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Original Research Paper

Existence of a Solution for a Multi Singular Pointwise Defined Fractional Differential System

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Abstract. In this paper, we will investigate the existence of some solutions for a multi singular fractional differential system with some boundary conditions.

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1 Introduction

Fractional differential equations, appears in many scientific problems such physics, chemistry, dynamic and engineering problems (see [4] and

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[9]). In recent decades many researches have been done in this field (see [13] and [1]). Also a great deal of papers have been written on considering the existence of a solution for fractional differential equations. Sometimes these equations are singular at some points (see [8] and [2]).

In 2011, Feng and Sun [5], considered the existence of a solution for the following singular system,

$$\begin{cases} D^{\alpha}u(t) + f(t, v(t)) = 0 \\ D^{\beta}v(t) + g(t, u(t)) = 0 \end{cases}$$

with boundary conditions $u(0) = u(1) = u'(0) = v(0) = v(1) = v'(0) = 0$, where $2 < \alpha, \beta \leq 3$, $f, g : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous, $\lim_{t \to 0^+} f(t, .) = +\infty$ and $\lim_{t \to 0^+} g(t, .) = +\infty$.
In 2014 Li [6], worked on the existence and uniqueness of solutions for singular fractional boundary value problem

$$D^{q}u(t) + f(t, u(t), D^{\sigma}u(t)) = 0,$$

with $u(0) = u'(1) = 0$ and $u'(1) = D^{\alpha}u(t)$, where $0 < t < 1$, $2 < q < 3, 0 < \sigma < 1, f : (0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, $f(t, x, y)$ may be singular at $t = 0$ and D^{α} is the standard Caputo derivative.
In 2017 Shabibi, Postolache and Rezapour investigated the singular fractional integro-differential system

$$\begin{cases} D^{\alpha_1}u_1 + f_1(t, u_1, \ldots, u_m, D^{\mu_1}u_1, \ldots, D^{\mu_m}u_m) \\ +g_1(t, u_1, \ldots, u_m, D^{\mu_1}u_1, \ldots, D^{\mu_m}u_m) = 0, \\ . \\ . \\ . \\ D^{\alpha_m}u_m + f_m(t, u_1, \ldots, u_m, D^{\mu_1}u_1, \ldots, D^{\mu_m}u_m) \\ +g_m(t, u_1, \ldots, u_m, D^{\mu_1}u_1, \ldots, D^{\mu_m}u_m) = 0, \end{cases}$$

with boundary conditions $u_i(0) = 0$, $u_i'(1) = 0$ and $\frac{d^k}{dt^k}[u_i(t)]_{t=0} = 0$ for $1 \leq i \leq m$ and $2 \leq k \leq n-1$, where $\alpha_i \geq 2$, $[\alpha_i] = n-1$, $0 < \mu_i < 1$, D is the Caputo fractional derivative, f_i is a Caratheodory function, g_i satisfies Lipschitz condition and $f_i(t, x_1, \ldots, x_{2m})$ is singular at $t = 0$ of for all $1 \leq i \leq m$ ([12]).

In 2018, the existence of solutions for the pointwise defined three steps crisis integro-differential equation

$$D^{\alpha}x(t) + f(t, x(t), x'(t), D^{\beta}x(t), \int_0^t h(\xi)x(\xi)d\xi, \phi(x(t))) = 0$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geq 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that $\|\phi(x) - \phi(y)\| \leq \theta_0\|x - y\| + \theta_1\|x' - y'\|$ for some non-negative real numbers θ_0 and $\theta_1 \in [0, \infty)$ and all $x, y \in X$, D^{α} is the Caputo fractional derivative of order α, $f(t, x_1(t), ..., x_5(t)) = f_1(t, x_1(t), ..., x_5(t))$ for all $t \in [0, \lambda)$, $f(t, x_1(t), ..., x_5(t)) = f_2(t, x_1(t), ..., x_5(t))$ for all $t \in [\lambda, \mu]$ and $f(t, x_1(t), ..., x_5(t)) = f_3(t, x_1(t), ..., x_5(t))$ for all $t \in (\mu, 1]$, $f_1(t, ., ., ., .)$ and $f_3(t, ., ., ., .)$ are continuous on $[0, \lambda)$ and $(\mu, 1]$ and $f_2(t, ., ., ., .)$ is multi-singular was investigated [3].

Motivated by the above works, we will investigate the existence of a solution of the following nonlinear fractional differential pointwise defined system

$$\begin{cases} D^{\alpha_1}x(t) + f_1(t, x(t), y(t), x'(t), y'(t), D^{\beta_1}x(t), D^{\beta_2}y(t), \\ \int_0^t h_1(\xi)x(\xi)d\xi, \int_0^t h_2(\xi)x(\xi)d\xi) = 0, \\ \\ D^{\alpha_2}y(t) + f_2(t, x(t), y(t), x'(t), y'(t), D^{\beta_1}x(t), D^{\beta_2}y(t), \\ \int_0^t h_1(\xi)x(\xi)d\xi, \int_0^t h_2(\xi)x(\xi)d\xi) = 0, \end{cases} \tag{1}$$

with boundary conditions $D^{\mu_1}x(\eta_1) = \lambda_1$, $D^{\mu_2}x(\eta_2) = \lambda_2$, where $\alpha \geq 2$, $x(1) = x^{(j)}(0) = 0$ and $y(1) = y^{(j)}(0) = 0$ for $j \geq 2$, where $\eta_i, \mu_i \in (0,1)$, $\lambda_i \geq 0$, D^{α_i} is the Caputo fractional derivative of order $\alpha_i \geq 2$, $n = [\alpha_i] + 1$, $h_i \in L^1$ and $f_i \in L^1$ is singular at some points $[0,1]$ for $i = 1, 2$. Recall that $D^{\alpha}x(t) + f(t) = 0$ is a pointwise defined equation on $[0,1]$ if there exists a set $E \subset [0,1]$ such that the measure of E^c is zero and the equation holds on E (see [2]). In this paper, we use $\|.\|_1$ for the norm of $L^1[0,1]$, $\|.\|$ for the sup norm of $Y = C[0,1]$, $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0,1]$ and $\|(x,y)\|_{**} = \max\{\|x\|_*, \|y\|_*\}$ for the norm of X^2.

2 Preliminaries

In this section, some of the definitions and primary theorems which are required in the sequel, are stated.

Definition 2.1. ([7]) The Riemann-Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a, \infty) \to \mathbb{R}$ is defined by $I^p_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s) ds$, provided that the right-hand side is pointwise define on (a, ∞). We denote $I^p_{0^+} f(t)$ by $I^p f(t)$.

Definition 2.2. ([7]) The Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds$, where $n = [\alpha] + 1$ and $f : (a, \infty) \to \mathbb{R}$ is a function.

Definition 2.3. ([10]) Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. One can check that $\psi(t) < t$ for all $t > 0$.

Definition 2.4. ([10]) Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 2.5. ([10]) Let (X, d) be a metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.

Lemma 2.6. *([10]) Let (X, d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0, \infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Lemma 2.7. *([11]) Let $n - 1 \leq \alpha < n$ and $x \in C(0, 1) \cap L^1(0, 1)$. Then, we have $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, \ldots, c_{n-1}$.*

3 Main Results

Now, we are ready for providing our results.

Lemma 3.1. *Let $\alpha \geq 2$, $[\alpha] = n-1$, $\mu, \eta \in (0,1)$, $\lambda \geq 0$ and $f \in L^1[0,1]$, then the solution of the problem $D^\alpha u(t) + f(t) = 0$ with the boundary conditions $D^\mu u(\eta) = \lambda$, $u(1) = u^{(j)}(0) = 0$ for $j \geq 2$ is $u(t) = \int_0^1 G(t,s)f(s)ds + H(t)$, where $G(t,s)$ and $H(t)$ are defined as follow*

$$G(t,s) = \begin{cases} \frac{-(t-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ -\frac{(1-t)\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(\mu-s)^{\alpha-\mu-1} & 0 \leq s \leq t \leq 1, \ s \leq \eta, \\ \\ \frac{-(t-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq \eta \leq s \leq t \leq 1, \\ \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ -\frac{(1-t)\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(\mu-s)^{\alpha-\mu-1} & 0 \leq t \leq s \leq \eta \leq 1, \\ \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1, \ \eta \leq s, \end{cases}$$

and

$$H(t) = -\frac{\lambda\Gamma(2-\mu)}{\eta^{1-\mu}}(1-t).$$

Proof. First by similar manner as [3], we conclude that lemma (2.6) is valid on $L^1[0,1]$. Now let $x(t)$ be a solution for the problem, since $x^{(j)}(0) = 0$ for $j \geq 2$, by using Lemma (2.6) we have

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds + c_0 + c_1 t. \tag{2}$$

By using $u(1) = 0$ we have

$$\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f(s)ds = c_0 + c_1. \tag{3}$$

By (2) we have

$$D^\mu u(\eta) = -\frac{1}{\Gamma(\alpha-\mu)}\int_0^\eta (\eta-s)^{\alpha-\mu-1}f(s)ds + \frac{c_1\eta^{1-\mu}}{\Gamma(2-\mu)},$$

and since $D^{\mu}u(\eta)=\lambda$ we have

$$-\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds+\frac{c_1\eta^{1-\mu}}{\Gamma(2-\mu)}=\lambda.$$

So

$$\frac{c_1\eta^{1-\mu}}{\Gamma(2-\mu)}=\lambda+\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds,$$

hence

$$c_1=\frac{\Gamma(2-\mu)}{\eta^{1-\mu}}\left[\lambda+\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds\right].$$

By putting in (3) we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}f(s)ds=c_0\\&+\frac{\Gamma(2-\mu)}{\eta^{1-\mu}}\left[\lambda+\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds\right],\end{aligned}$$

so

$$\begin{aligned}c_0 &= \frac{1}{\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}f(s)ds\\&\quad-\frac{\Gamma(2-\mu)}{\eta^{1-\mu}}\left[\lambda+\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds\right],\end{aligned}$$

hence

$$\begin{aligned}u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}f(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}f(s)ds\\&\quad-\frac{\Gamma(2-\mu)}{\eta^{1-\mu}}\left[\lambda+\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds\right]\\&\quad+\frac{\Gamma(2-\mu)}{\eta^{1-\mu}}\left[\lambda+\frac{1}{\Gamma(\alpha-\mu)}\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds\right]t\\&= -\frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}f(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}f(s)ds\\&\quad-\frac{\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(1-t)\int_0^{\eta}(\eta-s)^{\alpha-\mu-1}f(s)ds\\&\quad+\frac{\lambda\Gamma(2-\mu)}{\eta^{1-\mu}}(1-t),\end{aligned}$$

so

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s)ds \\ &\quad -\frac{\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(1-t)\int_0^\eta (\eta-s)^{\alpha-\mu-1} f(s)ds + H(t), \end{aligned}$$

where

$$H(t) = -\frac{\lambda\Gamma(2-\mu)}{\eta^{1-\mu}}(1-t).$$

If $t \le \eta \le 1$ then

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\left(\int_0^t + \int_t^\eta + \int_\eta^1\right)(1-s)^{\alpha-1} f(s)ds \\ &\quad -\frac{\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(1-t)\left(\int_0^t + \int_t^\eta\right)(\eta-s)^{\alpha-\mu-1} f(s)ds + H(t) \end{aligned}$$

and if $\eta \le t \le 1$ then

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\left(\int_0^\eta + \int_\eta^t\right)(t-s)^{\alpha-1} f(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha)}\left(\int_0^\eta + \int_\eta^t + \int_t^1\right)(1-s)^{\alpha-1} f(s)ds \\ &\quad -\frac{\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(1-t)\int_0^\eta (\eta-s)^{\alpha-\mu-1} f(s)ds + H(t), \end{aligned}$$

so we can write $u(t) = \int_0^1 G(t,s)f(s)ds + H(t)$, where

$$G(t,s)=\begin{cases} \frac{-(t-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ -\frac{(1-t)\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(\mu-s)^{\alpha-\mu-1} & 0\le s\le t\le 1,\ \ s\le\eta, \\ \\ \frac{-(t-s)^{\alpha-1}+(1-s)^{\alpha-1}}{\Gamma(\alpha)} & 0\le\eta\le s\le t\le 1, \\ \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ -\frac{(1-t)\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(\mu-s)^{\alpha-\mu-1} & 0\le t\le s\le\eta\le 1, \\ \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} & 0\le t\le s\le 1,\ \ \eta\le s. \end{cases}$$

$\square$ Now we can gain $\frac{\partial G}{\partial t}(t,s)$ as follow:

$$\frac{\partial G}{\partial t}(t,s)=\begin{cases} \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ +\frac{\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(\mu-s)^{\alpha-\mu-1} & 0\le s\le t\le 1,\ \ s\le\eta, \\ \\ \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} & 0\le\eta\le s\le t\le 1, \\ \\ \frac{\Gamma(2-\mu)}{\eta^{1-\mu}\Gamma(\alpha-\mu)}(\mu-s)^{\alpha-\mu-1} & 0\le t\le s\le\eta\le 1, \\ \\ 0 & 0\le t\le s\le 1,\ \ \eta\le s. \end{cases}$$

One can see G and $\frac{\partial}{\partial t}G$ are continuous respect to t. Consider the space $X=C^1[0,1]$ with the norm $\|.\|_*$ and the space X^2 with the norm $\|.\|_{**}$ where $\|(x,y)\|_{**}=\max\{\|x\|_*,\|y\|_*\}$, $\|x\|_*=\max\{\|x\|,\|x'\|\}$ and $\|.\|$ is the supremum norm on $C[0,1]$. Let f_1,f_2 be two maps on $[0,1]\times X^8$ such that are singular at some points of $[0,1]$. For $i=1,2$, let

$$H_i(t)=-\frac{\lambda_i\Gamma(2-\mu_i)}{{\eta_i}^{1-\mu_i}}(1-t),$$

$$G_{\alpha_i}(t,s)=\begin{cases}\frac{-(t-s)^{\alpha_i-1}+(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \\ -\frac{(1-t)\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(\mu_i-s)^{\alpha_i-\mu_i-1} & 0\le s\le t\le 1,\ \ s\le\eta_i,\\ \frac{-(t-s)^{\alpha_i-1}+(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} & 0\le\eta_i\le s\le t\le 1,\\ \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \\ -\frac{(1-t)\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(\mu_i-s)^{\alpha_i-\mu_i-1} & 0\le t\le s\le\eta_i\le 1,\\ \frac{(1-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} & 0\le t\le s\le 1,\ \ \eta_i\le s,\end{cases}$$

and define $F : X^2 \to X^2$ as

$$F(x,y)(t)=\begin{pmatrix}\phi_1(x,y)(t)\\ \phi_2(x,y)(t)\end{pmatrix},$$

where

$$\begin{aligned} &\phi_i(x,y)(t)=\\ &\int_0^1 G_{\alpha_i}(t,s)f_i(s,x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\\ &\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)ds+H_i(t)\\ = \quad &-\frac{1}{\Gamma(\alpha_i)}\int_t^1 (t-s)^{\alpha_i-1}f_i(s,x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\\ &\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)ds\\ &+\frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha-1}f_i(s,x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\\ &\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)ds\\ &-\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}f_i(s,x(s),\\ &y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\\ &\int_0^s h_2(\xi)y(\xi)d\xi)ds+H_i(t), \end{aligned}$$

so

$$F'(x,y)(t) = \begin{pmatrix} \phi_1'(x,y)(t) \\ \phi_2'(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned}
&\phi_i'(x,y)(t) = \\
&\int_0^1 \frac{\partial G_{\alpha_i}}{\partial t}(t,s) f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \\
&\int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi)ds + H_i'(t) \\
= \quad & -\frac{1}{\Gamma(\alpha_i - 1)} \int_t^1 (t-s)^{\alpha_i - 2} f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), \\
&D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi)ds \\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} \int_0^{\eta_i} (\eta_i - s)^{\alpha_i - \mu_i - 1} f_i(s, x(s), y(s), x'(s), y'(s), \\
&D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi)ds + H_i'(t),
\end{aligned}$$

for all $t \in [0,1]$. It's obvious that the singular pointwise defined equation (1) has a solution if and only if the map F has a fixed point. In the next theorem, we provide our main result about the existence of a solution for the problem (1).

Theorem 3.2. *For $i = 1,2$, let $\alpha_i \geq 2$, $[\alpha_i] = n-1$, $\beta_i\mu_i, \eta_i \in (0,1)$, $\lambda_i \geq 0$, $h_i \in L^1[0,1]$ with $\|h\|_i := m_i$, $f_i : [0,1] \times X^8 \to \mathbb{R}$ be mappings that are singular on some points $[0,1]$ such that*

$$|f_i(t, x_1, ..., x_8) - f_i(t, y_1, ..., y_8)| \leq \sum_{j=1}^{8} a_{i,j}(t)|x_j - y_j|^{\gamma_{i,j}}$$

for all $x_1, ..., x_8, y_1, ..., y_8 \in X$ and almost all $t \in [0,1]$ and

$$|f_i(t, x_1, ..., x_8)| \leq \sum_{k=1}^{k_0} b_{i,k}(t) T_{i,k}(x_1, ..., x_8) + M_i(x_1, ..., x_8)$$

where $k_0 \in \mathbb{N}$, $a_{i,j}, b_{i,k} : [0,1] \to \mathbb{R}^+$, $T_{i,k}, M_i : X^8 \to \mathbb{R}^+$ *for each* $1 \leq k \leq k_0$ *are nondecreasing mappings respect all their components with* $\lim_{z\to\infty} \frac{T_{i,k}(z,...,z)}{z} = p_{i,k}$ *and* $\lim_{z\to\infty} M_i(z,...,z) < \infty$ *for some* $p_{i,k} \in \mathbb{R}^+$ *and all* $1 \leq i \leq 2$, $1 \leq j \leq 8$ *and* $1 \leq k \leq k_0$.

Also, let

$$\max_{1\leq i\leq 2}\left\{\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}p_{i,k}\right\} \in [0,\frac{1}{\Delta})$$

where

$$\Delta = \max\left\{1, \frac{1}{\Gamma(2-\beta_1)}, \frac{1}{\Gamma(2-\beta_2)}, m_1, m_2\right\}$$

and $\hat{b}_{i,k}, \hat{a}_{i,j} \in L^1[0,1]$, $\hat{a}_{i,j}(s) = (1-s)^{\alpha_i-2}a_{i,j}(s)$. *If*

$$\max_{1\leq i\leq 2}\left\{\left(\sum_{j=1}^{8}\Delta_{i,j}\|\hat{a}_{i,j}\|_{[0,1]}\right)\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\right\} < 1$$

where $\Delta_{i,j} = \Delta^{\gamma_{i,j}}$ *then the pointwise defined system*

$$\begin{cases} D^{\alpha_1}x(t) + f_1(t, x(t), y(t), x'(t), y'(t), D^{\beta_1}x(t), D^{\beta_2}y(t), \\ \int_0^t h_1(\xi)x(\xi)d\xi, \int_0^t h_2(\xi)x(\xi)d\xi) = 0, \\ \\ D^{\alpha_2}y(t) + f_2(t, x(t), y(t), x'(t), y'(t), D^{\beta_1}x(t), D^{\beta_2}y(t), \\ \int_0^t h_1(\xi)x(\xi)d\xi, \int_0^t h_2(\xi)y(\xi)d\xi) = 0, \end{cases}$$

with boundary contions $D^{\mu_1}x(\eta_1) = \lambda_1$, $D^{\mu_2}y(\eta_2) = \lambda_2$, $x(1) = x^{(j)}(0) = 0$ *and* $y(1) = y^{(j)}(0) = 0$ *for* $j \geq 2$ *has a solution.*

Proof. First we will prove that F is continuous on X^2. Let $0 < \epsilon < 1$ be artibary and $\{(x_n, y_n)\}_{n\geq 1} \to (x, y)$ in X^2, then there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that $\|(x_n, y_n) - (x, y)\|_{**} < \epsilon$, hence $\|x_n - x\|_* < \epsilon$

and $\|y_n - y\|_* < \epsilon$, then we have

$$\begin{aligned}
&|\phi_i(x_n, y_n)(t) - \phi_i(x, y)(t)| \\
= &\Big| \int_0^1 G_{\alpha_i}(t,s)\Big(f_i(s, x_n(s), y_n(s), x_n'(s), y_n'(s), D^{\beta_1}x_n(s), D^{\beta_2}y_n(s), \\
&\int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) - f_i(s, x(s), y(s), x'(s), y'(s), \\
&D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi\Big) ds\Big| \\
\leq &\frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} \Big| f_i(s, x_n(s), y_n(s), x_n'(s), y_n'(s), D^{\beta_1}x_n(s), \\
&D^{\beta_2}y_n(s), \int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) \\
&-f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \\
&\int_0^s h_2(\xi)y(\xi)d\xi)\Big| ds \\
&+\frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha - 1} \Big| f_i(s, x_n(s), y_n(s), x_n'(s), y_n'(s), D^{\beta_1}x_n(s), \\
&D^{\beta_2}y_n(s), \int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) - f_i(s, x(s), y(s), \\
&x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi)\Big| ds \\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)}(1-t) \int_0^{\eta_i} (\eta_i - s)^{\alpha_i - \mu_i - 1} \Big| f_i(s, x_n(s), y_n(s), x_n'(s), \\
&y_n'(s), D^{\beta_1}x_n(s), D^{\beta_2}y_n(s), \int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) \\
&-f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \\
&\int_0^s h_2(\xi)y(\xi)d\xi)\Big| ds
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} \Bigg[a_{i,1}(s)|x_n(s) - x(s)|^{\gamma_{i,1}} + a_{i,2}(s) \times$$
$$|y_n(s) - y(s)|^{\gamma_{i,2}} + a_{i,3}(s)|x'_n(s) - x'(s)|^{\gamma_{i,3}} + a_{i,4}(s)|y'_n(s) - y'(s)|^{\gamma_{i,4}}$$
$$+a_{i,5}(s)|D^{\beta_1}(x_n - x)(s)|^{\gamma_{i,5}} + a_{i,6}(s)|D^{\beta_2}(y_n - y)(s)|^{\gamma_{i,6}}$$
$$+a_{i,7}(s)(\int_0^s |x_n - x|(\xi)d\xi)^{\gamma_{i,7}} + a_{i,8}(s)(\int_0^s |y_n - y|(\xi)d\xi)^{\gamma_{i,8}} \Bigg] ds$$
$$+\frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha - 1} \Bigg[a_{i,1}(s)|x_n(s) - x(s)|^{\gamma_{i,1}} + a_{i,2}(s)|y_n(s) - y(s)|^{\gamma_{i,2}}$$
$$+a_{i,3}(s)|x'_n(s) - x'(s)|^{\gamma_{i,3}} + a_{i,4}(s)|y'_n(s) - y'(s)|^{\gamma_{i,4}} + a_{i,5}(s) \times$$
$$|D^{\beta_1}(x_n - x)(s)|^{\gamma_{i,5}} + a_{i,6}(s)|D^{\beta_2}(y_n - y)(s)|^{\gamma_{i,6}}$$
$$+a_{i,7}(s)(\int_0^s |x_n - x|(\xi)d\xi)^{\gamma_{i,7}} + a_{i,8}(s)(\int_0^s |y_n - y|(\xi)d\xi)^{\gamma_{i,8}} \Bigg] ds$$
$$+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)}(1-t) \int_0^{\eta_i} (\eta_i - s)^{\alpha_i - \mu_i - 1} \Bigg[a_{i,1}(s)|x_n(s) - x(s)|^{\gamma_{i,1}}$$
$$+a_{i,2}(s)|y_n(s) - y(s)|^{\gamma_{i,2}} + a_{i,3}(s)|x'_n(s) - x'(s)|^{\gamma_{i,3}}$$
$$+a_{i,4}(s)|y'_n(s) - y'(s)|^{\gamma_{i,4}} + a_{i,5}(s)|D^{\beta_1}(x_n - x)(s)|^{\gamma_{i,5}}$$
$$+a_{i,6}(s)|D^{\beta_2}(y_n - y)(s)|^{\gamma_{i,6}} + a_{i,7}(s)(\int_0^s |x_n - x|(\xi)d\xi)^{\gamma_{i,7}}$$
$$+a_{i,8}(s)(\int_0^s |y_n - y|(\xi)d\xi)^{\gamma_{i,8}} \Bigg] ds.$$

Now since $|D^{\beta_i}x(s)| = \frac{|x'(s)|}{\Gamma(2-\beta_i)}$ and

$$\int_0^s h_i(\xi)|x(\xi)|d\xi \leq \|x\| \int_0^s h_i(\xi)d\xi = m_i\|x\|$$

we have

$$\begin{aligned}
&|\phi_i(x_n,y_n)(t)-\phi_i(x,y)(t)|\\
\leq\ & \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}\Bigg[a_{i,1}(s)|x_n(s)-x(s)|^{\gamma_{i,1}}+a_{i,2}(s)|y_n(s)-y(s)|^{\gamma_{i,2}}\\
&+a_{i,3}(s)|x_n'(s)-x'(s)|^{\gamma_{i,3}}+a_{i,4}(s)|y_n'(s)-y'(s)|^{\gamma_{i,4}}\\
&+a_{i,5}(s)(\frac{|x_n'(s)-x'(s)|}{\Gamma(2-\beta_1)})^{\gamma_{i,5}}+a_{i,6}(s)(\frac{|y_n'(s)-y'(s)|}{\Gamma(2-\beta_2)})^{\gamma_{i,6}}\\
&+a_{i,7}(s)(m_1\|x_n-x\|)^{\gamma_{i,7}}+a_{i,8}(s)(m_2\|y_n-y\|)^{\gamma_{i,8}}\Bigg]ds\\
&+\frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha-1}\Bigg[a_{i,1}(s)|x_n(s)-x(s)|^{\gamma_{i,1}}+a_{i,2}(s)|y_n(s)-y(s)|^{\gamma_{i,2}}\\
&+a_{i,3}(s)|x_n'(s)-x'(s)|^{\gamma_{i,3}}+a_{i,4}(s)|y_n'(s)-y'(s)|^{\gamma_{i,4}}\\
&+a_{i,5}(s)(\frac{|x_n'(s)-x'(s)|}{\Gamma(2-\beta_1)})^{\gamma_{i,5}}+a_{i,6}(s)(\frac{|y_n'(s)-y'(s)|}{\Gamma(2-\beta_2)})^{\gamma_{i,6}}\\
&+a_{i,7}(s)(m_1\|x_n-x\|)^{\gamma_{i,7}}+a_{i,8}(s)(m_2\|y_n-y\|)^{\gamma_{i,8}}\Bigg]ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\Bigg[a_{i,1}(s)|x_n(s)-x(s)|^{\gamma_{i,1}}\\
&+a_{i,2}(s)|y_n(s)-y(s)|^{\gamma_{i,2}}+a_{i,3}(s)|x_n'(s)-x'(s)|^{\gamma_{i,3}}\\
&+a_{i,4}(s)|y_n'(s)-y'(s)|^{\gamma_{i,4}}+a_{i,5}(s)(\frac{|x_n'(s)-x'(s)|}{\Gamma(2-\beta_1)})^{\gamma_{i,5}}\\
&+a_{i,6}(s)(\frac{|y_n'(s)-y'(s)|}{\Gamma(2-\beta_2)})^{\gamma_{i,6}}+a_{i,7}(s)(m_1\|x_n-x\|)^{\gamma_{i,7}}\\
&+a_{i,8}(s)(m_2\|y_n-y\|)^{\gamma_{i,8}}\Bigg]ds\\
\leq\ & \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}\Bigg[a_{i,1}(s)\|x_n-x\|^{\gamma_{i,1}}+a_{i,2}(s)\|y_n-y\|^{\gamma_{i,2}}\\
&+a_{i,3}(s)\|x_n'-x'\|^{\gamma_{i,3}}+a_{i,4}(s)\|y_n'-y'\|^{\gamma_{i,4}}+a_{i,5}(s)\frac{\|x_n'-x'\|^{\gamma_{i,5}}}{\Gamma(2-\beta_1)^{\gamma_{i,5}}}
\end{aligned}$$

$$+a_{i,6}(s)\frac{\|y_n'-y'\|^{\gamma_{i,6}}}{\Gamma(2-\beta_2)^{\gamma_{i,6}}}+a_{i,7}(s)m_1^{\gamma_{i,7}}\|x_n-x\|^{\gamma_{i,7}}$$

$$+a_{i,8}(s)m_2^{\gamma_{i,8}}\|y_n-y\|^{\gamma_{i,8}}\Bigg]ds$$

$$+\frac{1}{\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha-1}\Bigg[a_{i,1}(s)\|x_n-x\|^{\gamma_{i,1}}+a_{i,2}(s)\|y_n-y\|^{\gamma_{i,2}}$$

$$+a_{i,3}(s)\|x_n'-x'\|^{\gamma_{i,3}}+a_{i,4}(s)\|y_n'-y'\|^{\gamma_{i,4}}+a_{i,5}(s)\frac{\|x_n'-x'\|^{\gamma_{i,5}}}{\Gamma(2-\beta_1)^{\gamma_{i,5}}}$$

$$+a_{i,6}(s)\frac{\|y_n'-y'\|^{\gamma_{i,6}}}{\Gamma(2-\beta_2)^{\gamma_{i,6}}}+a_{i,7}(s)m_1^{\gamma_{i,7}}\|x_n-x\|^{\gamma_{i,7}}$$

$$+a_{i,8}(s)m_2^{\gamma_{i,8}}\|y_n-y\|^{\gamma_{i,8}}\Bigg]ds$$

$$+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\Bigg[a_{i,1}(s)\|x_n-x\|^{\gamma_{i,1}}$$

$$+a_{i,2}(s)\|y_n-y\|^{\gamma_{i,2}}+a_{i,3}(s)\|x_n'-x'\|^{\gamma_{i,3}}+a_{i,4}(s)\|y_n'-y'\|^{\gamma_{i,4}}$$

$$+a_{i,5}(s)\frac{\|x_n'-x'\|^{\gamma_{i,5}}}{\Gamma(2-\beta_1)^{\gamma_{i,5}}}+a_{i,6}(s)\frac{\|y_n'-y'\|^{\gamma_{i,6}}}{\Gamma(2-\beta_2)^{\gamma_{i,6}}}+a_{i,7}(s)m_1^{\gamma_{i,7}}\|x_n-x\|^{\gamma_{i,7}}$$

$$+a_{i,8}(s)m_2^{\gamma_{i,8}}\|y_n-y\|^{\gamma_{i,8}}\Bigg]ds \tag{4}$$

$$\leq \frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}\Bigg[a_{i,1}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,1}}+a_{i,2}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,2}}$$

$$+a_{i,3}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,3}}+a_{i,4}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,4}}$$

$$+a_{i,5}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,5}}+a_{i,6}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,6}}$$

$$+a_{i,7}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,7}}+a_{i,8}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,8}}\Bigg]ds$$

$$+\frac{1}{\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha-1}\Bigg[a_{i,1}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,1}}+a_{i,2}(s)\times$$

$$(\Delta\|y_n-y\|_*)^{\gamma_{i,2}}+a_{i,3}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,3}}+a_{i,4}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,4}}$$

$$+a_{i,5}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,5}}+a_{i,6}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,6}}$$

$$+a_{i,7}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,7}}+a_{i,8}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,8}}\Bigg]ds$$

$$
\begin{aligned}
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\times\\
&\bigg[a_{i,1}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,1}}+a_{i,2}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,2}}\\
&+a_{i,3}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,3}}+a_{i,4}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,4}}\\
&+a_{i,5}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,5}}+a_{i,6}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,6}}\\
&+a_{i,7}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,7}}+a_{i,8}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,8}}\bigg]ds\\
\leq\ & \frac{1}{\Gamma(\alpha_i)}\sum_{j=1}^{8}\left[\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\int_0^t(t-s)^{\alpha_i-1}a_{i,j}(s)ds\right]\\
&+\frac{1}{\Gamma(\alpha_i)}\sum_{j=1}^{8}\left[\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\int_0^1(1-s)^{\alpha_i-1}a_{i,j}(s)ds\right]\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\sum_{j=1}^{8}\left[\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}a_{i,j}(s)ds\right]\\
\leq\ & \frac{1}{\Gamma(\alpha_i)}\sum_{j=1}^{8}\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}+\frac{1}{\Gamma(\alpha_i)}\sum_{j=1}^{8}\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\sum_{j=1}^{8}\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\\
\leq\ & \epsilon^{\gamma}\sum_{j=1}^{8}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\left[\frac{2}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)(1-t)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right]
\end{aligned}
$$

where $\Delta=\max\{1,\frac{1}{\Gamma(2-\beta_1)},\frac{1}{\Gamma(2-\beta_2)},m_1,m_2\}$, hence $\gamma=\min\{\gamma_{i,j},\ \ j=1,...,8,\ i=1,2\}$

$$
\|\phi_i(x_n,y_n)-\phi_i(x,y)\|\leq\epsilon^{\gamma}\sum_{j=1}^{8}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\left[\frac{2}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right].
$$

Also we have

$$
\begin{aligned}
& |\phi_i'(x_n,y_n)(t)-\phi_i'(x,y)(t)| \\
= & \left| \int_0^1 \frac{\partial G_{\alpha_i}}{\partial t}(t,s)(f_i(s,x_n(s),y_n(s),x_n'(s),y_n'(s),D^{\beta_1}x_n(s),D^{\beta_2}y_n(s), \right. \\
& \int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) - f_i(s,x(s),y(s),x'(s),y'(s), \\
& \left. D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi))ds \right| \\
\leq & \frac{1}{\Gamma(\alpha_i-1)} \int_0^t (t-s)^{\alpha_i-2} \Big| f_i(s,x_n(s),y_n(s),x_n'(s),y_n'(s),D^{\beta_1}x_n(s), \\
& D^{\beta_2}y_n(s), \int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) - f_i(s,x(s),y(s), \\
& x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi)\Big| ds \\
& + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)} \int_0^{\eta_i} (\eta_i-s)^{\alpha_i-\mu_i-1} \Big| f_i(s,x_n(s),y_n(s),x_n'(s), \\
& y_n'(s),D^{\beta_1}x_n(s),D^{\beta_2}y_n(s), \int_0^s h_1(\xi)x_n(\xi)d\xi, \int_0^s h_2(\xi)y_n(\xi)d\xi) \\
& -f_i(s,x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \\
& \int_0^s h_2(\xi)y(\xi)d\xi)\Big| ds \\
\leq & \frac{1}{\Gamma(\alpha_i-1)} \int_0^t (t-s)^{\alpha_i-2} \Big[a_{i,1}(s)|x_n(s)-x(s)|^{\gamma_{i,1}} + a_{i,2}(s) \times \\
& |y_n(s)-y(s)|^{\gamma_{i,2}} + a_{i,3}(s)|x_n'(s)-x'(s)|^{\gamma_{i,3}} + a_{i,4}(s)|y_n'(s)-y'(s)|^{\gamma_{i,4}} \\
& +a_{i,5}(s)|D^{\beta_1}(x_n-x)(s)|^{\gamma_{i,5}} + a_{i,6}(s)|D^{\beta_2}(y_n-y)(s)|^{\gamma_{i,6}} \\
& +a_{i,7}(s)(\int_0^s |x_n-x|(\xi)d\xi)^{\gamma_{i,7}} + a_{i,8}(s)(\int_0^s |y_n-y|(\xi)d\xi)^{\gamma_{i,8}} \Big] ds \\
& + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)} \int_0^{\eta_i} (\eta_i-s)^{\alpha_i-\mu_i-1} \Big[a_{i,1}(s)|x_n(s)-x(s)|^{\gamma_{i,1}} \\
& +a_{i,2}(s)|y_n(s)-y(s)|^{\gamma_{i,2}} + a_{i,3}(s)|x_n'(s)-x'(s)|^{\gamma_{i,3}}
\end{aligned}
$$

$$
\begin{aligned}
&+a_{i,4}(s)|y_n'(s)-y'(s)|^{\gamma_{i,4}}+a_{i,5}(s)|D^{\beta_1}(x_n-x)(s)|^{\gamma_{i,5}}\\
&+a_{i,6}(s)|D^{\beta_2}(y_n-y)(s)|^{\gamma_{i,6}}+a_{i,7}(s)(\int_0^s|x_n-x|(\xi)d\xi)^{\gamma_{i,7}}\\
&\left.+a_{i,8}(s)(\int_0^s|y_n-y|(\xi)d\xi)^{\gamma_{i,8}}\right]ds\\
\leq\;& \frac{1}{\Gamma(\alpha_i-1)}\int_0^t(t-s)^{\alpha_i-2}\Big[a_{i,1}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,1}}\\
&+a_{i,2}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,2}}+a_{i,3}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,3}}\\
&+a_{i,4}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,4}}+a_{i,5}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,5}}\\
&+a_{i,6}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,6}}+a_{i,7}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,7}}\\
&+a_{i,8}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,8}}\Big]ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\Big[a_{i,1}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,1}}\\
&+a_{i,2}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,2}}+a_{i,3}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,3}}\\
&+a_{i,4}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,4}}+a_{i,5}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,5}}\\
&+a_{i,6}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,6}}+a_{i,7}(s)(\Delta\|x_n-x\|_*)^{\gamma_{i,7}}\\
&+a_{i,8}(s)(\Delta\|y_n-y\|_*)^{\gamma_{i,8}}\Big]ds\\
\leq\;& \frac{1}{\Gamma(\alpha_i-1)}\sum_{j=1}^{8}\left[\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\int_0^t(t-s)^{\alpha_i-2}a_{i,j}(s)ds\right]\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\sum_{j=1}^{8}\left[\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}a_{i,j}(s)ds\right]\\
\leq\;& \frac{1}{\Gamma(\alpha_i-1)}\sum_{j=1}^{8}\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\sum_{j=1}^{8}\epsilon^{\gamma_{i,j}}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\\
\leq\;& \epsilon^{\gamma}\sum_{j=1}^{8}\Delta^{\gamma_{i,j}}\|\hat{a}_{i,j}\|_{[0,1]}\left[\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right],
\end{aligned}
$$

so

$$\begin{aligned}
&\|\phi_i'(x_n, y_n) - \phi_i'(x, y)\| \\
&\leq \epsilon^\gamma \sum_{j=1}^{8} \Delta^{\gamma_{i,j}} \|\hat{a}_{i,j}\|_{[0,1]} \left[\frac{1}{\Gamma(\alpha_i - 1)} + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& \|\phi_i(x_n, y_n) - \phi_i(x, y)\|_* \\
= \; & \max\left\{ \|\phi_i(x_n, y_n) - \phi_i(x, y)\|, \|\phi_i'(x_n, y_n) - \phi_i'(x, y)\| \right\} \\
& \leq \left(\epsilon^\gamma \sum_{j=1}^{8} \Delta^{\gamma_{i,j}} \|\hat{a}_{i,j}\|_{[0,1]} \right) \times \\
& \max\left\{ \frac{2}{\Gamma(\alpha_i)} + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)}, \frac{1}{\Gamma(\alpha_i - 1)} + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} \right\} \\
& = \left(\epsilon^\gamma \sum_{j=1}^{8} \Delta^{\gamma_{i,j}} \|\hat{a}_{i,j}\|_{[0,1]} \right) \left(\frac{1}{\Gamma(\alpha_i - 1)} + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} \right),
\end{aligned}$$

then we have

$$\begin{aligned}
& \|F(x_n, y_n) - F(x, y)\|_{**} \\
= \; & \max\left\{ \|\phi_1(x_n, y_n) - \phi_1(x, y)\|_*, \|\phi_2(x_n, y_n) - \phi_2(x, y)\|_* \right\} \\
\leq \; & \epsilon^\gamma \max_{1 \leq i \leq 2} \left\{ \left(\sum_{j=1}^{8} \Delta^{\gamma_{i,j}} \|\hat{a}_{i,j}\|_{[0,1]} \right) \left(\frac{1}{\Gamma(\alpha_i - 1)} + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} \right) \right\}.
\end{aligned}$$

Now since $\epsilon > 0$ was arbitary, we have $\|F(x_n, y_n) - F(x, y)\|_{**}$ tends to zero as $\|(x_n, y_n) - (x, y)\|_{**} \to 0$, so we conclude that F is contiuous on X^2. Because of

$$\max_{1 \leq i \leq 2} \left\{ \left(\frac{1}{\Gamma(\alpha_i - 1)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} \right) \sum_{k=1}^{k_0} \|\hat{b}_{i,k}\|_{[0,1]} p_{i,k} \right\} \in [0, \frac{1}{\Delta})$$

we can choose $\epsilon > 0$ such that for all $i = 1, 2$

$$\epsilon + \left(\frac{1}{\Gamma(\alpha_i - 1)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i - 1)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}(p_{i,k}+\epsilon) \tag{5}$$
$$+\epsilon\left(\frac{1}{\Gamma(\alpha_i)} + \frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\right) \in [0, \frac{1}{\Delta}).$$

Now since $\lim_{z\to\infty} \frac{T_{i,k}(\Delta z,...,\Delta z)}{\Delta z} = p_{i,k}$ then there exists $r_1 > 0$ such that for $z \geq r_1$,

$$T_{i,k}(\Delta z, ..., \Delta z) \leq (p_{i,k} + \epsilon)\Delta z. \tag{6}$$

Also since $\lim_{z\to\infty} M_i(\Delta z, ..., \Delta z) < \infty$ hence $\lim_{z\to\infty} \frac{M_i(\Delta z,...,\Delta z)}{\Delta z} = 0$ so there exists $r_2 > 0$ such that $z \geq r_2$ implies

$$M_i(\Delta z, ..., \Delta z) \leq \epsilon\Delta z \tag{7}$$

and since $\lim_{z\to\infty} \frac{\lambda_i\Gamma(2-\mu_i)}{\Delta z\eta_i^{1-\mu_i}} = 0$, so there exists $r_3 > 0$ such that $z \geq r_3$ implies

$$\frac{\lambda_i\Gamma(2-\mu_i)}{\Delta z\eta_i^{1-\mu_i}} \leq \epsilon\Delta z. \tag{8}$$

Let $r = \max\{r_1, r_2, r_3\}$ then by (6), (7) and (8) and by putting $z = r$, we have

$$T_{i,k}(\Delta r, ..., \Delta r) \leq (p_{i,k} + \epsilon)\Delta r, \tag{9}$$

$$M_i(\Delta r, ..., \Delta r) \leq \epsilon\Delta r \tag{10}$$

and

$$\frac{\lambda_i\Gamma(2-\mu_i)}{\Delta z\eta_i^{1-\mu_i}} \leq \epsilon\Delta z. \tag{11}$$

Let $C = \{(x, y) \in X^2 : \|(x, y)\|_{**} \leq r\}$ and define $\alpha : X^2 \to \mathbb{R}$ as $\alpha((x, y), (u, v)) = 1$ when $(x, y), (u, v) \in C$, other wise put $\alpha((x, y), (u, v)) =$

0. Let $\alpha((x,y),(u,v)) \geq 1$ then $(x,y),(u,v) \in C$, hence $\|(x,y)\|_{**} \leq r$ and so $\|x\|_* \leq r$ and $\|y\|_* \leq r$, then for all $t \in [0,1]$, we have

$$
\begin{aligned}
& |\phi_i(x,y)(t)| \\
& \leq \bigg| \int_0^1 G_{\alpha_i}(t,s) f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \\
& \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi)) ds \bigg| + |H_i(t)| \\
\leq \;& \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} \bigg| f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \\
& \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi) \bigg| ds \\
& + \frac{1}{\Gamma(\alpha_i)} \int_0^1 (1-s)^{\alpha - 1} \bigg| f_i(s, x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \\
& \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi) \bigg| ds \\
& + \frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i - \mu_i)} (1-t) \int_0^{\eta_i} (\eta_i - s)^{\alpha_i - \mu_i - 1} \bigg| f_i(s, x(s), y(s), x'(s), \\
& y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi) \bigg| ds \\
& + \frac{\lambda_i \Gamma(2-\mu_i)}{\eta^{1-\mu_i}} (1-t) \\
\leq \;& \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} \Bigg[\sum_{k=1}^{k_0} b_{i,k}(s) T_{i,k}(x(s), y(s), x'(s), y'(s), D^{\beta_1}x(s), \\
& D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi) + M_i(x(s), y(s), x'(s), \\
& y'(s), D^{\beta_1}x(s), D^{\beta_2}y(s), \int_0^s h_1(\xi)x(\xi)d\xi, \int_0^s h_2(\xi)y(\xi)d\xi) \Bigg] ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha-1}\left[\sum_{k=1}^{k_0} b_{i,k}(s)T_{i,k}(x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),\right.\\
&D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)+M_i(x(s),y(s),x'(s),\\
&\left.y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\right]ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\times\\
&\left[\sum_{k=1}^{k_0} b_{i,k}(s)T_{i,k}(x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),\right.\\
&D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)+M_i(x(s),y(s),x'(s),\\
&\left.y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\right]ds\\
&+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta^{1-\mu_i}}(1-t)\\
\leq\ &\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}\int_0^t (t-s)^{\alpha_i-1} b_{i,k}(s)T_{i,k}(|x(s)|,|y(s)|,|x'(s)|,\\
&|y'(s)|,\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}M_i(|x(s)|,|y(s)|,|x'(s)|,|y'(s)|,\\
&\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}\int_0^1 (1-s)^{\alpha-1} b_{i,k}(s)T_{i,k}(|x(s)|,|y(s)|,|x'(s)|,|y'(s)|,\\
&\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha-1} M_i(|x(s)|,|y(s)|,|x'(s)|,\\
&|y'(s)|,\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\sum_{k=1}^{k_0}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}b_{i,k}(s)\times\\
&T_{i,k}(|x(s)|,|y(s)|,|x'(s)|,\|y'(s)|,\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\\
&\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}M_i(|x(s)|,|y(s)|,|x'(s)|,\\
&|y'(s)|,\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}}(1-t)\\
\leq\ &\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}\int_0^1(1-s)^{\alpha_i-1}b_{i,k}(s)T_{i,k}(\|x\|,\|y\|,\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\\
&\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds+\frac{1}{\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha_i-1}M_i(\|x\|,\|y\|,\\
&\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds\\
&+\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}\int_0^1(1-s)^{\alpha-1}b_{i,k}(s)T_{i,k}(\|x\|,\|y\|,\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\\
&\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds+\frac{1}{\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha-1}M_i(\|x\|,\|y\|,\\
&\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\sum_{k=1}^{k_0}\int_0^1(1-s)^{\alpha_i-\mu_i-1}b_{i,k}(s)T_{i,k}(\|x\|,\|y\|,\\
&\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}M_i(\|x\|,\|y\|,\|x'\|,\|y'\|,\\
&\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta^{1-\mu_i}}(1-t)\\
\leq\ &\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}\Bigg[T_{i,k}(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-1}b_{i,k}(s)ds\Bigg]\\
&+\frac{1}{\Gamma(\alpha_i)}M_i(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-1}ds\\
&+\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}\Bigg[T_{i,k}(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-1}b_{i,k}(s)ds\Bigg]\\
&+\frac{1}{\Gamma(\alpha_i)}M_i(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-1}ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\sum_{k=1}^{k_0}\Bigg[T_{i,k}(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-\mu_i-1}b_{i,k}(s)ds\Bigg]\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)M_i(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\\
&\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-1}ds\\
&+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}}(1-t)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0} T_{i,k}(\Delta r,...,\Delta r)\|\hat{b}_{i,k}\|_{[0,1]} + \frac{1}{\Gamma(\alpha_i+1)}M_i(\Delta r,...,\Delta r) \\
& + \frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0} T_{i,k}(\Delta r,...,\Delta r)\|\hat{b}_{i,k}\|_{[0,1]} + \frac{1}{\Gamma(\alpha_i+1)}M_i(\Delta r,...,\Delta r) \\
& + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}(1-t)\sum_{k=1}^{k_0} T_{i,k}(\Delta r,...,\Delta r)\|\hat{b}_{i,k}\|_{[0,1]} \\
& + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i+1)}(1-t)\eta_i^{\alpha_i-\mu_i}M_i(\Delta r,...,\Delta r) \\
& + \frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}}(1-t).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \|\phi_i(x,y)\| \\
\leq\ & \left(\frac{2}{\Gamma(\alpha_i)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}T_{i,k}(\Delta r,...,\Delta r) \\
& + \left(\frac{2}{\Gamma(\alpha_i+1)} + \frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\right)M_i(\Delta r,...,\Delta r) + \frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}} \\
\leq\ & \left(\frac{2}{\Gamma(\alpha_i)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}(p_{i,k}+\epsilon)\Delta r \\
& + \left(\frac{2}{\Gamma(\alpha_i+1)} + \frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\right)\epsilon\Delta r + \epsilon\Delta r \\
\leq\ & \left(\frac{\alpha_i}{\Gamma(\alpha_i)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}(p_{i,k}+\epsilon)\Delta r \\
& + \left(\frac{\alpha_i}{\Gamma(\alpha_i+1)} + \frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\right)\epsilon\Delta r + \epsilon\Delta r
\end{aligned}
$$

$$= \Bigg[\Bigg(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\Bigg)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}(p_{i,k}+\epsilon)$$
$$+\Bigg(\frac{1}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\Bigg)\epsilon+\epsilon\Bigg]\Delta r<\frac{1}{\Delta}\Delta r=r.$$

Also we have

$$\begin{aligned}
&|\phi_i'(x,y)(t)| \\
&\leq \Bigg|\int_0^1 \frac{\partial G_{\alpha_i}}{\partial t}(t,s)f_i(s,x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s), \\
&\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi))ds\Bigg|+|H_i'(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha_i-1)}\int_0^t (t-s)^{\alpha_i-2}\Bigg|f_i(s,x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s), \\
&D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\Bigg|ds \\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\Bigg|f_i(s,x(s),y(s),x'(s),y'(s), \\
&D^{\beta_1}x(s),D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\Bigg|ds+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta^{1-\mu_i}} \\
\leq\ & \frac{1}{\Gamma(\alpha_i-1)}\int_0^t (t-s)^{\alpha_i-2}\Bigg[\sum_{k=1}^{k_0} b_{i,k}(s)T_{i,k}(x(s),y(s),x'(s),y'(s), \\
&D^{\beta_1}x(s),D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi) \\
&+M_i(x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s), \\
&\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\Bigg]ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}\Bigg[\sum_{k=1}^{k_0}b_{i,k}(s)T_{i,k}(x(s),y(s),\\
&x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\\
&+M_i(x(s),y(s),x'(s),y'(s),D^{\beta_1}x(s),D^{\beta_2}y(s),\\
&\int_0^s h_1(\xi)x(\xi)d\xi,\int_0^s h_2(\xi)y(\xi)d\xi)\Bigg]ds+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta^{1-\mu_i}}\\
\le\ &\frac{1}{\Gamma(\alpha_i-1)}\sum_{k=1}^{k_0}\int_0^t(t-s)^{\alpha_i-2}b_{i,k}(s)T_{i,k}(|x(s)|,|y(s)|,|x'(s)|,|y'(s)|,\\
&\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{1}{\Gamma(\alpha_i-1)}\int_0^t(t-s)^{\alpha_i-2}M_i(|x(s)|,|y(s)|,|x'(s)|,|y'(s)|,\\
&\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\|y\|\int_0^s h_2(\xi)d\xi)ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\sum_{k=1}^{k_0}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}b_{i,k}(s)T_{i,k}(|x(s)|,|y(s)|,\\
&|x'(s)|,|y'(s)|,\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\\
&\|y\|\int_0^s h_2(\xi)d\xi)ds+\frac{\Gamma(2-\mu_i)}{\eta^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}M_i(|x(s)|,\\
&|y(s)|,|x'(s)|,|y'(s)|,\frac{|x'(s)|}{\Gamma(2-\beta_1)},\frac{|y'(s)|}{\Gamma(2-\beta_2)},\|x\|\int_0^s h_1(\xi)d\xi,\\
&\|y\|\int_0^s h_2(\xi)d\xi)ds+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}}\\
\le\ &\frac{1}{\Gamma(\alpha_i-1)}\sum_{k=1}^{k_0}\int_0^1(1-s)^{\alpha_i-2}b_{i,k}(s)T_{i,k}(\|x\|,\|y\|,\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\\
&\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds+\frac{1}{\Gamma(\alpha_i-1)}\int_0^1(1-s)^{\alpha_i-2}M_i(\|x\|,\|y\|,\\
&\|x'\|,\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\sum_{k=1}^{k_0}\int_0^1(1-s)^{\alpha_i-\mu_i-1}b_{i,k}(s)T_{i,k}(\|x\|,\|y\|,\|x'\|,\\
&\|y'\|,\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}M_i(\|x\|,\|y\|,\|x'\|,\|y'\|,\\
&\frac{\|x'\|}{\Gamma(2-\beta_1)},\frac{\|y'\|}{\Gamma(2-\beta_2)},m_1\|x\|,m_2\|y\|)ds+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta^{1-\mu_i}}\\
\leq\ &\frac{1}{\Gamma(\alpha_i-1)}\sum_{k=1}^{k_0}\Big[T_{i,k}(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-2}b_{i,k}(s)ds\Big]\\
&+\frac{1}{\Gamma(\alpha_i-1)}M_i(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1(1-s)^{\alpha_i-2}ds\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\Big[T_{i,k}(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^1((\eta_i-s)^{\alpha_i-\mu_i-1}b_{i,k}(s)ds\Big]\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}M_i(\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\Delta\|x\|_*,\Delta\|y\|_*,\\
&\Delta\|x\|_*,\Delta\|y\|_*)\int_0^{\eta_i}(\eta_i-s)^{\alpha_i-\mu_i-1}ds+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}}\\
\leq\ &\frac{1}{\Gamma(\alpha_i)}\sum_{k=1}^{k_0}T_{i,k}(\Delta r,...,\Delta r)\|\hat{b}_{i,k}\|_{[0,1]}+\frac{1}{\Gamma(\alpha_i)}M_i(\Delta r,...,\Delta r)\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\sum_{k=1}^{k_0}T_{i,k}(\Delta r,...,\Delta r)\|\hat{b}_{i,k}\|_{[0,1]}\\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i+1)}\eta_i^{\alpha_i-\mu_i}M_i(\Delta r,...,\Delta r)+\frac{\lambda_i\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}}
\end{aligned}
$$

$$\begin{aligned}
&\leq \left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}(p_{i,k}+\epsilon)\Delta r \\
&\quad +\left(\frac{1}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\right)\epsilon\Delta r+\epsilon\Delta r \\
&= \left[\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}(p_{i,k}+\epsilon)\right. \\
&\quad \left.+\left(\frac{1}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)\eta_i^{\alpha_i-1}}{\Gamma(\alpha_i-\mu_i+1)}\right)\epsilon+\epsilon\right]\Delta r<\frac{1}{\Delta}\Delta r=r.
\end{aligned}$$

So for all $i=1,2$ we have

$$\|\phi_i(x,y)\|_*=\max\{\|\phi_i(x,y)\|,\|\phi_i'(x,y)\|\}\leq r,$$

hence

$$\|F(x,y)\|_{**}=\max_{1\leq i\leq 2}\{\|\phi_i(x,y)\|_*\}\leq r,$$

therefore $F(x,y)\in C$. By the same reason we conclude that $F(u,v)\in C$ so we have
$\alpha(F(x,y),F(u,v))\geq 1$. Also it's obvious that $C\neq\phi$, and we know for $(x_0,y_0)\in C$, $F(x_0,y_0)\in C$, so $\alpha((x_0,y_0),F(x_0,y_0))\geq 1$. Now let $(x,y),(u,v)\in C$ then by (4) we have

$$\begin{aligned}
&\|\phi_i(x,y)-\phi_i(u,v)\| \\
&\leq \frac{1}{\Gamma(\alpha_i)}\int_0^1(1-s)^{\alpha_i-1}\left[a_{i,1}(s)\|x-u\|_*^{\gamma_{i,1}}+a_{i,2}(s)\|y-v\|_*^{\gamma_{i,2}}\right. \\
&\quad +a_{i,3}(s)\|x-u\|_*^{\gamma_{i,3}}+a_{i,4}(s)\|y-v\|_*^{\gamma_{i,4}}+a_{i,5}(s)\frac{\|x-u\|_*^{\gamma_{i,5}}}{\Gamma(2-\beta_1)^{\gamma_{i,5}}} \\
&\quad +a_{i,6}(s)\frac{\|y-v\|_*^{\gamma_{i,6}}}{\Gamma(2-\beta_2)^{\gamma_{i,6}}}+a_{i,7}(s)m_1^{\gamma_{i,7}}\|x-u\|_*^{\gamma_{i,7}} \\
&\quad \left.+a_{i,8}(s)m_2^{\gamma_{i,8}}\|y-v\|_*^{\gamma_{i,8}}\right]ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha-1}\Bigg[a_{i,1}(s)\|x-u\|_*^{\gamma_{i,1}} + a_{i,2}(s)\|y-v\|_*^{\gamma_{i,2}} \\
&+a_{i,3}(s)\|x-u\|_*^{\gamma_{i,3}} + a_{i,4}(s)\|y-v\|_*^{\gamma_{i,4}} + a_{i,5}(s)\frac{\|x-u\|_*^{\gamma_{i,5}}}{\Gamma(2-\beta_1)^{\gamma_{i,5}}} \\
&+a_{i,6}(s)\frac{\|y-v\|_*^{\gamma_{i,6}}}{\Gamma(2-\beta_2)^{\gamma_{i,6}}} + a_{i,7}(s)m_1^{\gamma_{i,7}}\|x-u\|_*^{\gamma_{i,7}} \\
&+a_{i,8}(s)m_2^{\gamma_{i,8}}\|y-v\|_*^{\gamma_{i,8}}\Bigg]ds \\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i} (\eta_i-s)^{\alpha_i-\mu_i-1}\Bigg[a_{i,1}(s)\|x-u\|_*^{\gamma_{i,1}} \\
&+a_{i,2}(s)\|y-v\|_*^{\gamma_{i,2}} + a_{i,3}(s)\|x-u\|_*^{\gamma_{i,3}} + a_{i,4}(s)\|y-v\|_*^{\gamma_{i,4}} \\
&+a_{i,5}(s)\frac{\|x-u\|_*^{\gamma_{i,5}}}{\Gamma(2-\beta_1)^{\gamma_{i,5}}} + a_{i,6}(s)\frac{\|y-v\|_*^{\gamma_{i,6}}}{\Gamma(2-\beta_2)^{\gamma_{i,6}}} + a_{i,7}(s)m_1^{\gamma_{i,7}}\|x-u\|_*^{\gamma_{i,7}} \\
&+a_{i,8}(s)m_2^{\gamma_{i,8}}\|y-v\|_*^{\gamma_{i,8}}\Bigg]ds \\
\leq\ & \frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha_i-1}\Bigg[a_{i,1}(s)\Delta_{i,1}\|(x,y)-(u,v)\|_{**}^{\gamma_{i,1}} + ... \\
&+a_{i,8}(s)\Delta_{i,8}\|(x,y)-(u,v)\|_{**}^{\gamma_{i,8}}\Bigg]ds \\
&+\frac{1}{\Gamma(\alpha_i)}\int_0^1 (1-s)^{\alpha-1}\Bigg[a_{i,1}(s)\Delta_{i,1}\|(x,y)-(u,v)\|_{**}^{\gamma_{i,1}} + ... \\
&+a_{i,8}(s)\Delta_{i,8}\|(x,y)-(u,v)\|_{**}^{\gamma_{i,8}}\Bigg]ds \\
&+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\int_0^{\eta_i} (\eta_i-s)^{\alpha_i-\mu_i-1}\Bigg[a_{i,1}(s)\Delta_{i,1}\|(x,y)-(u,v)\|_{**}^{\gamma_{i,1}} \\
&+... + a_{i,8}(s)\Delta_{i,8}\|(x,y)-(u,v)\|_{**}^{\gamma_{i,8}}\Bigg]ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha_i)}\|(x,y)-(u,v)\|_{**}^{\gamma_0}\sum_{j=1}^{8}\Delta_{i,j}\int_0^1(1-s)^{\alpha_i-2}a_{i,j}(s)ds \\
&\quad+\frac{1}{\Gamma(\alpha_i)}\|(x,y)-(u,v)\|_{**}^{\gamma_0}\sum_{j=1}^{8}\Delta_{i,j}\int_0^1(1-s)^{\alpha_i-2}a_{i,j}(s)ds \\
&\quad+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\|(x,y)-(u,v)\|_{**}^{\gamma_0}\sum_{j=1}^{8}\Delta_{i,j}\int_0^1(1-s)^{\alpha_i-2}a_{i,j}(s)ds \\
&= \sum_{j=1}^{8}\Delta_{i,j}\|\hat{a}_{i,j}\|_{[0,1]}\left(\frac{2}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\|(x,y)-(u,v)\|_{**}^{\gamma_0},
\end{aligned}$$

where

$$\gamma_0 := \gamma_{(x,y),(u,v)} = \begin{cases} \gamma & \|(x,y)-(u,v)\|_{**}\leq 1 \\ \\ 1 & \text{other wise} \end{cases}$$

and $\Delta_{i,j}=\Delta^{\gamma_{i,j}}$. By the similar way we conclude that

$$\begin{aligned}
&\|\phi_i'(x,y)-\phi_i'(u,v)\| \\
&\leq \sum_{j=1}^{8}\Delta_{i,j}\|\hat{a}_{i,j}\|_{[0,1]}\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\|(x,y)-(u,v)\|_{**}^{\gamma_0},
\end{aligned}$$

hence

$$\begin{aligned}
&\|\phi_i(x,y)-\phi_i(u,v)\|_{*} \\
&\leq \sum_{j=1}^{8}\Delta_{i,j}\|\hat{a}_{i,j}\|_{[0,1]}\max\left\{\frac{2}{\Gamma(\alpha_i)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)},\right. \\
&\left.\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right\}\|(x,y)-(u,v)\|_{**}^{\gamma_0}, \\
&=\left(\sum_{j=1}^{8}\Delta_{i,j}\|\hat{a}_{i,j}\|_{[0,1]}\right)\times \\
&\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\|(x,y)-(u,v)\|_{**}^{\gamma_0},
\end{aligned}$$

so $\|F(x,y)-F(u,v)\|_{**}\leq\lambda\|(x,y)-(u,v)\|_{**}^{\gamma_0}$, where

$$\lambda:=\max_{1\leq i\leq 2}\left\{\left(\sum_{j=1}^{8}\Delta_{i,j}\|\hat{a}_{i,j}\|_{[0,1]}\right)\left(\frac{1}{\Gamma(\alpha_i-1)}+\frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\right\}.$$

Therefore

$$\alpha\Bigg((x,y),(u,v)\Bigg)d\Bigg(F(x,y),F(u,v)\Bigg)\leq\psi\Bigg(d(F(x,y),F(u,v))\Bigg),$$

where $\psi:[0,\infty)\to\ [0,\infty)$ define as $\psi(t)=\lambda t^{\gamma_0}$ when $t\in[0,1)$ and $\psi(t)=\lambda t$ when $t\in[1,\infty)$. It's obvious that ψ is nondecreasing, also if $t\in[0,1)$ then

$$\sum_{i=1}^{\infty}\psi^i(t)=\lambda t^{\gamma_0}+\lambda^2t^{2\gamma_0}+...\leq\sum_{i=1}^{\infty}\lambda^it^{\gamma_0}=\frac{\lambda}{1-\lambda}t^{\gamma_0}<\infty$$

and if $t\in[1,\infty)$ we have $\sum_{i=1}^{\infty}\psi^i(t)=\frac{\lambda}{1-\lambda}t<\infty$, hence $\sum_{i=1}^{\infty}\psi^i(t)$ is convergence for all $t\geq 0$, so $\psi\in\Psi$. Now by using lemma (2.6) we conclude that the problem (1) has a solution in X^2. □

4 Example

The following example illustrates our last result.

Example 4.1. Consider the pointwise defined problem

$$\begin{cases} D^{\frac{5}{2}}x(t)+f_1(t,x(t),y(t),x'(t),y'(t),D^{\frac{1}{2}}x(t),D^{\frac{1}{2}}y(t),\\ \int_0^t\xi x(\xi)d\xi,\int_0^t\xi^2y(\xi)d\xi)=0,\\ \\ D^{\frac{7}{3}}y(t)+f_2(t,x(t),y(t),x'(t),y'(t),D^{\frac{1}{2}}x(t),D^{\frac{1}{2}}y(t),\\ \int_0^t h_1(\xi)\xi x(\xi)d\xi,\int_0^t\xi^2y(\xi)d\xi)=0,\end{cases}\tag{12}$$

where

$$f_1(t,x_1,...,x_8)=\sum_{j=1}^{8}\frac{1}{t^{\sigma_j}}|x_j|+1$$

$$f_2(t, x_1, ..., x_8) = \frac{c(t)}{p(t)} \sum_{j=1}^{8} |x_j| + \frac{1}{10} \sum_{j=1}^{8} \frac{|x_j|}{1 + |x_j|}$$

with boundary contions $D^{\frac{2}{3}}x(\frac{1}{4}) = 1$, $D^{\frac{1}{2}}y(\frac{1}{3}) = 0$, $x(1) = x''(0) = 0$ and $y(1) = y''(0) = 0$ where $c(t) = 1$ and $p(t) = 0$ whenever $t \in [0,1] \cap \mathbb{Q}$,$c(t) = 0$ and $p(t) = 1$ whenever $t \in [0,1] \cap \mathbb{Q}^c$, $\sigma_1, ..., \sigma_8 \in (0,1)$ and $\sum_{k=1}^{8} \frac{1}{1-\sigma_j} \leq \frac{1}{3}$.

then

$$m_1 = \int_0^1 h_1(\xi) d(\xi) = \int_0^1 \xi d(\xi) = \frac{1}{2},$$

$$m_2 = \int_0^1 h_1(\xi) d(\xi) = \int_0^1 \xi^2 d(\xi) = \frac{1}{3},$$

$$|f_1(t, x_1, ...x_8) - f_1(t, y_1, ..., y_8)| \leq \sum_{k=1}^{8} \frac{1}{t^{\sigma_i}} |x_k - y_k|,$$

$$|f_2(t, x_1, ...x_8) - f_2(t, y_1, ..., y_8)| \leq \frac{c(t)}{p(t)} \sum_{k=1}^{8} |x_k - y_k|$$

$$+ \frac{1}{10} \sum_{j=1}^{8} \frac{|x_j - y_j|}{(1 + |x_j|)(1 + |y_j|)} \leq (\frac{c(t)}{p(t)} + \frac{1}{10}) \sum_{k=1}^{8} |x_k - y_k|.$$

Let $k_0 = 8$, $b_{1,k} = a_{1,j} = \frac{1}{t^{\sigma_j}}$, $\gamma_{i,j} = 1$, $b_{2,k} = \frac{c(t)}{p(t)}$, $a_{2,j} = \frac{c(t)}{p(t)} + \frac{1}{10}$, $T_{1,k}(x_1, ..., x_8) = T_{2,k}(x_1, ..., x_8) = |x_k|$, $M_1(x_1, ..., x_8) = 1$, $M_2(x_1, ..., x_8) = \frac{1}{10} \sum_{j=1}^{8} \frac{|x_j|}{1+|x_j|}$ for $1 \leq j, k \leq 8$, then

$$\|\hat{b}_{1,j}\|_{[0,1]} = \|\hat{a}_{1,j}\|_{[0,1]} = \int_0^1 (1-s)^{\alpha_1 - 2} a_{i,j}(s) ds = \int_0^1 (1-s)^{\frac{5}{2} - 2} \frac{1}{s^{\sigma_j}} ds$$

$$\leq \frac{1}{1 - \sigma_j},$$

$\|\hat{b}_{2,j}\|_{[0,1]} = 0$, $\|\hat{a}_{2,j}\|_{[0,1]} = \frac{2}{50}$,

$T_{i,k}, M_i$ are nondecreasing respect to their components,
$p_{i,k} = \lim_{z \to \infty} \frac{T_{i,k}(z,...,z)}{z} = 1$ and $\lim_{z \to \infty} M_i(z, ..., z) < \infty$

for all $1 \leq i \leq 2$, $1 \leq j \leq 8$ and $1 \leq k \leq k_0$. One check we can calculate that

$$\begin{aligned}
\Delta_{i,j} = \Delta^{\gamma_{i,j}} = \Delta &= \max\left\{1, \frac{1}{\Gamma(2-\beta_1)}, \frac{1}{\Gamma(2-\beta_2)}, m_1, m_2\right\} \\
&= \max\left\{1, \frac{1}{\Gamma(2-\frac{1}{2})}, \frac{1}{\Gamma(2-\frac{1}{2})}, \frac{1}{2}, \frac{1}{2}\right\} = \frac{2}{\sqrt{\pi}},
\end{aligned}$$

$$\begin{aligned}
&\max_{1\leq i\leq 2}\left\{\left(\frac{1}{\Gamma(\alpha_i-1)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\sum_{k=1}^{k_0}\|\hat{b}_{i,k}\|_{[0,1]}p_{i,k}\right\} \\
&\leq \max\left\{\left(\frac{1}{\Gamma(\frac{5}{2}-1)} + \frac{\Gamma(2-\frac{2}{3})}{\frac{1}{4}^{1-\frac{2}{3}}\Gamma(\frac{5}{2}-\frac{2}{3})}\right)\sum_{k=1}^{8}\frac{1}{1-\sigma_j},\right. \\
&\left.\left(\frac{1}{\Gamma(\frac{7}{3}-1)} + \frac{\Gamma(2-\frac{1}{2})}{\frac{1}{3}^{1-\frac{1}{2}}\Gamma(\frac{7}{3}-\frac{1}{2})}\right)\times 0\right\} \in [0, \frac{1}{\Delta})
\end{aligned}$$

and

$$\begin{aligned}
&\max_{1\leq i\leq 2}\left\{\left(\sum_{k=1}^{k_0}\|\hat{a}_{i,k}\|_{[0,1]}\Delta_{i,j}\right)\left(\frac{1}{\Gamma(\alpha_i-1)} + \frac{\Gamma(2-\mu_i)}{\eta_i^{1-\mu_i}\Gamma(\alpha_i-\mu_i)}\right)\right\} \\
&\leq \max\left\{\left(\sum_{k=1}^{8}\frac{1}{1-\sigma_j}\Delta\right)\left(\frac{1}{\Gamma(\frac{5}{2}-1)} + \frac{\Gamma(2-\frac{2}{3})}{\frac{1}{4}^{1-\frac{2}{3}}\Gamma(\frac{5}{2}-\frac{2}{3})}\right),\right. \\
&\left.\left(\frac{16}{50}\Delta\right)\left(\frac{1}{\Gamma(\frac{7}{3}-1)} + \frac{\Gamma(2-\frac{1}{2})}{\frac{1}{3}^{1-\frac{1}{2}}\Gamma(\frac{7}{3}-\frac{1}{2})}\right)\right\} < 1,
\end{aligned}$$

so by using theorem (3.2), the problem (12) has a solution in X^2.

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