that $\tau(f) \leqslant \mathcal{U}(f)$ and $\tau^*(f) \geqslant \mathcal{U}^*(f)$. Consequently, $I_{\tau,\tau^*}(f,r,s) \leqslant I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f$ and $f \leqslant C_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant C_{\tau,\tau^*}(f,r,s)$ $\forall f \in I^X$, $(r,s) \in I_0 \oplus I_1$. However, an (r,s)-rfc set in a double fuzzy topological space need not be so in an extension of it. That is, the property of being (r,s)-rfc in a double fuzzy topological space is not invariant under taking extensions.

Now the following two questions arise naturally:

1. Can we find a class of (r,s)-rfc sets in a double fuzzy topological space (X,τ,τ^*) which remain (r,s)-rfc in all the extensions of (X,τ,τ^*)?
2. What kind of extensions will retain all the (r,s)-rfc sets as (r,s)-rfc?

This study attempts to answer these questions and related ones.

Notation: Let (X,τ,τ^*) be a double fuzzy topological space and $(r,s) \in I_0 \oplus I_1$. Denote by,

$C_{\tau,r,s}$ = the collection of all (r,s)-fuzzy closed sets,

$C'_{\tau,r,s}$ = the collection of all (r,s)-regular fuzzy closed sets,

$O_{\tau,r,s}$ = the collection of all (r,s)-fuzzy open sets and

$O'_{\tau,r,s}$ = the collection of all (r,s)-regular fuzzy open sets.

$\mathscr{D}_{\tau,r,s} = O_{\tau,r,s} \cap C_{\tau,r,s}$

With respect to the above notations, a straightforward observation given in [15] takes the following form:

Theorem 3.1. [15] *Let (X,τ,τ^*) be a double fuzzy topological space and $(r,s) \in I_0 \oplus I_1$. Then, $O'_{\tau,r,s} \subseteq O_{\tau,r,s}$ and $C'_{\tau,r,s} \subseteq C_{\tau,r,s}$.*

The following theorem is an easy consequence of Definition 2.4 and Definition 2.5.

Theorem 3.2. *Let* (X,τ,τ^*) *be a double fuzzy topological space. Then for any* $(r,s)\in I_0\oplus I_1$,

(i). $f\in C_{\tau,r,s}$ *if and only if* $C_{\tau,\tau^*}(f,r,s)=f$.

(ii). $f\in O_{\tau,r,s}$ *if and only if* $I_{\tau,\tau^*}(f,r,s)=f$.

The following Theorem shows that a fuzzy set is both (r,s)-rfc and (r,s)-rfo if and only if it is both (r,s)-fc and (r,s)-fo.

Theorem 3.3. $\mathscr{D}_{\tau,r,s}=O_{\tau,r,s}\cap C_{\tau,r,s}=O'_{\tau,r,s}\cap C'_{\tau,r,s}$ *for any* $(r,s)\in I_0\oplus I_1$.

Proof. By Theorem 3.1, we have $O'_{\tau,r,s}\subseteq O_{\tau,r,s}$ and $C'_{\tau,r,s}\subseteq C_{\tau,r,s}$.
For the reverse implication, let $f\in O_{\tau,r,s}\cap C_{\tau,r,s}$. Then by Theorem 3.2
$I_{\tau,\tau^*}\left(C_{\tau,\tau^*}\left(f,r,s\right),r,s\right)=I_{\tau,\tau^*}\left(f,r,s\right)=f$ and $C_{\tau,\tau^*}\left(I_{\tau,\tau^*}\left(f,r,s\right),r,s\right)=C_{\tau,\tau^*}\left(f,r,s\right)=f$.

This shows that, $f\in O'_{\tau,r,s}\cap C'_{\tau,r,s}$ as desired. □

As expected, (r,s)-fc sets and (r,s)-fo sets in a double fuzzy topological space will remain so in any extension of the space. Consequently we have,

Theorem 3.4. *Let* (X,τ,τ^*) *a double fuzzy topological space and* $(X,\mathcal{U},\mathcal{U}^*)$ *be an extension of it. Then,* $O_{\tau,r,s}\subseteq O_{\mathcal{U},r,s}$ *and* $C_{\tau,r,s}\subseteq C_{\mathcal{U},r,s}$ *for any* $(r,s)\in I_0\oplus I_1$.

Remark 3.5. *Converse of the above theorem is not true. Consider the* (g,α,β)*-extension* $(X,\mathcal{U},\mathcal{U}^*)$ *of the double fuzzy topological space* (X,τ,τ^*) *where* $\alpha>\tau(g)$ *and* $\beta<\tau^*(g)$. *Then we have* $g\notin O_{\tau,\alpha,\beta}$ *and* $g^c\notin C_{\tau,\alpha,\beta}$. *But, it follows easily that* $g\in O_{\mathcal{U},\alpha,\beta}$ *and* $g^c\in C_{\mathcal{U},\alpha,\beta}$.

Remark 3.6. *Moreover, the result given by Theorem 3.4 does not hold for* $O'_{\tau,r,s}$ *and* $C'_{\tau,r,s}$. *That is, it is not true in general that* $O'_{\tau,r,s}\subseteq O'_{\mathcal{U},r,s}$ *and* $C'_{\tau,r,s}\subseteq C'_{\mathcal{U},r,s}$. *The following example illustrates this.*

Example 3.7. Let $X=\{a,b\}$ and consider the following fuzzy subsets of X.
$f_1=\left(\frac{1}{3},\frac{2}{5}\right),f_2=\left(\frac{2}{3},\frac{3}{5}\right),f_3=\left(\frac{4}{7},\frac{4}{7}\right),f_4=\left(\frac{3}{7},\frac{3}{7}\right),f_5=\left(\frac{1}{2},\frac{9}{20}\right),$
$f_6=\left(\frac{1}{2},\frac{11}{20},\right),f_7=\left(\frac{1}{3},\frac{3}{10},\right),\ f_8=\left(\frac{2}{3},\frac{7}{10},\right)$ where $f=(p,q)\Rightarrow$

$$f(x)=\begin{cases} p, & \text{if } x=a \\ q, & \text{if } x=b \end{cases}$$

Now, define a double fuzzy topology (τ,τ^*) on X as follows:

$$\tau(f)=\begin{cases}1, & \text{if } f\in\{\underline{0},\underline{1}\}\\ \frac{1}{3}, & \text{if } f=f_1\\ \frac{3}{10}, & \text{if } f=f_4\\ \frac{1}{4}, & \text{if } f=f_5\\ \frac{3}{5}, & \text{if } f\in\{f_7,f_8\}\\ 0, & \text{elsewhere}\end{cases} \quad\text{and}\quad \tau^*(f)=\begin{cases}0, & \text{if } f\in\{\underline{0},\underline{1}\}\\ \frac{2}{3}, & \text{if } f=f_1\\ \frac{7}{10}, & \text{if } f=f_4\\ \frac{3}{4}, & \text{if } f=f_5\\ \frac{2}{5}, & \text{if } f\in\{f_7,f_8\}\\ 1, & \text{elsewhere}\end{cases}$$

Then, $I_{\tau,\tau^*}(f_3,\frac{3}{10},\frac{7}{10})=f_1\vee f_4\vee f_7\vee\underline{0}=f_4$ and

$$C_{\tau,\tau^*}\left(I_{\tau,\tau^*}\left(f_3,\frac{3}{10},\frac{7}{10}\right),\frac{3}{10},\frac{7}{10}\right)=C_{\tau,\tau^*}\left(f_4,\frac{3}{10},\frac{7}{10}\right)=f_2\wedge f_3\wedge f_8\wedge\underline{1}=f_3.$$

i.e., $f_3\in C'_{\tau,\frac{3}{10},\frac{7}{10}}$.

Again for the set f_5, $C_{\tau,\tau^*}(f_5,\frac{1}{4},\frac{3}{4})=f_2\wedge f_3\wedge f_6\wedge f_8\wedge\underline{1}=f_6$ and

$$I_{\tau,\tau^*}\left(C_{\tau,\tau^*}\left(f_5,\frac{1}{4},\frac{3}{4}\right),\frac{1}{4},\frac{3}{4}\right)=I_{\tau,\tau^*}\left(f_6,\frac{1}{4},\frac{3}{4}\right)=f_1\vee f_4\vee f_5\vee f_7\vee\underline{0}=f_5.$$

Consequently, $f_5\in O'_{\tau,\frac{1}{4},\frac{3}{4}}$.

Also, note that $C_{\tau,\tau^*}(f_1,\frac{1}{4},\frac{3}{4})=f_2\wedge f_3\wedge f_6\wedge f_8\wedge\underline{1}=f_6$ and

$$\begin{aligned}I_{\tau,\tau^*}\left(C_{\tau,\tau^*}\left(f_1,\frac{1}{4},\frac{3}{4}\right),\frac{1}{4},\frac{3}{4}\right) &= I_{\tau,\tau^*}\left(f_6,\frac{1}{4},\frac{3}{4}\right)=f_1\vee f_4\vee f_5\vee f_7\vee\underline{0}\\ &= f_5\neq f_1.\end{aligned}$$

Consequently, $f_1\notin O'_{\tau,\frac{1}{4},\frac{3}{4}}$. Further,

$$C_{\tau,\tau^*}\left(I_{\tau,\tau^*}\left(f_2,\frac{1}{4},\frac{3}{4}\right),\frac{1}{4},\frac{3}{4}\right)=C_{\tau,\tau^*}\left(f_5,\frac{1}{4},\frac{3}{4}\right)=f_6\neq f_2$$

so that $f_2\notin C'_{\tau,\frac{1}{4},\frac{3}{4}}$.

Moreover, $f_7,f_8\in C'_{\tau,\frac{1}{4},\frac{3}{4}}$ since $C_{\tau,\tau^*}\left(I_{\tau,\tau^*}\left(f_7,\frac{1}{4},\frac{3}{4}\right),\frac{1}{4},\frac{3}{4}\right)=$ $C_{\tau,\tau^*}\left(f_7,\frac{1}{4},\frac{3}{4}\right)=f_7$ and $C_{\tau,\tau^*}\left(I_{\tau,\tau^*}\left(f_8,\frac{1}{4},\frac{3}{4}\right),\frac{1}{4},\frac{3}{4}\right)=C_{\tau,\tau^*}\left(f_8,\frac{1}{4},\frac{3}{4}\right)=f_8$. Similarly, both $f_7,f_8\in O'_{\tau,\frac{1}{4},\frac{3}{4}}$.

Let $g=\left(\underline{\frac{1}{2}}\right)$, $\alpha=\frac{4}{7},\beta=\frac{3}{7}$ and $(X,\mathcal{U},\mathcal{U}^*)$ be the (g,α,β)-extension of (X,τ,τ^*). Then $\mathcal{U}$ and $\mathcal{U}^*$ are given by:

$$\mathcal{U}(f)=\begin{cases}1, & \text{if } f\in\{\underline{0},\underline{1}\}\\ \frac{1}{3}, & \text{if } f=f_1\\ \frac{3}{10}, & \text{if } f=f_4\\ \frac{1}{4}, & \text{if } f=f_5\\ \frac{4}{7}, & \text{if } f=g\\ \frac{3}{5}, & \text{if } f\in\{f_7,f_8\}\\ 0, & \text{elsewhere}\end{cases} \quad \text{and} \quad \mathcal{U}^*(f)=\begin{cases}0, & \text{if } f\in\{\underline{0},\underline{1}\}\\ \frac{2}{3}, & \text{if } f=f_1\\ \frac{7}{10}, & \text{if } f=f_4\\ \frac{3}{4}, & \text{if } f=f_5\\ \frac{3}{7}, & \text{if } f=g\\ \frac{2}{5}, & \text{if } f\in\{f_7,f_8\}\\ 1, & \text{elsewhere}\end{cases}$$

Now, $I_{\mathcal{U},\mathcal{U}^*}(f_3,\frac{3}{10},\frac{7}{10})=f_1\vee f_4\vee g\vee\underline{0}=g$ and

$$\begin{aligned}C_{\mathcal{U},\mathcal{U}^*}\left(I_{\mathcal{U},\mathcal{U}^*}\left(f_3,\frac{3}{10},\frac{7}{10}\right),\frac{3}{10},\frac{7}{10}\right) &= C_{\mathcal{U},\mathcal{U}^*}\left(g,\frac{3}{10},\frac{7}{10}\right)\\ &= f_2\wedge f_3\wedge g\wedge\underline{1}=g\neq f_3\end{aligned}$$

so that $f_3\notin C'_{\mathcal{U},\frac{3}{10},\frac{7}{10}}$. Hence, $C'_{\tau,\frac{3}{10},\frac{7}{10}}\not\subseteq C'_{\mathcal{U},\frac{3}{10},\frac{7}{10}}$.

Also, $C_{\mathcal{U},\mathcal{U}^*}(f_5,\frac{1}{4},\frac{3}{4})=f_2\wedge f_3\wedge f_6\wedge g\wedge\underline{1}=g$ and

$$\begin{aligned}I_{\mathcal{U},\mathcal{U}^*}\left(C_{\mathcal{U},\mathcal{U}^*}\left(f_5,\frac{1}{4},\frac{3}{4}\right),\frac{1}{4},\frac{3}{4}\right) &= I_{\mathcal{U},\mathcal{U}^*}\left(g,\frac{1}{4},\frac{3}{4}\right)\\ &= f_1\vee f_4\vee f_5\vee g\vee\underline{0}=g\neq f_5.\end{aligned}$$

Consequently, $f_5\notin O'_{\mathcal{U},\frac{1}{4},\frac{3}{4}}$ so that $O'_{\tau,\frac{1}{4},\frac{3}{4}}\not\subseteq O'_{\mathcal{U},\frac{1}{4},\frac{3}{4}}$.

For $r=\frac{1}{4}$ and $s=\frac{3}{4}$, the discussions above can be portrayed as in Figure 1 and Figure 2.

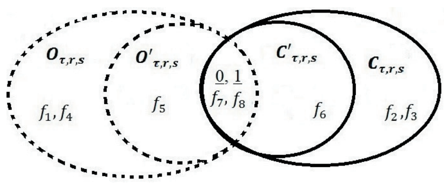

Figure 1. Inclusion structure of various sets in (X,τ,τ^*) of Example 3.7.

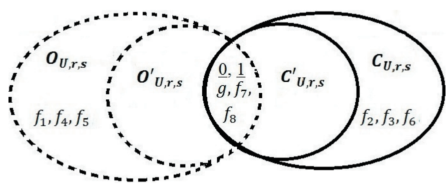

Figure 2. Inclusion structure of various sets in $(X,\mathcal{U},\mathcal{U}^*)$ of Example 3.7.

However, there may arise a situation in which $C'_{\tau,r,s} \subsetneqq C'_{\mathcal{U},r,s}$. For instance, take $r = \frac{4}{7}$ and $s = \frac{3}{7}$ in the above example. But, the reverse inclusion will never take place. That is, $C'_{\mathcal{U},r,s}$ will never be a proper subset of $C'_{\tau,r,s}$ as proved below.

Theorem 3.8. *Let* $(X,\mathcal{U},\mathcal{U}^*)$ *be an extension of a double fuzzy topological space* (X,τ,τ^*)*. Then,* $C'_{\mathcal{U},r,s} \not\subset C'_{\tau,r,s}$ *for any* $(r,s) \in I_0 \oplus I_1$*.*

Proof. Let $C'_{\mathcal{U},r,s} \subsetneqq C'_{\tau,r,s}$. Then,

$$\begin{aligned} f \in C'_{\tau,r,s} \setminus C'_{\mathcal{U},r,s} \\ &\Rightarrow C_{\mathcal{U},\mathcal{U}^*}\big(I_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s\big) \neq f \\ &\Rightarrow \exists f_1 \in C_{\mathcal{U},r,s} \text{ such that } C_{\mathcal{U},\mathcal{U}^*}\big(I_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s\big) = f_1 \lneqq f \\ &\Rightarrow \exists f_1 \in C_{\mathcal{U},r,s} \text{ such that } I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 \lneqq f \\ &\Rightarrow I_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) = I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 \\ &\Rightarrow C_{\mathcal{U},\mathcal{U}^*}\big(I_{\mathcal{U},\mathcal{U}^*}(f_1,r,s),r,s\big) = f_1 \\ &\Rightarrow f_1 \in C'_{\mathcal{U},r,s} \end{aligned}$$

Again, $f_1 \notin C_{\tau,r,s}$. For, $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 \lneqq f \Rightarrow I_{\tau,\tau^*}(f,r,s) \leqslant I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 \lneqq f$ so that $f_1 \in C_{\tau,r,s} \Rightarrow f = C_{\tau,\tau^*}\big(I_{\tau,\tau^*}(f,r,s),r,s\big) \leqslant f_1 \lneqq f$, a contradiction.

Thus $f_1 \in C'_{\mathcal{U},r,s} \setminus C'_{\tau,r,s}$, contradicting the assumption that $C'_{\mathcal{U},r,s} \subset C'_{\tau,r,s}$. □

Now, the following theorem gives a class of (r,s)-rfc sets which remain so under extensions.

Theorem 3.9. *Let* $(X,\mathcal{U},\mathcal{U}^*)$ *be an extension of a double fuzzy topological space* (X,τ,τ^*)*. Then for any* $(r,s) \in I_0 \oplus I_1$*,* $\mathscr{D}_{\tau,r,s} \subseteq \mathscr{D}_{\mathcal{U},r,s}$*.*

Proof. $f \in \mathscr{D}_{\tau,r,s} \Rightarrow f \in C_{\tau,r,s}$ and $f \in O_{\tau,r,s}$. Also, $f \in C_{\tau,r,s} \Rightarrow f^c \in O_{\tau,r,s}$ and $f \in O_{\tau,r,s} \Rightarrow f^c \in C_{\tau,r,s}$. Hence, $f, f^c \in O_{\tau,r,s} \cap C_{\tau,r,s}$.

Again from definition of extension we have, $\mathcal{U}(f) \geqslant \tau(f)$ and $\mathcal{U}^*(f) \leqslant \tau^*(f)$ for all $f \in I^X$. Therefore, $f, f^c \in O_{\mathcal{U},r,s} \cap C_{\mathcal{U},r,s}$. i.e., $f, f^c \in \mathscr{D}_{\mathcal{U},r,s}$. □

Corollary 3.10. $\mathscr{D}_{\tau,r,s} \subseteq C'_{\tau,r,s} \cap C'_{\mathcal{U},r,s}$.

Remark 3.11. *In general,* $\mathscr{D}_{\mathcal{U},r,s} \nsubseteq C'_{\tau,r,s} \cap C'_{\mathcal{U},r,s}$*. For instance, in Example 3.7* $\mathscr{D}_{\mathcal{U},\frac{1}{4},\frac{3}{4}} = \{\underline{0},\underline{1},f_7,f_8,g\} \nsubseteq \{\underline{0},\underline{1},f_7,f_8\} = C'_{\tau,\frac{1}{4},\frac{3}{4}} \cap C'_{\mathcal{U},\frac{1}{4},\frac{3}{4}}$*. However, Corollary 3.13 below provides a sufficient condition for this desired result.*

There are extensions of a double fuzzy topological space which keeps the set of all (r,s)-rfc sets intact. The following theorem accounts for one such situation.

Theorem 3.12. *Let (X,τ,τ^*) be a double fuzzy topological space and $(X,\mathcal{U},\mathcal{U}^*)$ be its (g,α,β)-extension such that $\tau(g) \leqslant \alpha < r$ and $\tau^*(g) \geqslant \beta > s$, $(r,s) \in I_0 \oplus I_1$. Then, $C'_{\tau,r,s} = C'_{\mathcal{U},r,s}$ and $O'_{\tau,r,s} = O'_{\mathcal{U},r,s}$.*

Proof. Since $\tau(f_1) \wedge \tau(f_2) \wedge \alpha \leqslant \alpha < r$, $\tau^*(f_1) \vee \tau^*(f_2) \vee \beta \geqslant \beta > s$ for all $f_1, f_2 \in I^X$, we have $O_{\tau,r,s} = O_{\mathcal{U},r,s}$. Therefore, $C_{\tau,r,s} = C_{\mathcal{U},r,s}$.

Hence, $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) = I_{\tau,\tau^*}(f,r,s)$ and $C_{\mathcal{U},\mathcal{U}^*}(f,r,s) = C_{\tau,\tau^*}(f,r,s)$, for all $f \in I^X$.

Again, $C_{\mathcal{U},\mathcal{U}^*}\big(I_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s\big) = C_{\tau,\tau^*}\big(I_{\tau,\tau^*}(f,r,s),r,s\big)$ and $I_{\mathcal{U},\mathcal{U}^*}\big(C_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s\big) = I_{\tau,\tau^*}\big(C_{\tau,\tau^*}(f,r,s),r,s\big)$. Hence the proof. □

Corollary 3.13. *Let $(X,\mathcal{U},\mathcal{U}^*)$ be the (g,α,β)-extension of a double fuzzy topological space (X,τ,τ^*) with $\tau(g) \leqslant \alpha < r$ and $\tau^*(g) \geqslant \beta > s$ for some $(r,s) \in I_0 \oplus I_1$. Then, $\mathscr{D}_{\mathcal{U},r,s} \subseteq C'_{\tau,r,s} \cap C'_{\mathcal{U},r,s}$.*

Remark 3.14. *Converse of the above theorem is not true in general. For, consider the double fuzzy topological space (X,τ,τ^*) defined in Example 3.7. Let $\alpha = \frac{7}{25}, \beta = \frac{3}{5}$ and $g(x) = \begin{cases} \frac{1}{2} & \text{if } x = a \\ \frac{3}{7} & \text{if } x = b. \end{cases}$ Then, the (g,α,β)-extension $(\mathcal{U},\mathcal{U}^*)$ of (τ,τ^*) is defined by*

$$\mathcal{U}(f) = \begin{cases} 1, & \text{if } f \in \{\underline{0},\underline{1}\} \\ \frac{1}{3}, & \text{if } f = f_1 \\ \frac{3}{10}, & \text{if } f = f_4 \\ \frac{1}{4}, & \text{if } f = f_5 \\ \frac{7}{25}, & \text{if } f = g \\ \frac{3}{5}, & \text{if } f \in \{f_7, f_8\} \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \mathcal{U}^*(f) = \begin{cases} 0, & \text{if } f \in \{\underline{0},\underline{1}\} \\ \frac{2}{3}, & \text{if } f = f_1 \\ \frac{7}{10}, & \text{if } f = f_4 \\ \frac{3}{4}, & \text{if } f = f_5 \\ \frac{3}{5}, & \text{if } f = g \\ \frac{2}{5}, & \text{if } f \in \{f_7, f_8\} \\ 1, & \text{elsewhere} \end{cases}$$

Here, $C'_{\tau,\frac{1}{4},\frac{3}{4}} = \{\underline{0},\underline{1},f_6,f_7,f_8\} = C'_{\mathcal{U},\frac{1}{4},\frac{3}{4}}$; but $r \lneqq \alpha$ and $s \gneqq \beta$.

Now, it is quite natural to ask, given $f \in I^X$, does there exists an extension of (X,τ,τ^*) which makes f an (r,s)-rfc set? The following theorem answers this in the affirmative.

Theorem 3.15. *Let (X,τ,τ^*) be a double fuzzy topological space and $f \in I^X$ be such that $f \in C_{\tau,r,s} \setminus C'_{\tau,r,s}$ for some $(r,s) \in I_0 \oplus I_1$. Then there exists an extension $(X,\mathcal{U},\mathcal{U}^*)$ of (X,τ,τ^*) such that $f \in C'_{\mathcal{U},r,s}$.*

Proof. Let $g = f, \alpha = \tau(f^c), \beta = \tau^*(f^c)$ and define a (g,α,β)-extension

$(X,\mathcal{U},\mathcal{U}^*)$ of (X,τ,τ^*). Then, clearly $\mathcal{U}(f) \geqslant r$ and $\mathcal{U}^*(f) \leqslant s$. Therefore, $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) = f$ and $C_{\mathcal{U},\mathcal{U}^*}\big(I_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s\big) = f$. i.e., $f \in C'_{\mathcal{U},r,s}$. □

Now, we characterize the situation in which (r,s)-rfc sets in a space are so in its extension.

Theorem 3.16. *Let (X,τ,τ^*) be a double fuzzy topological space and $(X,\mathcal{U},\mathcal{U}^*)$ be its extension. Also, let $f \in C'_{\tau,r,s}$ for some $(r,s) \in I_0 \oplus I_1$. Then, $f \in C'_{\mathcal{U},r,s}$ if and only if $\nexists f_1 \in C_{\mathcal{U},r,s}$ such that $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 < f$.*

Proof. Suppose $\nexists f_1 \in C_{\mathcal{U},r,s}$ such that $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 < f$. Then, clearly

$$\left\{f_2 \in I^X : I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_2, f_2 \in C_{\mathcal{U},r,s}\right\} = \left\{f_2 \in I^X : f \leqslant f_2, f_2 \in C_{\mathcal{U},r,s}\right\}.$$

Again, $f \in C'\tau,r,s \Rightarrow f \in C_{\tau,r,s}$ and hence $f \in C_{\mathcal{U},r,s}$. Therefore, $f \in C'_{\mathcal{U},r,s}$.

Conversely, $f \in C'_{\mathcal{U},r,s} \Rightarrow C_{\mathcal{U},\mathcal{U}^*}\big(I_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s\big) = f \Rightarrow \forall f_1 \in C_{\mathcal{U},r,s}$, $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f \leqslant f_1$ or $f_1 \leqslant I_{\mathcal{U},\mathcal{U}^*}(f,r,s)$. i.e., $\nexists f_1 \in C_{\mathcal{U},r,s}$ such that $I_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant f_1 < f$. □

4. Conclusion

This study has brought out the interrelations among various families of fuzzy sets in a double fuzzy topological space and any extension of it. These families form a lattice $Ł^{r,s}_{\tau,\mathcal{U}}$ under set inclusion as shown below.

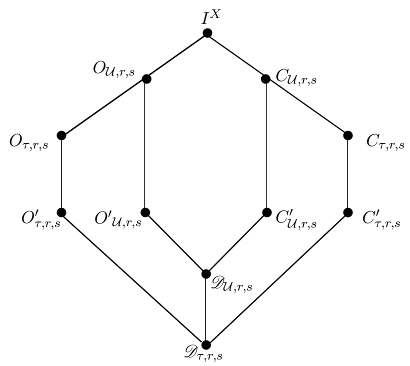

Figure 3. Hasse diagram of $Ł^{r,s}_{\tau,\mathcal{U}}$

References

[1] K. Attanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, 20 (1986), 87-96.

[2] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.*, 82 (1981), 14-32.

[3] J. P. Bajpai and S. S. Thakur, Intuitionistic fuzzy rgα-continuity, *Int. J. Contemp. Math. Sciences*, 6 (47) (2011), 2335-2351.

[4] D. Çoker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets Syst.*, 88 (1997), 81-89.

[5] J. Gutiérrez García and S. E. Rodabaugh, Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies, *Fuzzy Sets Syst.*, 156 (2005), 445-484.

[6] A. Ghareeb, Normality of double fuzzy topological spaces, *Appl. Math. Lett.*, 24 (2011), 533-540.

[7] E. P. Lee and Y. B. Im, Mated fuzzy topological spaces, *International Journal Fuzzy Logic and Intelligent Systems*, 11 (2) (2001), 161-165.

[8] N. Levine, Simple extensions of topologies, *Amer. Math. Monthly*, 71 (1964), 22-25.

[9] S. C. Mathew and T. P. Johnson, Generalized closed fuzzy sets and simple extensions of a fuzzy topology, *J. Fuzzy Math.*, 11 (1) (2003), 195-202.

[10] S. C. Mathew and T. P. Johnson, On simple extensions of fuzzy topologies, *J. Fuzzy Math.*, 12 (3) (2004), 581-590.

[11] F. M. Mohammed, M. S. M. Noorani, and A. Ghareeb, Generalized fuzzy b-closed and generalized $*$-fuzzy b-closed sets in double fuzzy topological spaces, *Egyptian Journal of Basic and Applied Sciences*, 3 (2016), 61-67.

[12] F. M. Mohammed and A. Ghareeb, More on generalized b-closed sets in double fuzzy topological spaces, *Songklanakarin J. Sci. Technol.*, 38 (1) (2016), 99-103.

[13] T. K. Mondal and S. K. Samanta, On intuitionistic gradation of openness, *Fuzzy Sets Syst.*, 131 (2002), 323-336.

[14] A. A. Ramadan, Smooth topological spaces, *Fuzzy Sets Syst.*, 48 (1992), 371-375.

[15] A. A. Ramadan, S. E. Abbas, and A. A. Abd El-latif, Compactness in intuitionistic fuzzy topological spaces, *Int. J. Math. Math. Sci.*, 1 (2005), 19-32.

[16] A. Vadivel and E. Elavarasan, Generalized regular fuzzy closed sets and maps in double fuzzy topological spaces, *J. Fuzzy. Math.*, 25 (3) (2017), 627-644.

[17] V. S. and S. C. Mathew, On the extensions of a double fuzzy topological space, *Journal of Advanced Studies in Topology*, 9 (1) (2018), 75-93.

Vivek S
Research Scholar
Department of Mathematics
St. Thomas College, Palai
Arunapuram P.O.
Kottayam - 686574,
Kerala, India.
E-mail: vivekmaikkattu@yahoo.com

Sunil C. Mathew
Associate Professor of Mathematics
Department of Mathematics
St. Thomas College, Palai
Arunapuram P.O.
Kottayam - 686574,
Kerala, India.
E-mail: sunilcmathew@gmail.com