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# The Category of Topological De Morgan Molecular Lattices

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**Abstract.** The concept of topological molecular lattices was introduced by Wang as a generalization of ordinary topological spaces, fuzzy topological spaces and $L$-fuzzy topological spaces in terms of closed elements, molecules, remote neighbourhoods and generalized order-homomorphisms. In our previous work, we introduced the concept of generalized topological molecular lattices in terms of open elements and investigated some properties of them. In this paper, we define and consider the category **TDML** whose objects are topological De Morgan molecular lattices and whose morphisms are continuous generalized order-homomorphisms such that its right adjoins preserve the pseudocomplement operation. We show that this category is complete and cocomplete. In particular, we characterize products, coproducts, equalizers and coequalizers. Also, we show that the category **TOP** of all topological spaces is a reflective and coreflective subcategory of **TDML**.

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# 1. Introduction

A completely distributive complete lattice is called a molecular lattice. In 1992, Wang introduced his important theory called topological molecular lattices as a generalization of ordinary topological spaces, fuzzy topological spaces and $L$-fuzzy topological spaces in terms of closed elements, molecules, generalized order-homomorphisms and remote neighbourhoods [13]. Then many authors characterized some topological notions in such spaces, such as convergence theories of molecular nets or ideals [5, 3], separation axioms [6, 8], generalized topological molecular lattices [7, 12, 14] and other notions.

In general topological lattice theory, since there are no concepts like complement or pseudocomplement, 'open' and 'closed' are not dual concepts. To be exact, when an open concept is given, we could not take for granted that a 'closed' concept will be surely found, and vice versa.

Throughout this paper for a molecular lattice $L$, a subset $\tau$ of $L$ is called a topology if it is closed under arbitrary joins, finite meets and $0, 1 \in \tau$, where 0 and 1 are the smallest and the greatest elements of $L$, respectively. Every element of a topology is called open. A subset $\lambda$ of $L$ is called a cotopology if it is closed under arbitrary meets, finite joins and $0, 1 \in \lambda$; and it is called a generalized cotopology on $L$ if it is only closed under arbitrary meets and $0, 1 \in \lambda$. Every element of a cotopology or a generalized cotopology is called closed.

In [12], we introduced the concept of a generalized topological molecular lattice (briefly, **gtml**) as a pair $(L, \tau)$, where $\tau$ is a topology on $L$. Let $a^*$ denote the pseudocomplement of an element $a$. Then for any molecular lattice $L$, the first De Morgan law $\left(\bigvee_{i\in I} a_i\right)^* = \bigwedge_{i\in I} a_i^*$ holds. Thus if $\tau$ is a topology on $L$, then the set $\tau^* := \{a^* \mid a \in \tau\}$ is a generalized cotopology on $L$, and hence we have two structures on a molecular lattice, topology and generalized cotopology which are not dual to each other.

In the following, we recall some definitions and properties of molecular lattices. For two molecular lattices $L_1$ and $L_2$, and a mapping $f : L_1 \to L_2$ which preserves arbitrary joins, let $\hat{f}$ denote the right adjoint of $f$, then $\hat{f} : L_2 \to L_1$ is defined by $\hat{f}(y) = \bigvee\{x \in L_1 \mid f(x) \leqslant y\}$ for every $y \in L_2$.

**Definition 1.1.** [13] *A map $f : L_1 \to L_2$ between molecular lattices is called a generalized order-homomorphism or an* **ml**-*map in this paper, if $f$ preserves arbitrary joins and its right adjoint is a complete homomorphism, i.e., $\hat{f}$ preserves arbitrary joins and arbitrary meets.*

**Lemma 1.2.** [1, 4] *Let $g : L_2 \to L_1$ be a complete homomorphism between molecular lattices. Then $g$ has a left adjoint $f : L_1 \to L_2$ and hence $f$ is an* **ml**-*map.*

**Definition 1.3.** [13] *An element $a$ of a lattice $L$ is called coprime, if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$, for every $b, c \in F$, and it is called completely coprime if, for every $S \subseteq F$, $a \leqslant \vee S$ implies $a \leqslant s$ for some $s \in S$.*

We denote by $M(L)$ and $\overline{M}(L)$ the set of all nonzero coprime elements and nonzero completely coprime elements of $F$, respectively. Nonzero coprime elements are also called molecules. If $L$ is a molecular lattice, then $L$ is $\vee$-generated by the set $M(L)$, i.e., every element of $L$ is a join of some elements of $M(L)$.

The category of all molecular lattices with **ml**-maps between them is denoted by **MOL** and the category of all topological molecular lattices in the sense of Wang with continuous **ml**-maps between them is denoted by **TML**. It is well known that these categories are complete and cocomplete and some categorical structures of them were introduced by many authors [2, 9, 11, 13, 14, 15, 16, 17]. In the following, readers are suggested to refer to [1] for some categorical notions.

**Definition 1.4.** [9] *Let $\{L_i\}_{i\in I}$ be a family of molecular lattices and $L_i^0 = L_i \backslash \{0\}$. An element $A$ in $\bigotimes_i L_i$ is defined as a subset in the direct product $\prod_i L_i^0$ subject to the following conditions:*

1. *$\downarrow A = A$, that is, if $\{y_i\}_{i\in I} \in \prod_i L_i^0$, $\{x_i\}_{i\in I} \in A$, and $\forall i \in I$, $x_i \geqslant y_i$, then $\{y_i\}_{i\in I} \in A$.*
2. *If $\emptyset \neq B_i \subset L_i^0$, and $\prod_i B_i \subset A$, then $b \in A$, where $b = \{b_i\}_{i\in I}$, $\forall i \in I$, $b_i = supB_i$.*

**Remark 1.5.** [9] *Let $\{L_i\}_{i\in I}$ be a family of molecular lattices. Then $\{(\bigotimes_i L_i, p_i) \mid i \in I\}$ is the product of $\{L_i\}_{i\in I}$ in* **MOL**, *where the order* *in $\bigotimes_i L_i$ is the usual inclusion relation in set theory. The $\wedge_{j\in J} A_j$ in $\bigotimes_i L_i$ is the intersection $\cap_{j\in J} A_j$ and the $\vee_{j\in J} A_j$ is as follows:*

$$\bigvee_{j\in J} A_j = \{\{b_i\}_{i\in I} \mid \exists B_i \subset L_i^0, i \in I \quad s.t. \quad \prod_i B_i \subset \bigcup_{j\in J} A_j \quad and \quad \vee B_i = b_i\}.$$

*The projection mapping $p_{i_0} : \bigotimes_i L_i \to L_{i_0}$ is defined as follows:*

$$p_{i_0}(A) = \bigvee\{x_{i_0} \mid \{x_i\}_{i\in I} \in A\} \tag{1}$$

*If $x_i \in L_i^0$ for each $i \in I$, then $\downarrow \{x_i\}_{i\in I} \in \bigotimes_i L_i$, where $\downarrow \{x_i\}_{i\in I}$ is the lower set in $\prod_i L_i^0$ generated by $\{x_i\}_{i\in I}$. The coprime elements of $\bigotimes_i L_i$ are as follows:*

$$M = \{\downarrow \{m_i\}_{i\in I} \in \bigotimes_i L_i \mid m_i \in M(L_i), i \in I\}.$$

## 2. The Category TDML

In this section, we define the category **TDML** and show that the category **TOP** of all topological spaces is a reflective and coreflective subcategory of **TDML**.

**Definition 2.1.** *An element $m$ of a molecular lattice $L$ is called $*$-coprime, if for every $x \in L$, either $m \leqslant x$ or $m \leqslant x^*$.*

We denote by $\widetilde{M}(L)$ the set of all nonzero $*$-coprime elements of $L$. It is easy to show that $\widetilde{M}(L) \subseteq \overline{M}(L) \subseteq M(L)$. For any topological space $(X, \tau)$, let $\mathcal{P}(X)$ denote the powerset of $X$. Then the pseudocomplement on $\mathcal{P}(X)$ is the subset complement, $(\mathcal{P}(X), \tau)$ is a topological molecular lattice and $\widetilde{M}(\mathcal{P}(X)) = \overline{M}(\mathcal{P}(X)) = M(\mathcal{P}(X)) = \{\{x\} \mid x \in X\}$. In general, we have $\widetilde{M}(L)$ is a join generating base for a molecular lattice $L$ if and only if $L$ is a molecular lattice isomorphic to $\mathcal{P}(\widetilde{M})$, moreover $\widetilde{M}(L) = \overline{M}(L) = M(L)$.

**Definition 2.2.** *A mapping $f : L_1 \to L_2$ between molecular lattices is said to be an $*$-generalized order-homomorphism or an $*$**ml**-map if it is an **ml**-map and its right adjoint $\hat{f}$ preserves $*$, i.e., $\hat{f}(a^*) = (\hat{f}(a))^*$ for every $a \in L_2$.*

By Lemma 1.2, we have the following result.

**Lemma 2.3.** *Let $g : L_2 \to L_1$ be a complete homomorphism preserving $*$ between molecular lattices. Then $g$ has a left adjoint $f : L_1 \to L_2$ and hence $f$ is an $*$**ml**-map.*

**Definition 2.4.** *An $*$**ml**-map $f : (L_1, \tau_1) \to (L_2, \tau_2)$ between* **gtml***s is said to be continuous if it is continuous with respect to topology $\tau_2$, i.e., $\hat{f}(b) \in \tau_1$ whenever $b \in \tau_2$.*

**Definition 2.5.** *A molecular lattice $L$ is said to be a De Morgan lattice if the second De Morgan low $\left(\bigwedge_{i\in I} a_i\right)^* = \bigvee_{i\in I} a_i^*$ holds; and if $\tau$ is also a topology on $L$, then the pair $(L, \tau)$ is called a topological De Morgan molecular lattice (briefly,* **tdml***).*

Thus for **tdml**s the set $\tau^*$ is a cotopology, and if an $*$**ml**-map $f : (L_1, \tau_1) \to (L_2, \tau_2)$ between **gtml**s (**tdml**s) is continuous, then it is continuous with respect to generalized cotopology (cotopology) $\tau_2^*$, i.e., $\hat{f}(a) \in \tau_1^*$ whenever $a \in \tau_2^*$.

A distributive pseudocomplemented lattice $L$ is said to be a stone algebra if $a^* \vee a^{**} = 1$ for every $a \in L$ [4, 10]. It is easy to show that every De Morgan molecular lattice is a stone algebra, but the converse is not true, in general. For example, consider the molecular lattice $[0, 1]$ which is a stone algebra but it is not a De Morgan lattice, because if $a_n = \frac{1}{n}$ for every natural number $n$, then $(\bigwedge a_n)^* = 1$ but $\bigvee a_n^* = 0$.

The category of all **gtml**s with continuous $*$**ml**-maps between them is denoted by **GTML** and the full subcategory of **GTML** of all **tdml**s is denoted by **TDML**.

**Lemma 2.6.** *Let $L$ be a molecular lattice and $\mathcal{B}L := \{x \in L \mid x \vee x^* = 1\}$. Then:*

1. *$\mathcal{B}L$ is a complete sublattice of $L$ and if $x \in \mathcal{B}L$, then $x^* \in \mathcal{B}L$, that is, the inclusion map $e : \mathcal{B}L \to L$ is a complete homomorphism preserving $*$.*
2. *$\mathcal{B}L$ is a Boolean algebra.*

For any set $X$ and $x \in X$ we do not distinguish $\{x\}$ from $x$.
Let $L$ be a molecular lattice. We define a mapping $\varphi : L \to \mathcal{P}(\widetilde{M}(L))$ by

$$\varphi(a) = \{x \in \widetilde{M}(L) \mid x \leqslant a\} \tag{2}$$

Then $\varphi$ is a complete homomorphism preserving $*$. The functor $cr$ : **GTML** $\to$ **TOP** is defined by $cr(L,\eta) = (cr(L), \varphi(\eta))$, where $\varphi(\eta) = \{\varphi(a) \mid a \in \eta\}$ and $cr(L) = \widetilde{M}(L)$. For any continuous $*$**ml**-map $f : (L_1, \eta_1) \to (L_2, \eta_2)$, the mapping $cr(f) : (\widetilde{M}(L_1), \varphi(\eta_1)) \to (\widetilde{M}(L_2), \varphi(\eta_2))$ is given by $cr(f)(x) = f(x)$. Then $f$ is continuous if and only if $cr(f)$ is continuous.

**Theorem 2.7.** $cr$ : **GTML** $\to$ **Top** *is the right adjoint of the embedding functor* $\mathcal{P}$ : **Top** $\to$ **GTML**, *that is,* **Top** *is a coreflective subcategory of* **GTML**.

**Proof.** Let $(L,\eta)$ be any given **gtml**. Consider $u : (\mathcal{P}(\widetilde{M}(L)), \varphi(\eta)) \to (L,\eta)$ defined by $u(A) = \vee\{m \in \widetilde{M}(L) \mid m \in A\}$. Since $\varphi \circ u = id$ and $u \circ \varphi \leqslant id$, it follows that $u$ is a left adjoint of $\varphi$ as defined in (2) and so it is a continuous $*$**ml**-map. In the following we prove that $u$ is universal. Let $(X,\tau)$ be a topological space, and $f : (\mathcal{P}(X),\tau) \to (L,\eta)$ be a continuous $*$**ml**-map. Define $\bar{f} : (X,\tau) \to (\widetilde{M}(L), \varphi(\eta))$ by $\bar{f}(x) = f(x)$. Then $\bar{f}$ is the unique continuous $*$**ml**-map satisfying the condition $u \circ \mathcal{P}(\bar{f}) = f$. Thus $cr$ is a right adjoint of $\mathcal{P}$. $\square$

For a molecular lattice $L$ define a mapping $\psi : \mathcal{B}L \to \mathcal{P}(M(\mathcal{B}L))$ by

$$\psi(a) = \{m \in M(\mathcal{B}L) \mid m \leqslant a\}.$$

Then $\varphi$ is a complete isomorphism preserving $*$.

The functor $r$ : **GTML** $\to$ **Top** is defined by $r(L,\eta) = (r(L), r(\eta))$, where $r(L) = M(\mathcal{B}L)$ and $r(\eta) = \{\psi(a) \mid a \in \eta \cap \mathcal{B}L\}$. For any continuous $*$**ml**-map $f : (L_1,\eta_1) \to (L_2,\eta_2)$, the its right adjoint $\hat{f}$ is a complete homomorphism preserving $*$.

Since $b \in \mathcal{B}L_2$ implies $\hat{f}(b) \in \mathcal{B}L_1$, it follows that the restriction mapping $g := \hat{f} \mid_{\mathcal{B}L_2} : (\mathcal{B}L_2, r(\eta_2)) \to (\mathcal{B}L_1, r(\eta_1))$ is a complete homomorphism preserving $*$.

Let $h : (\mathcal{B}L_1, r(\eta_1)) \to (\mathcal{B}L_2, r(\eta_2))$ be a left adjoint of $g$. Then $h$ is a continuous $*$**ml**-map and the mapping $r(f) : (M(\mathcal{B}L_1), r(\eta_1)) \to (M(\mathcal{B}L_2), r(\eta_2))$ defined by $r(f)(m) = h(m)$ is a continuous function.

**Theorem 2.8.** $r$ : **GTML** $\to$ **Top** *is the left adjoint of the embedding* *functor* $\mathcal{P}$ : **Top** $\to$ **GTML**, *that is,* **Top** *is a reflective subcategory of* **GTML**.

**Proof.** Let $(L, \eta)$ be any given **gtml**. Suppose that $K : (L, \eta) \to (\mathcal{P}(M(\mathcal{B}L)), r(\eta))$ be a left adjoint of the map $e \circ \psi^{-1} : \mathcal{P}(M(\mathcal{B}L)) \to L$. In the following we prove that $k$ is couniversal. Let $(X, \tau)$ be a topological space, and $f : (L, \eta) \to (\mathcal{P}(X), \tau)$ be a continuous $*$**ml**-map. We define $\bar{f} : (r(L), r(\eta)) \to (X, \tau)$ by $\bar{f}(m) = f(m)$. Then $\bar{f}$ is the unique continuous $*$**ml**-map satisfying the condition $\mathcal{P}(\bar{f}) \circ k = f$. Thus $r$ is a left adjoint of $\mathcal{P}$. $\square$

Since for every molecular lattice $L$, the Boolean algebra $\mathcal{B}L$ is a De Morgan molecular lattice, similar to the proofs of the above theorems we have the following result.

**Theorem 2.9.** **Top** *is a reflective and coreflective full subcategory of* **TDML**.

**Remark 2.10.** *By the previous theorems, the categories* **GTML** *and* **TDML** *have a non-trivial reflective and coreflective full subcategory. On the other hand,* **TOP** *does not have such a subcategory, so this phenomenon sharply distinguishes these categories from* **TOP** *on the categorical level.*

## 3. Equalizers and Produts

In this section, we give the structures of limits as equalizers and products in **TDML**. Moreover, we show that the category **GTML** has equalizers. Also, we show that the forgetful functor $U$ : **GTML** $\to$ **MOL** does not reflect products.

For an **ml**-map, since $f$ and $\hat{f}$ preserve arbitrary joins, we have $\hat{f}(0) = \hat{f}(\vee\phi) = \vee\phi = 0$ and similarly, $f(0) = 0$.

**Lemma 3.1.** *Let* $f : F \to G$ *be an* **ml**-*map. Then the following statements hold.*

1. $f \circ \hat{f} \leqslant id$, $\hat{f} \circ f \geqslant id$, $f \circ \hat{f} \circ f = f$ *and* $\hat{f} \circ f \circ \hat{f} = \hat{f}$, *where* $id$ *denotes the identity map.*

*2. $\hat{f}$ is unique, i.e., if $g \circ f \geqslant id$ and $f \circ g \leqslant id$, then $g = \hat{f}$.*

*3. $\hat{f}(1) = 1$. Also, $f(a) = 0$ if and only if $a = 0$.*

**Proof.** For parts (1) and (2) see [1, 4]. Since $f(1) \leqslant 1$, we have $1 \leqslant \hat{f}(f(1)) \leqslant \hat{f}(1)$ and hence $\hat{f}(1) = 1$. Now, let $f(a) = 0$. Then $a \leqslant \hat{f}(f(a)) = \hat{f}(0) = 0$ and hence $a = 0$. □

**Lemma 3.2.** *Let $f : L_1 \to L_2$ be a map between molecular lattice such that preserves arbitrary joins.*

*(a) If $f$ is an $*$**ml**-map, then $f$ preserves $*$-coprime elements.*

*(b) If $\widetilde{M}$ is a join generating base for $L_1$, then $f$ is an $*$**ml**-map if and only if $f$ preserves $*$-coprime elements.*

**Proof.** (a) Let $m \in \widetilde{M}(L_1)$ and $y \in L_2$. Then $m \leqslant \hat{f}(y)$ or $m \leqslant (\hat{f}(y))^* = \hat{f}(y^*)$. Thus either $f(m) \leqslant y$ or $f(m) \leqslant y^*$. If $f(m) = 0$, then $m = 0$, which is a contradiction. Thus $f(m)$ is an $*$-coprime element.

(b) Let $y \in L_2$, $m \in \widetilde{M}(L_1)$ and $f$ preserves $*$-coprime elements. Then

$$m \leqslant \hat{f}(y^*) \Leftrightarrow f(m) \leqslant y^* \Leftrightarrow f(m) \nleqslant y \Leftrightarrow m \nleqslant \hat{f}(y) \Leftrightarrow m \leqslant (\hat{f}(y))^*.$$

Thus $\hat{f}(y^*) = (\hat{f}(y))^*$. Conversely, by part (a), the result holds. □

By Lemma 3.2, we have the following result.

**Corollary 3.3.** *$f : \{0, 1\} \to L$ is an $*$**ml**-map if and only if $f(1) \in \widetilde{M}(L)$.*

**Definition 3.4.** *Let $L$ be a molecular lattice and $E$ be a complete join subsemilattice of $L$, i.e., $E \subseteq L$ and it is closed under arbitrary joins. Then we say that $E$ is an $*$-molecular sublattice (briefly, $*$**msubl**) of $L$ if $E$ is a molecular lattice and the inclusion map $e : E \to L$ is an $*$**ml**-map.*

**Example 3.5.** Let $L = \{0, x, y, 1\}$, where $x$ and $y$ are incomparable. Then $L$ is a molecular lattice and $E = \{0, x\}$ is an $*$**msubl** of $L$. If $A = \{0, 1\}$, then $A$ is a molecular lattice but it is not an $*$**msubl** of $L$, because by Corollary 3.3, the inclusion $e : A \to L$ is not an $*$**ml**-map.

**Definition 3.6.** *Let* $(L, \tau)$ *be a* **gtml** *and* $E$ *be an* $*$**msubl** *of* $L$*. If* $\delta = \hat{e}(\tau)$*, then* $(E, \delta)$ *is also a* **gtml** *which is called a* **gtmsubl** *of* $L$*.*

Notice that if $E$ is a **gtmsubl** of $L$, then the inclusion $*$**ml**-map $e : A \to L$ is continuous.

Now, we present a characterization of $*$**msubl**s. Let $L$ be a molecular lattice. In the following, we consider a mapping $J : L \to L$ which satisfies the following conditions:

(S1) $J(a) \leqslant a$ for all $a \in L$;

(S2) $J \circ J = J$;

(S3) $J$ preserves arbitrary joins;

(S4) $J(\wedge_{i\in I} J(a_i)) = J(\wedge_{i\in I} a_i)$ for all $a_i \in L$;

(S5) $J(J(a) \wedge x) = 0$ implies $J(a) \leqslant J(x^*)$ for all $a, x \in L$.

**Lemma 3.7.** *Let* $L$ *be a molecular lattice and* $E$ *be a complete join subsemilattice of* $L$*. Then* $E$ *is an* $*$**msubl** *of* $L$ *if and only if there exists a mapping* $J : L \to L$ *which satisfies the conditions* $(S1) - (S5)$*, such that* $E = ImJ$ *and* $a^c = J(a^*)$ *for each* $a \in E$*, where* $a^c$ *denotes the pseudocomplement of* $a$ *in* $E$*.*

**Proof.** Let $E$ be an $*$**msubl** of $L$. Since the inclusion $e : E \to L$ is an $*$**ml**-map, if we define a mapping $J : L \to L$ by $J(a) = \hat{e}(a)$ for each $a \in L$, then the result holds. Conversely, let $J : L \to L$ be a mapping which satisfies the conditions $(S1) - (S5)$, such that $E = ImJ$ and $a^c = J(a^*)$ for each $a \in E$. Since $J$ preserves joins, it follows that $E$ is a complete join subsemilattice of $L$ and hence is a complete lattice. Now, we show that the infimum in $E$ is as following:

$$\sqcap_{i\in I} J(a_i) = J(\wedge_{i\in I} a_i).$$

Since $\wedge_{i\in I} a_i \leqslant a_i$, we have $J(\wedge_{i\in I} a_i) \leqslant \sqcap_{i\in I} J(a_i)$. Conversely, let $x \in E$ and $x \leqslant a_i$ for every $i \in I$. Then $x = J(x) \leqslant J(a_i)$ and hence $x = J(x) \leqslant \wedge_{i\in I} J(a_i)$. Thus $x \leqslant J(\wedge_{i\in I} J(a_i)) = J(\wedge_{i\in I} a_i)$, as desired. On the other hand, we have

$$\vee_{i\in I} \sqcap_{j\in J} J(a_{ij}) = J(\vee_{i\in I} \wedge_{j\in J} a_{ij}) = J(\wedge_{j\in J} \vee_{i\in I} a_{ij}) = \sqcap_{j\in J} \vee_{i\in I} J(a_{ij}).$$

Thus $E$ is a molecular lattice. Let $x \in E$ and $x^c$ be the pseudocomplement of $x$ in $E$. Since $x \sqcap J(x^*) = J(x) \sqcap J(x^*) = J(x \wedge x^*) = J(0) = 0$, it follows that $J(x^*) \leqslant x^c$. Conversely, let $y \in E$ and $y \sqcap x = 0$. Then we have

$$0 = y \sqcap x = J(y) \sqcap J(x) = J(y \wedge x) = J(J(y) \wedge x).$$

Therefore $y = J(y) \leqslant J(x^*)$, and hence $x^c = J(x^*)$. Finally, we show that $e : E \to L$ is an $*$**ml**$-$map. For any $x \in L$ we have

$$\hat{e}(x) = \vee\{J(a) \mid J(a) \leqslant x\} \Rightarrow \hat{e}(x) = J(\hat{e}(x)) \leqslant x \Rightarrow \hat{e}(x) \leqslant J(x).$$

Since $J(x) \leqslant x$, it follows that $\hat{e}(x) = J(x)$. Thus for any $x \in L$ we have

$$\hat{e}(x^*) = J(x^*) = (J(x))^c = (\hat{e}(x))^c. \quad \square$$

**Lemma 3.8.** *Let $L$ be a molecular lattice and $E$ be a complete join subsemilattice of $L$. Let $\mathcal{S}$ be the collection of all mappings $J : L \to L$ which satisfy the conditions $(S1) - (S5)$ and $ImJ \subseteq E$. Then $\mathcal{S}$ with respect to pointwise order has a maximal element $\gamma$ such that for each $J \in \mathcal{S}$, $ImJ \subseteq Im\gamma \subseteq E$.*

**Proof.** Let $\gamma : L \to L$ defined by $\gamma(a) = \vee\{J(a) \mid J \in \mathcal{S}\}$ for every $a \in L$. Then we have:

1. $\gamma(a) \leqslant a$;
2. $\gamma(\gamma(a)) = \gamma(\vee_{J\in\mathcal{S}} J(a)) = \vee_{I\in\mathcal{S}} I(\vee_{J\in\mathcal{S}} J(a)) = \vee_{I\in\mathcal{S}} I(a) = \gamma(a)$;
3. $\gamma(\vee_{i\in I} a_i) = \vee_{J\in\mathcal{S}} J(\vee_{i\in I} a_i) = \vee_{J\in\mathcal{S}} \vee_{i\in I} J(a_i) = \vee_{i\in I} \vee_{J\in\mathcal{S}} J(a_i) = \vee_{i\in I} \gamma(a_i)$;
4. $\gamma(\wedge_{i\in I}\gamma(a_i)) = \vee_{J\in\mathcal{S}}\{J(\wedge_{i\in I}\gamma(a_i))\} = J(\wedge_{i\in I}\gamma(a_i)) =$
   $J(\wedge_{i\in I}\vee_{J\in\mathcal{S}}\{J(a_i)\}) = J(\vee_{J\in\mathcal{S}}\{\wedge_{i\in I} J(a_i)\}) = \vee_{J\in\mathcal{S}}\{J(\wedge_{i\in I} J(a_i))\}$
   $= \vee_{J\in\mathcal{S}}\{J(\wedge_{i\in I} a_i)\} = \gamma(\wedge_{i\in I} a_i)$;
5. If $\gamma(\gamma(a) \wedge x) = 0$, then $J(\gamma(a) \wedge x) = 0$ for every $J \in S$. Thus $\gamma(a) \leqslant J(x^*) \leqslant \gamma(x^*)$.

It is easy to see that for every $J \in \mathcal{S}$, $J \leqslant \gamma$ and $Im\gamma \subseteq E$. On the other hand, we have $J(a) = J(J(a)) \leqslant \gamma(J(a)) \leqslant J(a)$. Thus $J(a) = \gamma(J(a))$, and hence $ImJ \subseteq Im\gamma$. $\square$

By Lemmas 3.7 and 3.8, we have the following results.

**Corollary 3.9.** *Let $L$ be a molecular lattice and $A$ be a complete join subsemilattice of $L$. Let $\mathcal{S}$ be the collection of all* $*$**msubl***s $B$ of $L$ such that $B \subseteq A$. Then $\mathcal{S}$ has a maximal element.*

**Corollary 3.10.** *Let $(L, \tau)$ be a* **gtml** *and $A$ be a complete join subsemilattice of $L$. Let $\mathcal{S}$ be the collection of all* **gtmsubl***s $B$ of $L$ such that $B \subseteq A$. Then $\mathcal{S}$ has a maximal element. Moreover, if $(L, \tau)$ is a* **tdml***, then so is the maximal element.*

**Theorem 3.11.** *The equalizer of $(L_1, \tau_1)\overset{f}{\underset{g}{\rightrightarrows}}(L_2, \tau_2)$ in* **GTML** *is the pair $(E, e)$, where $E$ is the maximal* **gtmsubl** *of $L_1$ such that $E \subseteq E_{fg} := \{x \in F \mid f(x) = g(x)\}$ and $e : E \to L_1$ is the continuous inclusion* $*$**ml***-map.*

**Proof.** Let $h : N \to L_1$ be an $*$**ml**-map such that $f \circ h = g \circ h$. We define a mapping $J : L_1 \to L_1$ by $J(a) = h \circ \hat{h}(a)$. Since $\hat{h} \circ h \circ \hat{h} = \hat{h}$, it is easy to show that $J$ satisfies the conditions $(S1) - (S5)$. Thus $Imh = ImJ \subseteq E_{fg}$ and hence $h(x) \in E$. Now, we define $\alpha : N \to E$ by $\alpha(x) = h(x)$. Then $e \circ \alpha = h$ and for every $x \in E$ we have

$$\hat{\alpha}(x^c) = \hat{\alpha}(\hat{e}(x)^c) = (\hat{\alpha} \circ \hat{e})(x^*) = \hat{h}(x^*) = (\hat{h}(x))^* = (\hat{\alpha}(x))^*.$$

Thus $\alpha$ is an $*$**ml**-map. Finally, it is easy to check that $\alpha$ is continuous and unique. $\square$

By Theorem 3.11 and Corollary 3.10, we have the following result.

**Theorem 3.12.** *Let $(L_1, \tau_1)\overset{f}{\underset{g}{\rightrightarrows}}(L_2, \tau_2)$ be morphisms in* **TDML***. Then $e : (E, \delta) \to (L_1, \tau)$ is an equalizer of $(L_1, \tau)\overset{f}{\underset{g}{\rightrightarrows}}(L_2, \eta)$ in* **TDML** *if and only if $e$ is an equalizer in* **GTML***.*

**Corollary 3.13.** *If $\widetilde{M}(L_1)$ is a join generating base for $L_1$, then the equalizer of $(L_1, \tau_1)\overset{f}{\underset{g}{\rightrightarrows}}(L_2, \tau_2)$ in* **GTML** *is the pair $(E, e)$, where $E$ is* *the* **gtml** *generated by the set* $\widetilde{M}_{fg} := \{m \in \widetilde{M}(L_1) \mid f(x) = g(x)\}$ *and* $e : E \to L_1$ *is the continuous inclusion* $*$**ml**-*map.*

**Proof.** Let $\gamma : L_1 \to L_1$ defined by $\gamma(a) = \vee\{m \in \widetilde{M}_{fg} \mid m \leqslant a\}$. 'It is easy to check that $\gamma$ satisfies the conditions $(S1)-(S5)$ and $Im\gamma = E$. On the other hand, for every mapping $J : L_1 \to L_1$ which satisfies the conditions $(S1) - (S5)$ and $ImJ \subseteq E_{fg}$, we have $J(a) = J(\vee\{m \in \widetilde{M}(L_1) \mid m \leqslant a\}) = \vee\{J(m) \mid m \in \widetilde{M}(L_1), m \leqslant a\}$. Since $J(m) \leqslant m$, for every $m \in \widetilde{M}$, either $J(m) = m$ or $J(m) = 0$. Thus $ImJ \subseteq Im\gamma$, which shows that $E$ is the maximal **gtmsubl** of $L_1$ such that $E \subseteq E_{fg}$, as desired. $\square$

**Example 3.14.** Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be arbitrary continuous functions. Then the equalizer of $\mathcal{P}(X) \overset{\mathcal{P}(f)}{\underset{\mathcal{P}(g)}{\rightrightarrows}} \mathcal{P}(Y)$ in **TDML** is the pair $(E, e)$, where $E$ is the **tdml** generated by the set $\{\{x\} \mid x \in X, f(x) = g(x)\}$ and $e : E \to \mathcal{P}(X)$ is the continuous inclusion $*$**ml**-map. Thus $E = \mathcal{P}(E_{fg})$, where $E_{fg} := \{x \in X \mid f(x) = g(x)\}$ is the equalizer of $f$ and $g$ in **TOP**. This of course amounts to the familiar fact that the reflector $\mathcal{P}$ preserves limits.

The following Lemma is an immediate consequence of Definition 1.4 and Remark 1.5.

**Lemma 3.15.** *Let* $p_k : \bigotimes_{i\in I} L_i \to L_k$ *be the projection mapping defined in* (1)*, for some* $k \in I$ *and* $\{x_i\}_{i\in I} \in \bigotimes_{i\in I} L_i$. *Then the following statements hold.*

1. $\hat{p_k}(z) = \downarrow \{y_i\}_{i\in I}$ *for every* $z \in L_k$*, where*

$$y_i = \begin{cases} 1, & \text{if } i \neq k, \\ z, & \text{if } i = k. \end{cases}$$

2. $(\downarrow \{x_i\}_{i\in I})^* = \bigvee_{i\in I} \hat{p_i}(x_i^*)$.

3. $\downarrow \{x_i\}_{i\in I} = \bigcap_{i\in I} \hat{p_i}(x_i)$.

4. $\hat{p_i}(z^*) = (\hat{p_i}(z))^*$ *for any* $i \in I$ *and* $z \in L_i$.

5. $(\hat{p}_i(z) \cap \hat{p}_j(w))^* = \hat{p}_i(z^*) \vee \hat{p}_j(w^*)$ *for any* $i, j \in I$, $z \in L_i$ *and* $w \in L_j$.

By Lemma 3.15, it is easy to show that if $L_i$ is a De Morgan molecular lattice for every $i \in I$, then so is $\bigotimes_{i\in I} L_i$. Thus we have the following result.

**Theorem 3.16.** $\{((\bigotimes_{i\in I} L_i, \tau), p_i) \mid i \in I\}$ *is the product of* $\{(L_i, \tau_i)\}_{i\in I}$ *in the category* **TDML**, *where* $\tau = \{a \in \bigotimes_{i\in I} L_i \mid \forall i \in I, \hat{p}_i(a) \in \tau_i\}$.

**Proof.** Let $\{f_i : (L, \tau') \to (L_i, \tau_i)\}_{i\in I}$ be a family of **TDML**-morphisms. By the property of product in **MOL**, there exists a unique morphism $f : (L, \tau') \to (\bigotimes_{i\in I} L_i, \tau)$ such that $p_i \circ f = f_i$ for every $i \in I$ and $f(a) = \downarrow \{f_i(a)\}_{i\in I}$. By Lemma 3.15, we have:

$$\hat{f}((\downarrow \{x_i\}_{i\in I})^*) = \hat{f}(\bigvee_{i\in I} \hat{p}_i(x_i^*)) = \bigvee_{i\in I} \hat{f}(\hat{p}_i(x_i^*)) = \bigvee_{i\in I} \hat{f}_i(x_i^*) = \bigvee_{i\in I} (\hat{f}_i(x_i))^*$$

$$= (\bigwedge_{i\in I} \hat{f}_i(x_i))^* = (\bigwedge_{i\in I} \hat{f}(\hat{p}_i(x_i)))^* = (\hat{f}(\bigwedge_{i\in I} \hat{p}_i(x_i)))^* = (\hat{f}(\downarrow \{x_i\}_{i\in I}))^*.$$

Thus $f$ is an $*$**ml**-map. It is easy to check that $f$ is continuous and unique. □

**Theorem 3.17.** *The forgetful functor* $U$ : **GTML** $\to$ **MOL** *does not reflect product.*

**Proof.** Consider the molecular lattice $L = \{0, a, b, c, 1\}$, where $a$ and $b$ are incomparable, $a < c, b < c$. Let $f, g : (L, \tau) \to (L, \tau)$ be the continuous $*$**ml**-mappings defined by $g(0) = 0$, $g(a) = b$, $g(b) = a$, $g(c) = c$, $g(1) = 1$ and $f = id$, where $id$ is the identity map and $\tau = \{0, 1\}$. Let $(L \bigotimes L, p_1, p_2)$ be the product in **MOL**. Then there exists a unique **ml**-map $h : L \to L \bigotimes L$ such that $p_1 \circ h = f$ and $p_2 \circ h = g$. Now, we have

$$\hat{h}((\downarrow (b, b))^*) = \hat{h}(\downarrow (a, 1)) \vee \hat{h}(\downarrow (1, a)) = a \vee b = c \neq (\hat{h}(\downarrow (b, b)))^* = 0^* = 1.$$

Thus $h$ is not an $*$**ml**-map and hence $(L \bigotimes L, p_1, p_2)$ is not a product in **GTML**. □

By Theorems 3.12 and 3.16, we have the following result.

**Theorem 3.18.** **TDML** *is a complete category.*

**Question.** Does **GTML** have products?

## 4. Coequalizers and Coproduts

In this section, we show that **GTML** and **TDML** have coequalizers and coproducts.

Recall that $\{(\prod_i L_i, q_i) \mid i \in I\}$ is the coproduct of $\{L_i\}_{i\in I}$ in the category **MOL**, where the order in $\prod_i L_i$ is the pointwise order and the mapping $q_{i_0} : L_{i_0} \to \prod_i L_i$ is defined by $q_{i_0}(x) = \{x_i\}_{i\in I}$, and

$$x_i = \begin{cases} 0, & \text{if } i \neq i_0, \\ x, & \text{if } i = i_0. \end{cases}$$

**Theorem 4.1.** $\{((\prod_{i\in I} L_i, \tau), q_i) \mid i \in I\}$ *is the coproduct of* $\{(L_i, \tau_i)\}_{i\in I}$ *in category* **GTML**, *where* $\tau = \{a \in \prod_{i\in I} L_i \mid \forall i \in I, \hat{q}_i(a) \in \tau_i\}$.

**Proof.** Let $\{f_i : (L_i, \tau_i) \to (L, \tau')\}_{i\in I}$ be a family of **GTML**-morphisms. By the property of coproduct in **MOL**, there exists a unique morphism $f : (\prod_{i\in I} L_i, \tau) \to (L, \tau')$ such that $f \circ q_i = f_i$ for every $i \in I$, and $f(\{x_i\}_{i\in I}) = \bigvee_{i\in I} f_i(x_i)$. Since $\hat{q}_i$ preserves the operation $*$ and $\hat{q}_i(\{b_i\}_{i\in I}) = b_i$, it follows that $q_i$ is an $*$**ml**-map for every $i \in I$. Now, we have

$$\hat{f}(a^*) = \{\hat{f}_i(a^*)\}_{i\in I} = \{\hat{f}_i(a)\}^*_{i\in I} = (\hat{f}(a))^*.$$

Thus $f$ is an $*$**ml**-map. It is easy to check that $f$ is continuous and unique. □

If $L_i$ is a De Morgan molecular lattice for every $i \in I$, then so is $\prod_{i\in I} L_i$. Thus we have the following result.

**Corollary 4.2.** *The structure of coproduct in* **TDML** *is the same coproduct in* **GTML**.

Let $(L_1, \tau) \overset{f}{\underset{g}{\rightrightarrows}} (L_2, \eta)$ be a pair of **GTML**-morphisms. Then $\hat{f}$ and $\hat{g}$ are complete homomorphism preserving $*$. So the set $Q_{fg} := \{a \in L_2 \mid \hat{f}(a) = \hat{g}(a)\}$ is a complete sublattice of $L_2$ such that $a^* \in Q_{fg}$ whenever

$a \in Q_{fg}$. Thus the inclusion map $e : Q_{fg} \to L_2$ is a complete homomorphism preserving $*$. By Lemma 2.3, $e$ has a left adjoint $q : L_2 \to Q_{fg}$.

**Theorem 4.3.** *The coequalizer of* $(L_1, \tau) \underset{g}{\overset{f}{\rightrightarrows}} (L_2, \eta)$ *in* **GTML** *is* $(Q_{fg}, q, \delta)$, *where* $q : (L_2, \eta) \to (Q_{fg}, \delta)$ *is the left adjoint of the inclusion map* $e : Q_{fg} \to L_2$ *and* $\delta = \{a \mid \hat{q}(a) \in \eta\}$.

**Proof.** Since $\hat{f} \circ e = \hat{g} \circ e$, it follows that $q \circ f = q \circ g$. Let $h : L_2 \to N$ be an $*$**ml**-map such that $h \circ f = h \circ g$. Then $\hat{g} \circ \hat{h} = \hat{f} \circ \hat{h}$, so $\hat{h}(a) \in E$ for every $a \in N$. Thus the mapping $\alpha : N \to E$ defined by $\alpha(a) = \hat{h}(a)$ is a complete homomorphism preserving $*$ and $e \circ \alpha = \hat{h}$. By Lemma 2.3, $\alpha$ has a left adjoint $r : E \to N$ such that $\hat{q} \circ \hat{r} = \hat{h}$ and consequently $r \circ q = h$. It is easy to check that $r$ is continuous and unique. □

Let $(L_1, \tau) \underset{g}{\overset{f}{\rightrightarrows}} (L_2, \eta)$ be a pair of **TDML**-morphisms. Then $Q_{fg}$ is a De Morgan molecular lattice. Thus we have the following result.

**Corollary 4.4.** *The structure of coequalizer in* **TDML** *is the same coequalizer in* **GTML**.

By the previous statements, we have the following result.

**Theorem 4.5.** **TDML** *and* **GTML** *are cocomplete categories.*

## 5. Conclusion

In 1992, Wang introduced the concept of topological molecular lattices in terms of closed elements as a generalization of ordinary topological spaces in tools of molecules, remote neighbourhoods and generalized order-homomorphisms. In this paper, we have introduced the cocomplete category **GTML** whose objects are generalized topological molecular lattices in terms of open elements and whose morphisms are continuous generalized order-homomorphisms such that its right adjoins preserve the pseudocomplement operation. Also, we have defined the concept of a topological De Morgan molecular lattice and shown that the category **TDML** of topological De Morgan molecular lattices as a full subcategory of **GTML** is both complete an cocomplete. In particular, we have investigated the structures of colimits as coequalizers, coproducts in **GTML** and **TDML**; and the structures of limits as equalizers and products in **TDML**. Moreover, we have shown that the category **GTML** has equalizers.

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