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Original Research Paper

Gaussian-Radial Basis Functions for Solving Fractional Parabolic Partial Integro-Differential Equations

F. S. Aghaei Maybodi
Yazd University

M. H. Heydari*
Shiraz University of Technology

F. M. Maalek Ghaini
Yazd University

Abstract. In this paper, we solve the Caputo's fractional parabolic partial integro-differential equations (FPPI-DEs) by Gaussian-radial basis functions (G-RBFs) method. The main idea for solving these equations is based on the radial basis functions (RBFs) which also provides approaches to higher dimensional spaces. In the suggested method, FPPI-DEs are reduced to nonlinear algebraic systems. We propose to apply the collocation scheme using G-RBFs to approximate the solutions of FPPI-DEs. Numerical examples are provided to show the convenience of the numerical scheme based on the G-RBFs. The results reveal that the presented method is very efficient and convenient for solving such problems.

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*Corresponding Author

1 Introduction

The fractional calculus has a long history since 30 September 1695, when the derivative of order $\alpha = 1/2$ was described by Leibniz [16]. The theory of derivatives and integrals of non-integer orders goes back to Leibniz, Liouville, Grünwald, Letnikov and Riemann. The fractional mathematics has various applications in formulation of many scientific events [16, 14]. The more than 3 hundered years old problem of geometric and physical interpretation of fractional integration and differentiation of an arbitrary real order of a function was finally given in the sense of Riemann-Liouville fractional differentiation and integration and the Caputo's fractional derivative [3].

In recent years many problems in mathematics, physics and engineering have been numerically solved by RBFs method, for instance, see [9]. We focus on different kinds of RBFs as one of the most important tools in engineering and sciences [16]. There are many different classes of these fractional partial integro-differential equations of parabolic, hyperbolic and elliptic types [1].

In this study, we focus on the integro-differential equations in the form of

$$ {}_0^cD_t^{\beta}u(x,t) = g(x,t) + u_{xx}(x,t) + \int_0^t k(x,t,\tau)u(x,\tau)d\tau, \tag{1} $$

where $(x,t) \in [a,b]\times[0,T]$, subject to the initial and boundary conditions

$$ u(x,0) = f(x), \quad x \in [a,b]\,, \tag{2} $$
$$ u(a,t) = h(t), \quad u(b,t) = l(t), \quad t \in [0,T]\,, \tag{3} $$

where f, g, l and k are given continuous functions. This class of equations can describe some phenomena in compression of viscoelastic media and nuclear reactor dynamics [13].

The motivation of this paper is to extend the application of the collocation method based on the G-RBFs to solve FPPI-DEs. In recent years, the subject of FPPI-DEs has been received more attention.
Riemann– Liouville fractional integration operator has bugs in modeling many real phenomena, whereas Caputo's fractional derivative solves

some of these difficulties, namely nonzero Riemann– Liouville's derivative of constant functions, but Caputo's fractional derivative of constant functions is zero. In fact, Caputo's fractional derivative is a generalization of ordinary derivative and applies for problems relating to viscoelasticity and oscillation. An example for the Caputo's fractional derivatives is the Abel's problem [7].

Methods based on RBFs are practical tools for approximating multivariate functions and scattered data [10] on irregular areas, and so are used in solving FPPI-DEs [2].

Fractional equations have been solved via the proposed methods by a predictor-corrector approach and extrapolation by Diethelm [5], Adomian decomposition method [6], variational iteration method, Homotopy perturbation method, Green's function method, Homotopy analysis method, new perturbative Laplace's method, Jafari's method, Elsayed's method, Generalizations of differential transformation method, block pulse function method, Integral mean value method, wavelet basis method and other new methods [8].

The article is organized as follows: In Section 2, we express some definitions regarding RBFs, fractional calculus and Gauss-Legendre quadrature method. In Section 3, we describe RBFs Collocation method for solving FPPI-DEs. In Section 4, we apply our method to various numerical examples. Finally, an epilogue is given in Section 5.

2 Basic Definitions

2.1 Fractional Calculus

In this sub-section, we give some basic definitions and theorems of the fractional calculus theory, which will be used later in this paper.

Definition 2.1. *([15]). A real function $f(t),\ t>0$, is said to be in the space $C_\mu[0,\infty),\ \mu\in\mathbb{R}$, if there exists a real number $p\ (>\mu)$ such that $f(t)=t^p f_1(t)$, and $f_1(t)\in C[0,\infty)$. Morever $f(t)$ is said to be in the space $C_\mu^n[0,\infty)$ if $f^{(n)}(t)\in C_\mu[0,\infty),\ n\in\mathbb{N}$.*

Definition 2.2. *([15]). The Riemann-Liouville fractional integral operator of order $\beta\geq 0$ of a function $f\in C_\mu[0,\infty),\ \mu\geq -1$, on the* *interval* $[0,t]$ *is defined as:*

$$ {}_{0}^{RL}I_t^{\beta} f(t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau, & \beta > 0, \\ f(t), & \beta = 0. \end{cases} $$

In the sequel, we use the abbreviation I^{β} *instead of* ${}_{0}^{RL}I_t^{\beta}$. *Note that the operator* I^{β} *has the following properties:*

$$ \begin{aligned} &(i)\ I^{\beta}I^{\alpha} = I^{\beta}I^{\alpha}, \\ &(ii)\ I^{\beta}I^{\alpha} = I^{\alpha+\beta}, \\ &(iii)\ I^{\beta}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+1)} t^{\beta+\gamma}, \end{aligned} $$

where $\alpha,\ \beta \geq 0,\ t > 0$ *and* $\gamma > -1$.

Definition 2.3. *([15]). The Riemann-Liouville fractional derivative of order* $\beta > 0$ *is defined as:*

$$ {}_{0}^{RL}D_t^{\beta} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\beta} f(t), \qquad (n-1 < \beta \leq n), $$

where n *is a natural number and* $f \in C_1^n[0,\infty)$.
The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations [14]. Therefore, we shall now introduce a modified fractional differential operator ${}_{0}^{c}D_t^{\beta}$ *proposed by Caputo.*

Definition 2.4. *([14]). The fractional derivative of order* $\beta > 0$ *in the Caputo's sense, on the interval* $[0,s]$ *is defined as:*

$$ {}_{0}^{C}D_s^{\beta} f(s) = \frac{1}{\Gamma(m-\beta)} \int_0^s (s-\tau)^{m-\beta-1} f^{(m)}(\tau)d\tau,\ m-1 < \beta \leq m, $$

where m *is a natural number,* $s > 0$ *and* $f \in C_1^m[0,\infty)$. *We remind that the Caputo's fractional derivative has the following useful property:*

$$ I^{\beta}({}_{0}^{C}D_s^{\beta} f(s)) = f(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{s^k}{k!},\ s > 0,\ m-1 < \beta \leq m, $$

where m *is a natural number,* $s > 0$ *and* $f \in C_1^m[0,\infty)$.

2.2 Definitions of the RBFs

In this subsection we review some definitions and properties of the RBFs method [12]. The well-conditionness of the interpolation problem for the scattered data by RBFs method is guaranteed by positive definiteness of the RBFs, because the function ϕ used in constructing the RBFs is positive definite, then the corresponding interpolation matrix is also positive definite, and so nonsingular.

Definition 2.5. *A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is called a radial function whenever a function of one variable $\psi : [0,\infty) \to \mathbb{R}$ exists so that $\phi(x) = \psi(r)$ where $r = \|x\|$, and $\|.\|$ is a norm on $\mathbb{R}^d$, usually Euclidean norm.*

Definition 2.6. *A real symmetric matrix A is called positive definite, if its associated quadratic form is non-negative, i.e.,*

$$C^t AC = \sum_{i,j}^{N} c_i A_{ij} c_j > 0, \textit{ for any } \ C = [c_1, c_2, \ldots, c_N] \in \mathbb{R}^d.$$

If the quadratic form is zero only for $C = 0$, then A is called positive definite.

Definition 2.7. *A continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$ is called positive definite on R^d, if a matrix $A \in \mathbb{R}^{n\times n}$ with entries $a_{ij} = \phi \left\| x_i - x_j \right\|, 1 \leqslant i, j \leqslant N$ is positive definite, for any N pairwise different points $X = \{x_1, \ldots, x_N\} \in \mathbb{R}^d$ and $C = [c_1, \ldots, c_N] \in \mathbb{R}^d$, where we have:*

$$C^t \phi C = \sum_{i,j=1}^{N} c_i c_j \phi(x_i - x_j) > 0.$$

Definition 2.8. *A symmetric function $\phi \in \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is conditionally positive definite of order m, if for the set $X =\{x_1, \ . \ . \ . \ , x_N\} \subset \ R^d$ of distinct points, and all vectors $\lambda \in \mathbb{R}^N$ satisfying $\sum\limits_{i=1}^{N} \lambda_i p(x_i) = 0$ for any polynomial p of degree at most $m-1$, the quadratic form $\lambda^t \phi \lambda = \sum\limits_{i,j} \lambda_i \lambda_j \phi(x_i - x_j) \geq 0$, whenever $\lambda \neq 0$.*

Table 1: Some well-known RBFs.

Number	Name of RBF	Abbreviation	Smooth	Definition
1	Gaussian	GA	Infinitely smooth	$\exp(-cr^2)$
2	Hyperbolic secant	HS	Infinitely smooth	$\operatorname{sech}(cr)$
3	Cauchy	CA	Infinitely smooth	$\frac{1}{1+cr}$
4	Inverse quadratic	IQ	Infinitely smooth	$\dfrac{1}{c^2+r^2}$
5	Multiquadratic	MQ	Infinitely smooth	$\sqrt{c^2+r^2}$
6	Inverse Multiquadratic	IMQ	Infinitely smooth	$\dfrac{1}{\sqrt{c^2+r^2}}$
7	Generated Multiquadratic	GMQ	Infinitely smooth	$(c^2+r^2)^{\frac{\alpha}{2}}, \alpha>0, \alpha\notin 2\mathbb{N}$
8	Linear spline	LS	Piecewise smooth	r
9	Cubic spline	CS	Piecewise smooth	r^3
10	Quintic spline	QS	Piecewise smooth	r^5
11	Power Splines	PS	Piecewise smooth	$r^{2k-1}, k\in\mathbb{N}$
12	Thin plate spline	TPS	Piecewise smooth	$r^2\ln r$
13	Generated thin plate spline	GTPS	Piecewise smooth	$\begin{cases} r^k\log r, & \text{k=odd,} \\ r^k, & \text{k=even.}\end{cases}$
14	Wendland	W	Piecewise smooth	$(1-r)_+^m p(r)$, p: a polynomial

2.3 Introduction of the Famous RBFs

The most famous examples of conditionally positive definite RBFs of order m according to Table 1 are GA, GMQ, TPS, PS, MQ and IMQ, respectively given by

$i)$ $\phi(r)=\exp(-cr^2),\ c\geq 0, m\geq 0,$

$ii)$ $\phi(r)=(a^2+r^2)^{k/2}, -d<k<0, k\in 2\mathbb{Z}+1, a\neq 0, m\geq 0$

$iii)$ $\phi(r)=(-1)^{k+1}r^{2k}\log r,\ k\in\mathbb{N}, m\geq k+1,$

$iv)$ $\phi(r)=(-1)^{\lceil k/2\rceil}r^k,\ k>0\ k\in 2\mathbb{N}+1, m\geq\lceil k/2\rceil,$

$v)$ $\phi(r)=(-1)^{\lceil k/2\rceil}(a^2+r^2)^{k/2},\ k>0\ k\in 2\mathbb{N}+1, m\geq\lceil k/2\rceil,$

where $\lceil k/2\rceil$ shows the smallest integer greater than $k/2$.
Note that RBFs are mainly divided into two categories as follows [11]:
(i) Infinitely smooth RBFs,
(ii) Piecewise smooth RBFs.
In the case of infinitely smooth RBFs, the existence of a shape parameter affects both accuracy of the solution and the condition number of

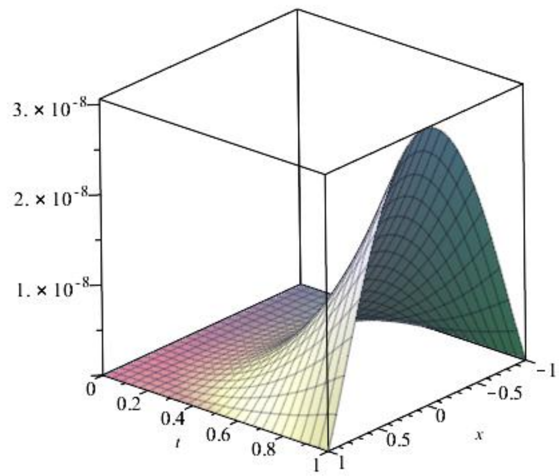

Figure 1: Graph of the absolute error function with $c = 0.01$ and $\beta = 0.25$ in Example 4.1.

the collocation matrix, while piecewise smooth RBFs are free of shape parameter. Because of some characteristic features of the RBF functions due to their radial nature, their implementation process in high dimensional domains and irregular domains is easier than traditional method. In two cases we review the matrix A resulted from interpolation with RBFs.

RBF Interpolation Method

Suppose that the approximation $Sf(x)$ of $f(x)$ on an arbitary point set $X = \{x_i\}_{i=1}^{N}$ can be written as a linear combination of N basis functions in the following form $Sf(x) = \sum_{i=1}^{N} \alpha_i \phi(x - x_i)$, where N is the number of data points, d is the dimension of the problem and α_i's are the coefficients to be determined. In order to determine the α_i's we require that $Sf(x_j) = f(x_j)$, or in matrix notation:

$$Sf(x) = \sum_{i=1}^{N} \alpha_i \phi(x - x_i), \quad A\Lambda = f, \quad A = [a_{ij}]_{i,j=1}^{N},$$

where $a_{ij} = \phi \left\| x_i - x_j \right\|, i, j = 1, 2, 3, ..., N$, $f = (f_1, f_2, ..., f_N)^T$, $\Lambda = (\alpha_1, \alpha_2, ..., \alpha_N)^T$. Note that the interpolant $Sf(x)$ is unique if and only if the matrix A is nonsingular. The most famous examples according to the RBF's table are GA, IQ, IMQ and LS.

Augmented RBFs Interpolation Method

RBFs interpolation of a continuous function $f : R^d \to R$ on a set $X = \{x_1, x_2, ..., x_N\}$ starts with choosing a radial function $\phi : \mathbb{R}^d \to \mathbb{R}$. We assume ϕ is strictly conditionally positive definite of order m, and so the interpolant has the form

$$Sf(x) = \sum_{i=1}^{N} \alpha_i \phi(x - x_i) + \sum_{j=1}^{L} \lambda_j p_j(x), x \in R^d$$

where

$$L = \dim \Pi_m^d = \binom{d+m-1}{d},$$

and $\{p_1, ..., p_L\}$ is a basis for Π_m^d. The interpolation problem is to find $\alpha_i, i = 1, ..., N$ and $\lambda_j, j = 1, ..., L$ such that the interpolant $Sf(x)$ satisfies

$$\begin{aligned} Sf(x_i) = f_i,\ l \qquad & i = 1, ..., N, \\ \sum_{j=1}^{L} \lambda_j p_j(x_i) = 0, \quad & j = 1, ..., L. \end{aligned} \tag{4}$$

In fact we can write this system in matrix form as:

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{5}$$

where A and P are N $\times$ N and N $\times$ L matrices, respectively with the elements $A = [a_{ij}]^{N}{}_{i,j=1}, a_{ij} = \phi \|x_j - x_i\|, i, j = 1, 2, ..., N,$
$f = (f_1, f_2, ..., f_N)^T, \lambda = (\lambda_1, \lambda_2, ..., \lambda_N)^T$, P_{ij}=$\mathrm{P}_j(x_i), i = 1, 2, ..., N, j = 1, 2, ..., L$ and $\alpha \in \mathbb{R}^N, \lambda \in \mathbb{R}^L$ are the vectors of coefficients of $Sf(x)$ and the components of f are the data $f(x_i), i = 1, 2, 3, ..., N$. For any set of N distinct points $X = \{x_i\}_{i=1}^N$, the $(N + L) \times (N + L)$ system (4) has a unique solution, whenever P has full rank L. In this of case X is said to be un-solvent with respect to $\dim \Pi_m^d$. Let $[b_{ij}] = \begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix}^{-1}$.
From system (5), we can obtain $\alpha_i = \sum\limits_{k=1}^{N} b_{ik} f_k, i = 1, ..., N,$ $\lambda_j = \sum\limits_{k=1}^{N} b_{(j+n)k} f_k, j = 1, ..., L$. Note that the matrix A in this case is singular. The most famous examples according to Table 1 are TPS, PS, MQ and IMQ. If we use a positive definite RBF in constructing A, then A will also be positive definite, and as a result the system (5) always has a unique solution [4].

2.4 Gauss-Legendre Nodes and Weights

Let $L_{M+1}(\xi)$ be the Legendre polynomial of degree $M+1$ on $[-1, 1]$, i.e.

$$
\begin{aligned}
&L_0(\xi) = 1, \\
&L_1(\xi) = \xi, \\
&L_{M+1}(\xi) = \frac{2M+1}{M+1} \xi L_M(\xi) - \frac{M}{m+1} L_{M-1}(\xi), \quad M = 1, 2, \ldots.
\end{aligned}
$$

Then, the Gauss-Legendre nodes $-1 < \xi_0 < \xi_1 < ... < \xi_M < 1$ are the zeros of $L_{M+1}(\xi)$. We approximate the integral of $f(\xi)$ on $[-1, 1]$ as : $\int_{-1}^{1} f(\xi) d\xi \simeq \sum\limits_{i=0}^{M} w_i f(\xi_i)$, where ξ_i's are the Gauss-Legendre nodes and $w_i = \frac{2}{(1-\xi_i{}^2)(L'_{M+1}(\xi_i))}$ for $i = 0, 1, ..., M$ are the Gauss-Legendre weights. The integration is exact whenever $f(\xi)$ is a polynomial of degree $\leq 2M + 1$.

3 Description of the Proposed Method

In this section, we describe the RBFs method for the smooth solution of Eq. (1), subject to the initial and boundary conditions (2) and (3). The time fractional derivative defined in Eq. (1) is a weakly singular integral operator. So, using integration by parts, we transform this operator into the following non-singular equivalent one

$$ {}_{0}^{c}D_t^{\beta}u(x,t) = \frac{1}{\Gamma(2-\beta)}\int_0^t (t-\tau)^{1-\beta}u_{\tau\tau}(x,\tau)d\tau + \frac{1}{\Gamma(2-\beta)}t^{1-\beta}u_t(x,0). \tag{7}$$

By substituting Eq. (7) into Eq. (1), we obtain the following fractional integro-differential equation:

$$\begin{aligned}\frac{1}{\Gamma(2-\beta)}\int_0^t (t-\tau)^{1-\beta}u_{\tau\tau}(x,\tau)d\tau = g(x,t) - \frac{1}{\Gamma(2-\beta)}t^{1-\beta}u_t(x,0)\\ + u_{xx}(x,t) + \int_0^t k(x,t,\tau)u(x,\tau)d\tau. \end{aligned}\tag{8}$$

We generate the collocation points x_i and t_j by

$$\begin{aligned} x_i &= a + \frac{b-a}{N_1}i, \qquad && i = 0,1,\ldots,N_1,\\ t_j &= \frac{T}{N_2}j, && j = 0,1,\ldots,N_2. \end{aligned}$$

Now, we approximate $u(x,t)$ in Eq. (8) by the RBFs in the following form:

$$u(x,t) \simeq \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\phi_j(t), \tag{9}$$

where α_{ij} for $i = 0,1,\ldots,N_1$ and $j = 0,1,\ldots,N_2$ are unknown coefficients, and $\phi_i(x) = \phi(x-x_i)$ and $\phi_j(t) = \phi(t-t_j)$. By substituting Eq.

(9) into Eq. (8), we achieve

$$\begin{aligned}\frac{1}{\Gamma(2-\beta)}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\int_0^t (t-\tau)^{1-\beta}\phi_j''(\tau)d\tau = g(x,t)\\ -\frac{1}{\Gamma(2-\beta)}t^{1-\beta}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\phi_j'(0)\\ +\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i''(x)\phi_j(t)\\ +\int_0^t k\left(x,t,\tau\right)\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\phi_j(\tau)d\tau. \end{aligned}\tag{10}$$

Now, we transfer the integration interval $[0,t]$ to the fixed interval $[-1,1]$ for using some appropriate properties of quadrature formula in the proposed method. To this end, the following transformation has been considered

$$\tau(t,\xi) = \frac{t}{2}\xi + \frac{t}{2}, \quad -1 \leq \xi \leq 1.$$

Employing this transformation, integrals in Eq. (10) take the following forms:

$$\int_0^t (t-\tau)^{1-\beta}\phi_j''(\tau)d\tau = \frac{t}{2}\int_{-1}^1 (t-\tau(t,\xi))^{1-\beta}\phi_j''\left(\tau(t,\xi)\right)d\xi \triangleq \Theta_1(t), \tag{11}$$

and

$$\begin{aligned}&\int_0^t k\left(x,t,\tau\right)\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\phi_j(\tau)d\tau =\\ &\frac{t}{2}\int_{-1}^1 k\left(x,t,\tau(t,\xi)\right)\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\phi_j(\tau(t,\xi))d\xi \triangleq \Theta_2(x,t).\end{aligned}\tag{12}$$

Applying an $(\hat{m}+1)$-point Gauss-Legendre quadrature formula relative with the nodal points ξ_r in the interval [-1,1] and the corresponding weights w_r for numerically computing the integrals in Eqs. (11) and (12) yields

$$\begin{aligned}\Theta_1(t) \simeq& \frac{t}{2}\sum_{r=0}^{\hat{m}} w_r \left(t-\tau(t,\xi_r)\right)^{1-\beta} \phi_j''\left(\tau(t,\xi_r)\right) \triangleq \widetilde{\Theta}_1(t),\\ \Theta_2(x,t) \simeq& \frac{t}{2}\sum_{r=0}^{\hat{m}} w_r k\left(x,t,\tau(t,\xi_r)\right) \sum_{i=0}^{N_1}\sum_{j=0}^{N_2} \alpha_{ij}\phi_i(x)\phi_j(\tau(t,\xi_r))\\ &\triangleq \widetilde{\Theta}_2(x,t). \end{aligned} \tag{13}$$

From Eqs. (10)-(13), we can define the following residual function for the the problem (1):

$$\begin{aligned}\mathbf{R}(x,t) \triangleq& \frac{1}{\Gamma(2-\beta)}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\widetilde{\Theta}_1(t) - \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i''(x)\phi_j(t)\\ &+\frac{1}{\Gamma(2-\beta)}t^{1-\beta}\sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x)\phi_j'(0) - g(x,t) - \widetilde{\Theta}_2(x,t) = 0. \end{aligned} \tag{14}$$

Since we need to find the unknown coefficients α_{ij} for $i = 0, 1, \ldots, N_1$ and $j = 0, 1, \ldots, N_2$, we follow the procedure by forming a system of $(N_1+1)(N_2+1)$ equations. To this end, from Eq. (14), we generate $(N_1-1)N_2$ linear algebraic equations as follows:

$$\mathbf{R}(x_i,t_j) = 0, \qquad \text{for } i = 1,2,\ldots,N_1-1,\ j = 1,2,\ldots,N_2. \tag{15}$$

Also, from Eqs. (2) and (3), we generate $N_1 + 2N_2 + 1$ linear algebraic equations as follows:

$$\begin{aligned}\Lambda_1(x_r) &\triangleq \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(x_r)\phi_j(0) - f(x_r) = 0, \qquad r = 1,2,\ldots,N_1-1,\\ \Lambda_2(t_r) &\triangleq \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(a)\phi_j(t_r) - h(t_r) = 0, \qquad r = 0,1,\ldots,N_2,\\ \Lambda_3(t_r) &\triangleq \sum_{i=0}^{N_1}\sum_{j=0}^{N_2}\alpha_{ij}\phi_i(b)\phi_j(t_r) - l(t_r) = 0, \qquad r = 0,1,\ldots,N_2. \end{aligned} \tag{16}$$

Thus, Eqs. (15) and (16) including $(N_1+1)(N_2+1)$ equations can provide the unknown coefficients α_{ij} for $i=0,1,\ldots,N_1$ and $j=0,1,\ldots,N_2$. Consequently, we can approximately determine $u(x,t)$ using Eq. (9).

4 Numerical Examples

In this section, some numerical examples are provided to demonstrate the applicability and accuracy of the proposed method. Also, for the numerical quadrature rule, we used the 40-point Gauss-Legendre quadrature formula. It is worth mentioning that all numeric computations are performed via MAPLE 18 with 60 decimal digits on a X64-based PC with Intel (R) CPU Corei 4, 2.50 GHz with 8.0 GB of RAM.

Example 4.1. Consider Eq. (1) on the domain $[-1,1]\times[0,1]$, where

$$\begin{aligned} g(x,t) =&(1-x^2)t^{1-\beta}\mathbf{E}_{2,2-\beta}\left(-t^2\right)+\frac{(x^2-1)(x\sin(t)-\exp(xt)+\cos(t))}{x^2+1}\\ &+2\sin(t), \end{aligned}$$

and

$$k(x,t,\tau)=-\exp(x(t-\tau)),$$

in which $\mathbf{E}_{\eta,\zeta}(z)$ is the Mittag-Leffler function with two parameters which is defined in [14] by

$$\mathbf{E}_{\eta,\zeta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(j\eta+\zeta)}.$$

The initial and boundary conditions can be computed by using the exact solution that is $u(x,t)=(1-x^2)\sin(t)$.
This problem is solved by the proposed method for $N_1=5$ and $N_2=10$. The absolute errors obtained by the proposed method with two values of shape parameter c for three values of β in some selected points $(x_i,t_i)\in[-1,1]\times[0,1]$ are reported in Table 2. Graph of the absolute error function for the case $c=0.01$ and $\beta=0.25$ is shown in Figure 2. From the obtained results, it can be concluded that the proposed method can provide numerical solutions with high accuracy for this problem in all cases. Meanwhile, as it can be seen in Table 2, two selected shape

parameters lead to almost the same results. Note that the best value for the shape parameter can be obtained by employing an appropriate optimization method. It should be mentioned that throughout the paper, we have used the first 10 terms of the above series that represents the Mittag-Leffler function.

Table 2: Absolute errors obtained by the proposed method with two values of the shape parameter for three values of β in Example 4.1.

	$c = 0.01$			$c = 0.05$		
(x_i, t_i)	$\beta = 0.25$	$\beta = 0.55$	$\beta = 0.75$	$\beta = 0.25$	$\beta = 0.65$	$\beta = 0.85$
$(-1.0, 1.0)$	1.0000E-13	1.1000E-13	6.0000E-13	2.0000E-19	1.6000E-18	2.3000E-18
$(-0.8, 0.8)$	5.0046E-09	5.0051E-09	3.3214E-07	4.1102E-07	2.2823E-07	8.4240E-08
$(-0.6, 0.6)$	4.5640E-09	4.5647E-09	3.0433E-07	2.6709E-07	8.5414E-08	2.1074E-07
$(-0.4, 0.4)$	2.0217E-09	2.0222E-09	1.4232E-07	1.3902E-07	3.1601E-08	1.1966E-07
$(-0.2, 0.2)$	2.6904E-10	3.0000E-13	2.4377E-08	6.4970E-08	2.1880E-08	1.5021E-08
$(0.2, 0.2)$	2.6944E-10	2.6985E-10	2.4378E-08	6.4942E-08	2.1871E-08	1.5024E-08
$(0.4, 0.4)$	2.0212E-09	2.0216E-09	1.4225E-07	1.3858E-07	3.1386E-08	1.1974E-07
$(0.6, 0.6)$	4.5565E-09	4.5578E-09	3.0382E-07	2.6521E-07	8.4370E-08	2.1108E-07
$(0.8, 0.8)$	4.9854E-09	4.9863E-09	3.3080E-07	4.0733E-07	2.2600E-07	8.4943E-08
$(1.0, 1.0)$	2.6918E-10	3.0000E-13	8.0000E-13	4.0000E-19	1.2300E-18	7.4700E-19

Example 4.2. Consider Eq. (1) on the domain $[-1, 1] \times [0, 1]$, where

$$g(x,t) = \left(1 - x^2\right) t^{1-\beta} \mathbf{E}_{1,2-\beta}(t) + \frac{\left(x^2 - 1\right)\left(x\cos(xt) + \sin(xt) - x\exp(t)\right)}{x^2 + 1} + 2\exp(t),$$

and

$$k(x,t,\tau) = -\sin(x(t-\tau)),$$

subject to the initial condition $u(x, 0) = 1 - x^2$ and the homogeneous boundary conditions. The exact solution of this problem is $u(x,t) = \left(1 - x^2\right)\exp(t)$. We have solved this problem by applying the proposed method for $N_1 = 5$ and $N_2 = 8$. The absolute errors obtained by the proposed method with two values of shape parameter c for three values of β in some selected points $(x_i, t_i) \in [-1, 1] \times [0, 1]$ are reported in Table 3. Graph of the absolute error function for the case $c = 0.1$ and $\beta = 0.5$ is shown in Figure 3. The obtained results confirm that the proposed approach is an effective tool in solving such problems.

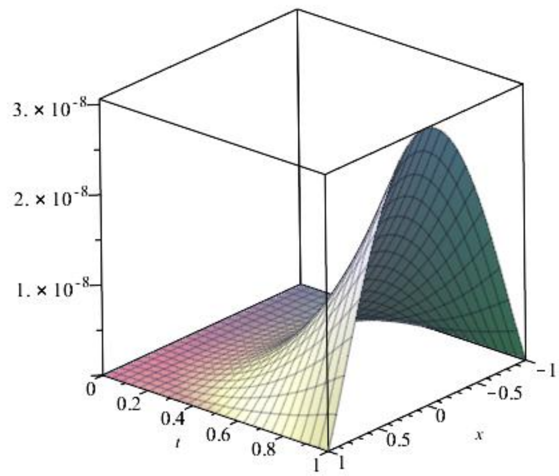

Figure 2: Graph of the absolute error function with $c = 0.01$ and $\beta = 0.25$ in Example 4.1.

Table 3: Absolute errors obtained by the proposed method with two values of the shape parameter for three values of β in Example 4.2.

	$c = 0.1$			$c = 0.001$		
(x_i, t_i)	$\beta = 0.25$	$\beta = 0.5$	$\beta = 0.75$	$\beta = 0.2$	$\beta = 0.6$	$\beta = 0.8$
$(-1.0, 1.0)$	4.0000E-21	4.0000E-21	6.0000E-13	1.8000E-20	2.0000E-05	3.0000E-04
$(-0.8, 0.8)$	2.1597E-05	2.1597E-05	3.3214E-07	2.1578E-05	3.4734E-05	5.2657E-06
$(-0.6, 0.6)$	1.4976E-05	1.4976E-05	3.0433E-07	1.5196E-05	2.6032E-05	4.3968E-05
$(-0.4, 0.4)$	9.0994E-06	9.0994E-06	1.4232E-07	9.7459E-06	7.2540E-06	3.2746E-05
$(-0.2, 0.2)$	5.9745E-06	5.9745E-06	2.4377E-08	7.4113E-06	1.6648E-05	3.5335E-04
$(0.2, 0.2)$	7.1760E-06	9.0000E-21	4.6565E-06	1.1352E-05	1.6648E-05	3.0948E-05
$(0.4, 0.4)$	9.5700E-06	9.0557E-06	8.8880E-06	4.5254E-05	2.7254E-05	9.8460E-06
$(0.6, 0.6)$	1.4952E-05	1.4821E-05	1.5452E-05	7.5032E-05	2.3968E-05	2.5232E-05
$(0.8, 0.8)$	2.1262E-05	2.1330E-05	2.2173E-05	1.8266E-05	1.5266E-05	2.9834E-05
$(1.0, 1.0)$	3.0000E-21	9.0000E-21	6.6000E-21	3.4000E-05	2.0000E-05	3.9200E-05

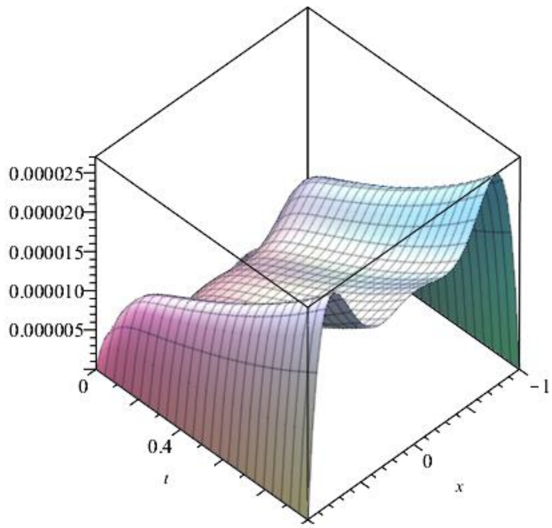

Figure 3: Graph of the absolute error function with $c = 0.1$ and $\beta = 0.5$ in Example 4.2.

Example 4.3. Consider Eq. (1) on the domain $[0, 1] \times [0, 1]$, where

$$g(x,t) = \frac{2x(1-x)}{\Gamma(3-\beta)} t^{2-\beta} + \frac{\exp(t^2 x)(t^4 x^3 - t^4 x^2 - 2t^2 x^2 + 2t^2 x - 2)}{t^3 x^2} + 2t^2,$$

and

$$k(x,t,\tau) = -\exp(xt\tau),$$

subject to the homogeneous initial and boundary conditions. The exact solution of this problem is $u(x,t) = t^2 x\,(1-x)$. The proposed method is applied for solving this problem where $N_1 = 9$ and $N_2 = 7$. The obtained numerical results reported in Table 4 and Figure 4. The yielded results confirm that the presented method is high accuracy in solving this example. Moreover, from Table 4, it can be concluded that there is no significant difference between the numerical results obtained via $c = 0.5$ and the ones obtained via $c = 0.001$.

Table 4: Absolute errors obtained by the proposed method with two values of the shape parameter for three values of β in Example 4.3.

	$c = 0.05$			$c = 0.001$		
(x_i, t_i)	$\beta = 0.2$	$\beta = 0.5$	$\beta = 0.75$	$\beta = 0.2$	$\beta = 0.6$	$\beta = 0.8$
$(0.2, 0.2)$	4.9927E-10	3.7641E-09	2.6513E-08	9.5000E-11	8.2100E-09	1.8500E-08
$(0.4, 0.4)$	2.0210E-09	1.7756E-08	1.1626E-07	6.4300E-10	3.9110E-08	1.2360E-07
$(0.6, 0.6)$	4.4404E-09	3.4003E-08	2.0437E-07	1.3560E-09	7.1120E-08	2.3800E-07
$(0.8, 0.8)$	7.5879E-09	3.6194E-08	1.9048E-07	1.4150E-09	6.7900E-08	2.1560E-07
$(1.0, 1.0)$	3.7100E-26	4.0000E-27	9.6000E-26	1.6000E-11	1.4000E-10	1.3900E-08

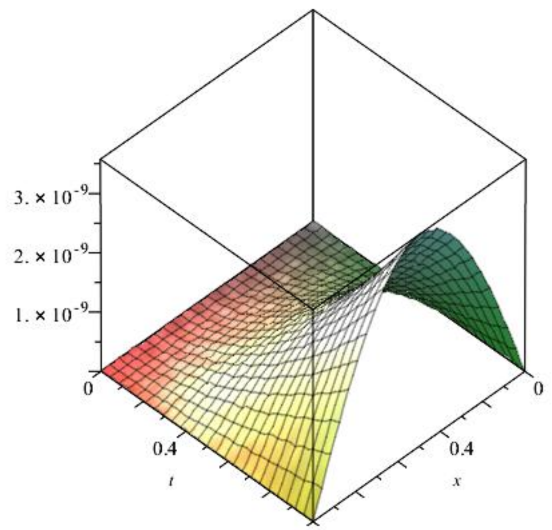

Figure 4: Graph of the absolute error function with $c = 0.001$ and $\beta = 0.2$ in Example 4.3.

Example 4.4. Consider Eq. (1) on the domain $[0, \frac{\pi}{2}] \times [0, 1]$, where

$$g(x,t) = \frac{6\sin^2(x)}{\Gamma(4-\beta)} t^{3-\beta} - (2t^3 \cos^2(x) - 2t^3 \sin^2(x)) \times \frac{\sin^2(x)(-t^3x^3 - 3t^2x^2 - 6xt + 6\exp(xt) - 6)}{x^4},$$

and

$$k(x,t,\tau) = -\exp(xt\tau),$$

subject to the homogeneous initial condition, and the boundary conditions $u(0,t) = 0$ and $u\left(\frac{\pi}{2}, t\right) = t^3$. The exact solution of this problem is $u(x,t) = t^3 \sin^2(x)$. The obtained results by applying the proposed method with $N_1 = 5$ and $N_2 = 8$ for two selections of the Gauss-Legendre points are reported in Tables 5 and 5. These results show that the presented method can lead to numerical solutions with high accuracy provided the shape parameter is selected appropriately. Also, it can be seen that the numerical results obtained with $c = 0.5$ are more accurate than the ones obtained with $c = 0.98$. We emphasize that the best value for the shape parameter can be exacted by utilizing an appropriate optimization method. Meanwhile, by comparison the numerical results summarized in Tables 5 and 6, it can be seen that there is no significant difference between the numerical results obtained via 20 Gauss-Legendre points and the ones obtained via 40 Gauss-Legendre points.

Table 5: Absolute errors obtained by the proposed method with two values of the shape parameter and three values of β via 20-point Gauss-Legendre quadrature formula in Example 4.4.

	$c = 0.5$			$c = 0.98$		
(x_i, t_i)	$\beta = 0.30$	$\beta = 0.47$	$\beta = 0.70$	$\beta = 0.30$	$\beta = 0.47$	$\beta = 0.70$
$(0.1\pi, 0.2)$	1.5935E-07	1.9263E-07	2.9610E-07	1.7318E-05	1.5271E-05	1.1084E-05
$(0.2\pi, 0.4)$	8.7320E-07	3.0967E-07	1.3649E-06	8.1439E-05	6.9177E-05	4.9439E-05
$(0.3\pi, 0.6)$	1.0212E-05	7.9706E-06	1.2360E-06	1.9132E-04	1.6526E-04	1.2282E-04
$(0.4\pi, 0.8)$	5.2105E-05	4.8137E-05	3.6355E-05	3.2403E-04	2.9590E-04	2.4810E-04
$(0.5\pi, 1.0)$	9.0000E-50	7.0000E-53	7.0000E-52	1.3000E-49	2.0000E-48	1.0000E-47

Table 6: Absolute errors obtained by the proposed method with two values of the shape parameter and three values of β via 40-point Gauss-Legendre quadrature formula in Example 4.4.

	$c = 0.5$			$c = 0.98$		
(x_i, t_i)	$\beta = 0.30$	$\beta = 0.47$	$\beta = 0.70$	$\beta = 0.30$	$\beta = 0.47$	$\beta = 0.70$
$(0.1\pi, 0.2)$	1.5272E-07	1.6604E-07	1.9927E-07	1.7324E-05	1.5297E-05	1.1180E-05
$(0.2\pi, 0.4)$	9.6468E-07	6.7590E-07	7.3124E-08	8.1530E-05	6.9540E-05	5.0866E-05
$(0.3\pi, 0.6)$	1.0570E-05	9.3811E-06	6.7962E-06	1.9167E-04	1.6665E-04	1.2832E-04
$(0.4\pi, 0.8)$	5.2748E-05	5.0622E-05	4.6000E-05	3.2466E-04	2.9832E-04	2.5751E-04
$(0.5\pi, 1.0)$	0.0000E-00	5.0000E-53	1.1000E-52	1.0000E-49	6.0000E-49	4.0000E-47

5 Conclusion

In this paper, a numerical method based on the radial basis functions (RBFs) has been developed and applied for the numerical solution of a class of linear fractional parabolic partial integro-differential equations (FPPI-DEs). The established method transforms such problems into equivalent systems of algebraic equations by expanding the solution of the problem in terms of the RBFs and applying Gauss-Legendre integration formula. Several test problems have been provided to show the accuracy of the presented approach. The achieved results confirm that only a few number of the RBFs are sufficient to obtain a high accurate numerical solution for such problems. The presented method can easily be developed for other classes of fractional partial integro-differential equations.

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F. S. Aghaei Maybodi
Faculty of Mathematics
Assistant Professor of Mathematics
Yazd University
Yazd, Iran
E-mail: fmaybodi1981@gmail.com

M. H. Heydari
Department of Mathematics
Assistant Professor of Mathematics
Shiraz University of Technology
Shiraz, Iran
E-mail: heydari@sutech.ac.ir

F. M. Maalek Ghaini
Faculty of Mathematics
Professor of Mathematics
Yazd University
Yazd, Iran
E-mail: maalek@yazd.ac.ir