Journal of Mathematical Extension
Vol. 15, No. 1, (2021), 215-269
ISSN: 1735-8299
URL: https://doi.org/10.30495/JME.2021.1117
Original Research Paper

The Investigation of a Solution Existence for a Bi-Dealing Singular Fractional Intergro-Differential Equation

A. Mansouri*
Arak Branch, Islamic Azad University

M. Shabibi
Meharn Branch, Islamic Azad University

Abstract. This article investigates the existence of a solution for a singular fractional differential equation. For this, the researchers have changed the main differential equation into an integral equation, then through determining some assumptions that could control its main singular points; the existence of a solution has been proved for the equation by applying a fixed point theorem, as well. The significance of the proposed paper is regarded as the equation's novelty on its boundary condition which is a generalization of similar ones. Likewise the condition, is a generalization for the similar cases and it conduces to consider a singular equation with infinite singular points. Having two dealings on its dominate is of high significance for the equation which should be remarkable.

AMS Subject Classification: 34A08; 37C25; 46F30

Keywords and Phrases: Caputo derivative, point-wise defined equation, singularity, fractional differential equation, bi-dealing

Received: December 2018; Accepted: September 2019

*Corresponding author

1. Introduction

An evaluation of the mathematical model of a scientific observation leads into a differential equation that sometimes equations are of the form fractional order. We can mention engineering sciences, dynamic, chemistry and physics among which the equation occurs ([2]). Many studies have investigated the existence and behavior of these equations in recent decats ([3], [4]). Sometimes we lead to a differential system that has singularity in some points, recently many works has been published on the existence of the solutions for these singular systems ([9]).

In 2010, Agarwal, O'Regan and Stanek ([1]) studied the existence of solutions for the problem $D^{\alpha}u(t) + f(t, u(t)) = 0$ with boundary conditions $u'(0) = \cdots = u^{(n-1)} = 0$ and $u(1) = \int_0^1 u(s)d\mu(s)$, where $n \geqslant 2$, $\alpha \in (n-1, n)$, $\mu(s)$ is a functional of bounded variation with $\int_0^1 d\mu(s) < 1$, and f may has singularity at $t = 0$.

In 2015, Y. Liu and P. J. Y. Wong investigated the existence of solution for the fractional problem ${}^{c}D^{\alpha}x(t) = f(t, x(t), D^{\beta}x(t))$ with boundary conditions $x(0) + x'(0) = y(x)$, $\int_0^1 x(t)dt = m$ and $x''(0) = x^{(3)} = \cdots = x^{(n-1)}(0) = 0$, where $0 < t < 1$, m is a real number, $n \geqslant 2$, $\alpha \in (n-1, n)$, $0 < \beta < 1$, D^{α} and D^{β} are the Caputo fractional derivatives, $y \in C([0,1], \mathbb{R}) \to \mathbb{R}$ and $f : (0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with $f(t, x, y)$ may be singular at $t = 0$ ([7]).

In 2016 M. Shabibi, M. Postolache, Sh. Rezapour and S. M. Vaezpour investigated the solution of the multi-singular pointwise defined fractional integro-differential equation $D^{\mu}x(t) + f(t, x(t), x'(t), D^{\beta}x(t), I^{p}x(t)) = 0$ with boundary conditions $x'(0) = x(\xi)$ and $x(1) = \int_0^{\eta} x(s)ds$ when $\mu \in [2, 3)$ and $x'(0) = x(\xi)$, $x(1) = \int_0^{\eta} x(s)ds$ and $x^{(j)}(0) = 0$ for $j = 2, \ldots, [\mu] - 1$ when $\mu \in [3, \infty)$, where $0 \leqslant t \leqslant 1$, $x \in C^1[0,1]$, $\mu \in [2, \infty)$, $\beta, \xi, \eta \in (0,1)$, $p > 1$, D^{μ} is the Caputo fractional derivative of order μ and $f : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ is a function such that $f(t, ., ., ., ., .)$ is singular at some points $t \in [0,1]$([8]).

In 2018 D. Baleanu, Kh. Ghafarnezhad, Sh. Reazapour and M. Shabibi reviewd the existence of solution for the pointwise defined three steps crisis integro-differential equation

$$D^{\alpha}x(t) + f(t, x(t), x'(t), D^{\beta}x(t), \int_0^t h(\xi)x(\xi)d\xi, \phi(x(t))) = 0$$

with boundary conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geqslant 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that $\|\phi(x) - \phi(y)\| \leqslant \theta_0\|x - y\| + \theta_1\|x' - y'\|$ for some non-negative real numbers θ_0 and $\theta_1 \in [0, \infty)$ and all $x, y \in X$, D^{α} is the Caputo fractional derivative of order α, $f(t, x_1(t), ..., x_5(t)) = f_1(t, x_1(t), ..., x_5(t))$ for all $t \in [0, \lambda)$, $f(t, x_1(t), ..., x_5(t)) = f_2(t, x_1(t), ..., x_5(t))$ for all $t \in [\lambda, \mu]$ and $f(t, x_1(t), ..., x_5(t)) = f_3(t, x_1(t), ..., x_5(t))$ for all $t \in (\mu, 1]$, $f_1(t, ., ., ., ., .)$ and $f_3(t, ., ., ., ., .)$ are continuous on $[0, \lambda)$ and $(\mu, 1]$ and $f_2(t, ., ., ., ., .)$ is multi-singular ([5]).

Using idea of these papers, we investigate the existence of solutions for the folowing bi-dealing singular fractional intergro-differential equation

$$D^{\alpha}x(t) + f(t, x(t), x'(t), D^{\beta}x(t), \int_0^t h(s)x(s)ds, \phi x(t)) = 0 \tag{1}$$

with boundary conditions $x'(\mu) = x'(0) + \int_0^{\lambda} g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for$j \geqslant 2$ where $\alpha \geqslant 2$, $\mu, \lambda \in (0,1)$, $g \in L^1[0, \lambda]$, $g(t) > 0$ for a.e. $t \in [0, \lambda]$, $\phi : [0,1] \to R^+$ is such that for all $x, y \in C^1[0,1]$, $|\phi x(t) - \phi y(t)| \leqslant b_1|x(t) - y(t)| + b_2|x'(t) - y'(t)|$ for some $b_1, b_2 \in [0, \infty)$, $h \in L^1[0,1]$, D^{α} is the Caputo fractional derivative of order α and $f(t, ., ., ., ., .)$ deals as a singular function on some set $E \subset [0,1]$ and deals as a continuous function on $E^c \subset [0,1]$. Recall that $D^{\alpha}x(t) = f(t)$ is a pointwise defined equation on $[0,1]$ if there exists set $D \subset [0,1]$ such that the measure of D^c is zero and the equation holds on D (see [8]). Here we use $\|.\|_1$ for the norm of $L^1[0,1]$, $\|.\|$ for the sup norm of $Y = C[0,1]$ and $\|x\|_* = \max\{\|x\|, \|x'\|\}$ for the norm of $X = C^1[0,1]$. The Riemann-Liouville integral of order p with the lower limit $a \geqslant 0$ for a function $f : (a, \infty) \to \mathbb{R}$ is defined by $I^p_{a^+}f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1}f(s)ds$, provided that the right-hand side is pointwise define on (a, ∞) ([12]). we denote $I^p_{0^+}f(t)$ by $I^p f(t)$. The Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^cD^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds$, where $n = [\alpha] + 1$ and

$f : (a, \infty) \to \mathbb{R}$ is a function ([12]). Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ (see [11]). One can check that $\psi(t) < t$ for all $t > 0$ ([11]). Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x, y) \geqslant 1$ implies $\alpha(Tx, Ty) \geqslant 1$ ([11]). Let (X, d) be a metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x, y)d(Tx, Ty) \leqslant \psi(d(x, y))$ for all $x, y \in X$ ([11]). We need next results.

Lemma 1.1. ([11]) *Let (X, d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0, \infty)$ a map and $T : X \to X$ an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$, then T has a fixed point.*

Lemma 1.2. ([14]) *Let X be a Banach space with $C \subseteq X$ closed and convex. Let Ω be a relatively open subset of C with $0 \in \Omega$ and let $F : \Omega \to C$ be a continuous and compact mapping. Then either*

i) *the mapping F has a fixed point in $\bar{\Omega}$, or*

ii) *there exist $y \in \partial\Omega$ and $\lambda \in (0, 1)$ with $y = \lambda F y$.*

Lemma 1.3. ([6]) *Let $n-1 \leqslant \alpha < n$ and $x \in C(0, 1) \cap L^1(0, 1)$. Then, we have $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, \ldots, c_{n-1}$.*

2. Main Results

Lemma 2.1. *Let $\alpha \geqslant 2$, $n = [\alpha] + 1$, $\mu, \lambda \in (0, 1)$, $g \in L^1[0, \lambda]$, $g(t) > 0$ for a.e. $t \in [0, \lambda]$ and $y \in L^1[0, 1]$. A map x is a solution for the pointwise defined equation $D^\alpha x(t) + f(t) = 0$ with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $2 \leqslant j \leqslant n$, if and only if $x(t) = \int_0^1 G(t, s)y(s)ds$ for all $t \in [0, 1]$, where*
$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + \alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)}$, *when* $0 \leqslant s \leqslant t \leqslant 1$ *and* $s \leqslant \mu, \lambda$,
$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1} + \alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda \Gamma(\alpha)}$, *when* $0 \leqslant s \leqslant t \leqslant 1$ *and* $\lambda \leqslant s \leqslant \mu$,
$G(t, s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s) + [(A_\lambda + B_\lambda)t - B_\lambda](1-s)^{\alpha-1}}{A_\lambda \Gamma(\alpha)}$, *when* $0 \leqslant s \leqslant t \leqslant 1$ *and* $\mu \leqslant s \leqslant \lambda$,

$G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, *when* $0 \leqslant s \leqslant t \leqslant 1$ *and* $s \geqslant \mu, \lambda$,
$G(t,s) = \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}+(t-1)(\mu-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, *when* $0 \leqslant t \leqslant s \leqslant 1$ *and* $s \leqslant \mu, \lambda$,
$G(t,s) = \frac{(1-t)H(s)+[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, *when* $0 \leqslant t \leqslant s \leqslant 1$ *and* $s \geqslant \mu, \lambda$,
$G(t,s) = \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, *when* $0 \leqslant t \leqslant s \leqslant 1$ *and* $s \geqslant \mu, \lambda$,
$H(s) = \int_s^\lambda (t-s)^{\alpha-1} g(t)dt$, $A_\lambda = \int_0^\lambda (1-t)g(t)dt$ *and*
$B_\lambda = \int_0^\lambda tg(t)dt$.

Proof. First by similar manner as [5] we conclude that lemma (1.3) is valid on $L^1[0,1]$. Now let $x(t)$ be a solution for the problem, since $x^{(j)}(0) = 0$ for $j \geqslant 2$, by using Lemma (1.3) we have $x(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_0 + c_1 t$. By boundary condition $x(1) = 0$, we have so $c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds - c_0$. On the other hand
$g(t)x(t) = \frac{-g(t)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_0 g(t) + c_1 t g(t)$, so

$$\int_0^\lambda g(t)x(t)dt = \frac{-1}{\Gamma(\alpha)}\int_0^\lambda\int_0^t g(t)(t-s)^{\alpha-1}y(s)dsdt + c_0\int_0^\lambda g(t)dt + c_1\int_0^\lambda tg(t)dt.$$

Now since

$$\int_0^\lambda\int_0^t g(t)(t-s)^{\alpha-1}y(s)dsdt = \int_0^\lambda\int_s^\lambda g(t)(t-s)^{\alpha-1}y(s)dtds = \int_0^\lambda\left(\int_s^\lambda (t-s)^{\alpha-1}g(t)dt\right)y(s)ds,$$

we coclude

$$\int_0^\lambda g(t)x(t)dt = \frac{-1}{\Gamma(\alpha)}\int_0^\lambda\left(\int_s^\lambda (t-s)^{\alpha-1}g(t)dt\right)y(s)ds + c_0\int_0^\lambda g(t)dt + c_1\int_0^\lambda tg(t)dt.$$

Also we have

$$x'(\mu) = \frac{-1}{\Gamma(\alpha)}\int_0^\mu (\mu-s)^{\alpha-2}y(s)ds + c_1$$

and $x'(0) = c_1$, so by $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$, we conclude that

$$\begin{aligned}\frac{-1}{\Gamma(\alpha-1)}\int_0^\mu(\mu-s)^{\alpha-2}y(s)ds &= \frac{-1}{\Gamma(\alpha)}\int_0^\lambda(\int_s^\lambda(t-s)^{\alpha-1}g(t)dt)y(s)ds \\ &\quad +c_0\int_0^\lambda g(t)dt + c_1\int_0^\lambda tg(t)dt \\ &= \frac{-1}{\Gamma(\alpha)}\int_0^\lambda(\int_s^\lambda(t-s)^{\alpha-1}g(t)dt)y(s)ds + c_0\int_0^\lambda g(t)dt \\ &\quad +\frac{1}{\Gamma(\alpha)}(\int_0^\lambda tg(t)dt)\int_0^1(1-s)^{\alpha-1}y(s)ds - c_0\int_0^\lambda tg(t)dt,\end{aligned}$$

so

$$c_0(\int_0^\lambda g(t)dt - \int_0^\lambda tg(t)dt) = \frac{1}{\Gamma(\alpha)}\int_0^\lambda(\int_s^\lambda(t-s)^{\alpha-1}g(t)dt)y(s)ds$$
$$-\frac{1}{\Gamma(\alpha)}(\int_0^\lambda tg(t)dt)\int_0^1(1-s)^{\alpha-1}y(s)ds - \frac{1}{\Gamma(\alpha-1)}\int_0^\mu(\mu-s)^{\alpha-2}y(s)ds,$$

therefore

$$\begin{aligned}c_0 &= \frac{1}{\Gamma(\alpha)\int_0^\lambda(1-t)g(t)dt}\int_0^\lambda(\int_s^\lambda(t-s)^{\alpha-1}g(t)dt)y(s)ds \\ &\quad -\frac{\int_0^\lambda tg(t)dt}{\Gamma(\alpha)\int_0^\lambda(1-t)g(t)dt}\int_0^1(1-s)^{\alpha-1}y(s)ds \\ &\quad -\frac{1}{\Gamma(\alpha-1)\int_0^\lambda tg(t)dt}\int_0^\mu(\mu-s)^{\alpha-2}y(s)ds,\end{aligned}$$

hence

$$\begin{aligned}c_1 &= \int_0^1(1-s)^{\alpha-1}y(s)ds - c_0 \\ &= \int_0^1(1-s)^{\alpha-1}y(s)ds \\ &\quad -\frac{1}{\Gamma(\alpha)\int_0^\lambda(1-t)g(t)dt}\int_0^\lambda(\int_s^\lambda(t-s)^{\alpha-1}g(t)dt)y(s)ds \\ &\quad +\frac{\int_0^\lambda tg(t)dt}{\Gamma(\alpha)\int_0^\lambda(1-t)g(t)dt}\int_0^1(1-s)^{\alpha-1}y(s)ds \\ &\quad +\frac{1}{\Gamma(\alpha-1)\int_0^\lambda tg(t)dt}\int_0^\mu(\mu-s)^{\alpha-2}y(s)ds.\end{aligned}$$

So we have

$$
\begin{aligned}
x(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\
&\quad - \frac{\int_0^\lambda t g(t) dt}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad - \frac{1}{\Gamma(\alpha) \int_0^\lambda t g(t) dt} \int_0^\mu (\mu-s)^{\alpha-1} y(s) ds, \\
&\quad + \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad - \frac{t}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^\lambda \left(\int_s^\lambda (t-s)^{\alpha-1} g(t) dt \right) y(s) ds \\
&\quad + \frac{t \int_0^\lambda t g(t) dt}{\Gamma(\alpha) \int_0^\lambda (1-t) g(t) dt} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad + \frac{t}{\Gamma(\alpha-1) \int_0^\lambda t g(t) dt} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds.
\end{aligned}
$$

Put $H(s) = \int_s^\lambda (t-s)^{\alpha-1} g(t) dt$, $A_\lambda = \int_0^\lambda (1-t) g(t) dt$ and $B_\lambda = \int_0^\lambda t g(t) dt$, so
$A_\lambda + B_\lambda = \int_0^\lambda g(t) dt$. Hence

$$
\begin{aligned}
x(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) y(s) ds \\
&- \frac{B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\mu (\mu-s)^{\alpha-1} y(s) ds, \\
&+ \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{t}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) y(s) ds \\
&+ \frac{t B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + \frac{t}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} y(s) ds.
\end{aligned}
$$

Therefore

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_0^\lambda H(s)y(s)ds \\ &+\frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ &+\frac{t-1}{A_\lambda\Gamma(\alpha-1)}\int_0^\mu (\mu-s)^{\alpha-2}y(s)ds. \end{aligned}$$

If $t \leqslant \lambda \leqslant \mu < 1$ then

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1-t}{A_\lambda\Gamma(\alpha)}(\int_0^t + \int_t^\lambda)H(s)y(s)ds \\ &+\frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}(\int_0^t + \int_t^\lambda + \int_\lambda^\mu + \int_\mu^1)(1-s)^{\alpha-1}y(s)ds \\ &+\frac{t-1}{A_\lambda\Gamma(\alpha-1)}(\int_0^t + \int_t^\lambda + \int_\lambda^\mu)(\mu-s)^{\alpha-2}y(s)ds, \end{aligned}$$

if $t \leqslant \mu \leqslant \lambda < 1$ then we have

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1-t}{A_\lambda\Gamma(\alpha)}(\int_0^t + \int_t^\mu + \int_\mu^\lambda)H(s)y(s)ds \\ &+\frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}(\int_0^t + \int_t^\mu + \int_\mu^\lambda + \int_\lambda^1)(1-s)^{\alpha-1}y(s)ds \\ &+\frac{t-1}{A_\lambda\Gamma(\alpha-1)}(\int_0^t + \int_t^\mu)(\mu-s)^{\alpha-2}y(s)ds, \end{aligned}$$

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}(\int_0^\mu + \int_\mu^t)(t-s)^{\alpha-1}y(s)ds \\ &+\frac{1-t}{A_\lambda\Gamma(\alpha)}(\int_0^\mu + \int_\mu^t + \int_t^\lambda)H(s)y(s)ds \\ &+\frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}(\int_0^\mu + \int_\mu^t + \int_t^\lambda + \int_\lambda^1)(1-s)^{\alpha-1}y(s)ds \\ &+\frac{t-1}{A_\lambda\Gamma(\alpha-1)}\int_0^\mu (\mu-s)^{\alpha-2}y(s)ds, \end{aligned}$$

when $0 < \lambda \leqslant t \leqslant \mu < 1$ then

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\Big(\int_0^\lambda + \int_\lambda^t\Big)(t-s)^{\alpha-1}y(s)ds + \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_0^\lambda H(s)y(s)ds \\ &+ \frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}\Big(\int_0^\lambda + \int_\lambda^t + \int_t^\mu + \int_\mu^1\Big)(1-s)^{\alpha-1}y(s)ds \\ &+ \frac{t-1}{A_\lambda\Gamma(\alpha-1)}\Big(\int_0^\lambda + \int_\lambda^t + \int_t^\mu\Big)(\mu-s)^{\alpha-2}y(s)ds. \end{aligned}$$

In the case $0 < \lambda \leqslant \mu \leqslant t < 1$ we have

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\Big(\int_0^\lambda + \int_\lambda^\mu + \int_\mu^t\Big)(t-s)^{\alpha-1}y(s)ds \\ &+ \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_0^\lambda H(s)y(s)ds \\ &+ \frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}\Big(\int_0^\lambda + \int_\lambda^\mu + \int_\mu^t + \int_t^1\Big)(1-s)^{\alpha-1}y(s)ds \\ &+ \frac{t-1}{A_\lambda\Gamma(\alpha-1)}\Big(\int_0^\lambda + \int_\lambda^\mu\Big)(\mu-s)^{\alpha-2}y(s)ds. \end{aligned}$$

Finally if $0 < \mu \leqslant \lambda \leqslant t < 1$ we can write $x(t)$ as

$$\begin{aligned} x(t) &= \frac{-1}{\Gamma(\alpha)}\Big(\int_0^\mu + \int_\mu^\lambda + \int_\lambda^t\Big)(t-s)^{\alpha-1}y(s)ds \\ &+ \frac{1-t}{A_\lambda\Gamma(\alpha)}\Big(\int_0^\mu + \int_\mu^\lambda\Big)H(s)y(s)ds \\ &+ \frac{(A_\lambda+B_\lambda)t-B_\lambda}{A_\lambda\Gamma(\alpha)}\Big(\int_0^\mu + \int_\mu^\lambda + \int_\lambda^t + \int_t^1\Big)(1-s)^{\alpha-1}y(s)ds \\ &+ \frac{t-1}{A_\lambda\Gamma(\alpha-1)}\int_0^\mu (\mu-s)^{\alpha-2}y(s)ds. \end{aligned}$$

So we have $x(t) = \int_0^1 G(t,s)y(s)ds$ where

$G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s)+[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}+\alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant s \leqslant t \leqslant 1$ and $s \leqslant \mu, \lambda$,

$G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}+\alpha(t-1)(\mu-s)^{\alpha-2}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant s \leqslant t \leqslant 1$ and $\lambda \leqslant s \leqslant \mu$,

$G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)H(s)+[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant s \leqslant t \leqslant 1$ and $\mu \leqslant s \leqslant \lambda$,

$G(t,s) = \frac{-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant s \leqslant t \leqslant 1$ and $s \geqslant \mu, \lambda$,

$G(t,s) = \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}+(t-1)(\mu-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant t \leqslant s \leqslant 1$ and $s \leqslant \mu, \lambda$,

$G(t,s) = \frac{(1-t)H(s)+[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant t \leqslant s \leqslant 1$ and $s \geqslant \mu, \lambda$ and

$G(t,s) = \frac{[(A_\lambda+B_\lambda)t-B_\lambda](1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant t \leqslant s \leqslant 1$ and $s \geqslant \mu, \lambda$,. $\square$

Note that, for the Green function $G(t,s)$, $\frac{\partial G}{\partial t}(t,s)$ is given as

$\frac{\partial G}{\partial t}(t,s) = \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{-H(s)+(A_\lambda+B_\lambda)(1-s)^{\alpha-1}+\alpha(\mu-s)^{\alpha-2}}{A_\lambda\Gamma(\alpha)}$,

when $0 \leqslant s \leqslant t \leqslant 1$ and $s \leqslant \mu, \lambda$,

$\frac{\partial G}{\partial t}(t,s) = \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(A_\lambda+B_\lambda)(1-s)^{\alpha-1}+\alpha(\mu-s)^{\alpha-2}}{A_\lambda\Gamma(\alpha)}$,

when $0 \leqslant s \leqslant t \leqslant 1$ and $\lambda \leqslant s \leqslant \mu$,

$\frac{\partial G}{\partial t}(t,s) = \frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{-H(s)+(A_\lambda+B_\lambda)(1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$,

when $0 \leqslant s \leqslant t \leqslant 1$ and $\mu \leqslant s \leqslant \lambda$,

$\frac{-(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(A_\lambda+B_\lambda)(1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$,

when $0 \leqslant s \leqslant t \leqslant 1$ and $s \geqslant \mu, \lambda$,
$\frac{(A_\lambda+B_\lambda)(1-s)^{\alpha-1}+(\mu-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$,
when $0 \leqslant t \leqslant s \leqslant 1$ and $s \leqslant \mu, \lambda$,
$\frac{-H(s)+(A_\lambda+B_\lambda)(1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$,
when $0 \leqslant t \leqslant s \leqslant 1$ and $s \geqslant \mu, \lambda$ and
$\frac{\partial G}{\partial t}(t,s) = \frac{(A_\lambda+B_\lambda)(1-s)^{\alpha-1}}{A_\lambda\Gamma(\alpha)}$, when $0 \leqslant t \leqslant s \leqslant 1$ and $s \geqslant \mu, \lambda$,

Also we can see that G and $\frac{\partial}{\partial t}G$ are continuous respect to t. Consider the space $X = C^1[0,1]$ with the norm $\|.\|_*$, where $\|x\|_* = \max\{\|x\|, \|x'\|\}$ and $\|.\|$ is the supremum norm on $C[0,1]$. Let f be a map on $[0,1] \times X^5$ such that f is singular at some points of $[0,1]$. Define the map
$F : X \to X$ by

$$
\begin{aligned}
& F_x(t) \\
= & \int_0^1 G(t,s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds \\
= & \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)dxi, \phi x(s))ds \\
& + \frac{1-t}{A_\lambda\Gamma(\alpha)} \int_0^\lambda H(s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds \\
& + \frac{(A_\lambda+B_\lambda)t - B_\lambda}{A_\lambda\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{t-1}{A_\lambda\Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds,
\end{aligned}
$$

for all $t \in [0,1]$. Also $F'_x(t)$ is given as

$$
\begin{aligned}
&F'_x(t)\\
= &\int_0^1 \frac{\partial G}{\partial t}(t,s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds\\
= &\frac{-1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s),\\
&\int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds\\
&-\frac{1}{A_\lambda \Gamma(\alpha)}\int_0^\lambda H(s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds\\
&+\frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s), D^\beta x(s),\\
&\int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds\\
&+\frac{1}{A_\lambda \Gamma(\alpha-1)}\int_0^\mu (\mu-s)^{\alpha-2} f(s, x(s), x'(s), D^\beta x(s),\\
&\int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds.
\end{aligned}
$$

Note that, the singular pointwise defined equation (1) has a solution $x_0 \in X$ if and only if x_0 a fixed point of the map F.

Theorem 2.1. *Let* $\alpha \geqslant 2$, $\beta, \mu, \lambda \in (0,1)$ $g \in L^1[0,\lambda]$, $g(t) > 0$ *for a.e.* $t \in [0,\lambda]$, $\phi : [0,1] \to \mathbb{R}^+$ *is such that for all* $x, y \in C^1[0,1]$, $|\phi x(t) - \phi y(t)| \leqslant b_1|x(t) - y(t)| + b_2|x'(t) - y'(t)|$ *for some* $b_1, b_2 \in [0,\infty)$, $h \in L^1[0,1]$ *and* $f : [0,1] \times X^5 \to \mathbb{R}$ *be a mapping which is singular on some set* $E \subset [0,1]$ *such that for* $t \in E$ *we have*

$$|f(t, x_1, x_2, ..., x_5) - f(t, y_1, y_2, ..., y_5)| \leqslant \sum_{i=1}^{k_0} a_i(t)\Lambda_i(|x_1 - y_1|, ..., |x_5 - y_5|)$$

and is continuous on $E^c \subset [0,1]$ *such that for* $t \in E^c$ *we have*

$$|f(t, x_1, x_2, ..., x_5) - f(t, y_1, y_2, ..., y_5)| \leqslant \sum_{j=1}^{5} l_j |x_j - y_j|$$

for all $x_1, ..., x_5, y_1, ..., y_5 \in X$, *where* $k_0 \in \mathbb{N}$, $a_i : [0,1] \to \mathbb{R}^+$, $\hat{a}_i \in L^1(E)$, $\hat{a}_i(s) = (1-s)^{\alpha-2} a_i(s)$, $l_i \in [0,\infty)$, $\Lambda_i : X^5 \to \mathbb{R}^+$ *is a nondecreasing mapping respect to all their components such that* $\lim_{z \to 0^+} \frac{\Lambda_i(z,...,z)}{z} = q_i$ *for some* $q_i \in [0,\infty)$, *and all* $1 \leqslant i \leqslant k_0$. *Also let for almost all* $t \in [0,1]$, $f(t,0,0,0,0,0) = 0$, *if*

$$\begin{aligned} & \max\{[\sum_{i=1}^{k_0} (q_i + \epsilon)\Delta \|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}] \\ + \quad & M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\ & (\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}), \\ & (\sum_{i=1}^{k_0} (q_i + \epsilon)\Delta \|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}) \\ + \quad & M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\ & (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)})\} < 1, \end{aligned}$$

then

$$D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(s)x(s)ds, \phi x(t)) = 0$$

with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ *and* $x(1) = x^{(j)}(0) = 0$ *forj* $\geqslant 2$ *has a solution.*

Proof. First we show that F is continuous on X. Let $x_1, x_2 \in X$ and $t \in [0,1]$, then we have

$$\begin{aligned} & |F_{x_1}(t) - F_{x_2}(t)| \\ \leqslant \quad & |\int_0^1 G(t,s) f(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s))ds \\ & - \int_0^1 G(t,s) f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))ds| \\ \leqslant \quad & |\int_0^1 G(t,s)(f(s, x_1(s), x_1'(s), D^\beta x_1(s), \int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s)) \end{aligned}$$

$$
\begin{aligned}
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s)))ds| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_{E\cap[0,t]}(t-s)^{\alpha-1}|f(s,x_1(t),x_1'(s),D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s)) \\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds \\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)|f(s,x_1(t),x_1'(s),D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s)) \\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds \\
+\ & \frac{|t\int_0^\lambda g(\xi)d\xi-\int_0^\lambda \xi g(\xi)d\xi|}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}|f(s,x_1(s),x_1'(s),D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s)) \\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds \\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}|f(s,x_1(t),x_1'(s),D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s)) \\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s)))ds| \\
\\
+\ & \frac{1}{\Gamma(\alpha)}\int_{E^c\cap[0,t]}(t-s)^{\alpha-1}|f(s,x_1(t),x_1'(s),D^\beta x_1(s), \\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s)) \\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds
\end{aligned}
$$

$$+ \quad \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0,\lambda]} H(s)|f(s, x_1(t), x_1'(s), D^\beta x_1(s),$$

$$\int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s))$$

$$-f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))|ds$$

$$+ \quad \frac{|t\int_0^\lambda g(\xi)d\xi - \int_0^\lambda \xi g(\xi)d\xi|}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1}|f(s, x_1(t), x_1'(s), D^\beta x_1(s),$$

$$\int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s))$$

$$-f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s))|ds$$

$$+ \quad \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0,\mu]} (\mu-s)^{\alpha-2}|f(s, x_1(t), x_1'(s), D^\beta x_1(s)$$

$$\int_0^s h(\xi)x_1(\xi)d\xi, \phi x_1(s))$$

$$-f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi)x_2(\xi)d\xi, \phi x_2(s)))ds|$$

$$\leqslant \quad \frac{1}{\Gamma(\alpha)} \int_{E \cap [0,t]} (t-s)^{\alpha-1}[\sum_{i=1}^{k_0} a_i(s)\Lambda_i(|x_1(s)-x_2(s)|,$$

$$|x_1'(s)-x_2'(s)|, |D^\beta(x_1-x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi|,$$

$$|\phi x_1(s)-\phi x_2(s)|)]ds$$

$$+ \quad \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0,\lambda]} H(s)[\sum_{i=1}^{k_0} a_i(s)\Lambda_i(|x_1(s)-x_2(s)|, |x_1'(s)-x_2'(s)|,$$

$$|D^\beta(x_1-x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi|, |\phi x_1(s)-\phi x_2(s)|)]ds$$

$$
\begin{aligned}
+ \quad & \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s)-x_2(s)|, \\
& |x_1'(s)-x_2'(s)|, |D^\beta(x_1-x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi)) d\xi|, \\
& |\phi x_1(s) - \phi x_2(s)|)] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E \cap [0,\mu]} (\mu-s)^{\alpha-2} [\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s)-x_2(s)|, \\
& |x_1'(s)-x_2'(s)|, |D^\beta(x_1-x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi)) d\xi|, \\
& |\phi x_1(s) - \phi x_2(s)|)] ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{E^c \cap [0,t]} (t-s)^{\alpha-1} [l_1 |x_1(s)-x_2(s)| + l_2 |x_1'(s)-x_2'(s)| \\
& + l_3 |D^\beta(x_1-x_2)(s)| + l_4 |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi)) d\xi| \\
& + l_5 |\phi x_1(s) - \phi x_2(s)|] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0,\lambda]} H(s) [l_1 |x_1(s)-x_2(s)| + l_2 |x_1'(s)-x_2'(s)| \\
& + l_3 |D^\beta(x_1-x_2)(s)| + l_4 |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi)) d\xi| \\
& + l_5 |\phi x_1(s) - \phi x_2(s)|] ds \\
+ \quad & \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 |x_1(s)-x_2(s)| + l_2 |x_1'(s)-x_2'(s)| \\
& + l_3 |D^\beta(x_1-x_2)(s)| + l_4 |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi)) d\xi| \\
& + l_5 |\phi x_1(s) - \phi x_2(s)|] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0,\mu]} (\mu-s)^{\alpha-2} [l_1 |x_1(s)-x_2(s)| + l_2 |x_1'(s)-x_2'(s)| \\
& + l_3 |D^\beta(x_1-x_2)(s)| + l_4 |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi)) d\xi| \\
& + l_5 |\phi x_1(s) - \phi x_2(s)|] ds
\end{aligned}
$$

$$
\begin{aligned}
\leqslant\ & \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_{E\cap[0,t]}(t-s)^{\alpha-1}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|,\\
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_{E\cap[0,\lambda]}H(s)a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|,\\
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds\\
+\ & \frac{\int_0^\lambda|t-\xi|g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_E(1-s)^{\alpha-1}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|,\\
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|,\\
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds\\
+\ & \frac{1}{\Gamma(\alpha)}\int_{E^c\cap[0,t]}(t-s)^{\alpha-1}[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\|+\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)}\\
& +l_4\|x_1-x_2\|\int_0^s|h(\xi)|d\xi+l_5b_1\|x_1-x_2\|+l_5b_2\|x_1'-x_2'\|]ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E^c\cap[0,\lambda]}H(s)[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\|+\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)}\\
& +l_4\|x_1-x_2\|\int_0^s|h(\xi)|d\xi+l_5b_1\|x_1-x_2\|+l_5b_2\|x_1'-x_2'\|]ds\\
+\ & \frac{\int_0^\lambda|t-\xi|g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\int_{E^c}(1-s)^{\alpha-1}[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\|\\
& +\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)}+l_4\|x_1-x_2\|\int_0^s|h(\xi)|d\xi+l_5b_1\|x_1-x_2\|\\
& +l_5b_2\|x_1'-x_2'\|]ds
\end{aligned}
$$

$$\begin{aligned}
&+ \quad \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_{E^c\cap[0,\mu]}(\mu-s)^{\alpha-2}[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\| \\
&\quad +\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)}+l_4\|x_1-x_2\|\int_0^s|h(\xi)|d\xi+l_5b_1\|x_1-x_2\| \\
&\quad +l_5b_2\|x_1'-x_2'\|]ds.
\end{aligned}$$

Note that $\|D^\beta x\| \leqslant \frac{\|x'\|}{\Gamma(2-\beta)}$, also we have

$$H(s)=\int_s^\lambda(t-s)^{\alpha-1}g(t)dt \leqslant \int_s^\lambda(\lambda-s)^{\alpha-1}g(t)dt \leqslant (\lambda-s)^{\alpha-1}\|g\|_{[0,\lambda]}$$

where $\|g\|_{[0,\lambda]}=\int_0^\lambda g(t)dt$. Let $\Delta=\max\{1,\frac{1}{\Gamma(2-\beta)},m,b_1+b_2\}$, then by the last inequality for all $x_1,x_2\in X$ and $t\in[0,1]$ we have

$$\begin{aligned}
&|F_{x_1}(t)-F_{x_2}(t)| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_E(1-s)^{\alpha-1}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|, \\
&\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi,b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds \\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_E(\lambda-s)^{\alpha-1}\|g\|_{[0,\lambda]}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|, \\
&\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi,b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds \\
+\ & \frac{\int_0^\lambda|t-\xi|g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_E(1-s)^{\alpha-1}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|, \\
&\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi,b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds \\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\int_E(\mu-s)^{\alpha-2}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|, \\
&\frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)},\|x_1-x_2\|\int_0^s|h(\xi)|d\xi,b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds
\end{aligned}$$

$$
\begin{aligned}
+ & \frac{1}{\Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 \|x_1 - x_2\| + l_2 \|x_1' - x_2'\| + \frac{l_3 \|x_1' - x_2'\|}{\Gamma(2-\beta)} \\
& + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| + l_5 b_2 \|x_1' - x_2'\|] ds \\
+ & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c} (\lambda - s)^{\alpha-1} \|g\|_{[0,\lambda]} [l_1 \|x_1 - x_2\| + l_2 \|x_1' - x_2'\| \\
& + \frac{l_3 \|x_1' - x_2'\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
& + l_5 b_2 \|x_1' - x_2'\|] ds \\
+ & \frac{\int_0^\lambda |t - \xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 \|x_1 - x_2\| + l_2 \|x_1' - x_2'\| \\
& + \frac{l_3 \|x_1' - x_2'\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
& + l_5 b_2 \|x_1' - x_2'\|] ds \\
+ & \frac{1-t}{A_\lambda \Gamma(\alpha - 1)} \int_{E^c} (\mu - s)^{\alpha-2} [l_1 \|x_1 - x_2\| + l_2 \|x_1' - x_2'\| \\
& + \frac{l_3 \|x_1' - x_2'\|}{\Gamma(2-\beta)} + l_4 \|x_1 - x_2\| \int_0^s |h(\xi)| d\xi + l_5 b_1 \|x_1 - x_2\| \\
& + l_5 b_2 \|x_1' - x_2'\|] ds \\
\leqslant & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i (\Delta \|x_1 - x_2\|_*, ..., \Delta \|x_1 - x_2\|_*) \int_E (1-s)^{\alpha-1} a_i(s) ds \\
+ & \frac{(1-t) \|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i (\Delta \|x_1 - x_2\|_*, ..., \Delta \|x_1 - x_2\|_*) \times \\
& \int_E (1-s)^{\alpha-1} a_i(s) ds \\
+ & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i (\Delta \|x_1 - x_2\|_*, ..., \Delta \|x_1 - x_2\|_*) \int_E (1-s)^{\alpha-1} a_i(s) ds \\
+ & \frac{1-t}{A_\lambda \Gamma(\alpha - 1)} \sum_{i=1}^{k_0} \Lambda_i (\Delta \|x_1 - x_2\|_*, ..., \Delta \|x_1 - x_2\|_*) \times \\
& \int_E (1-s)^{\alpha-2} a_i(s) ds
\end{aligned}
$$

$$+ \quad \frac{1}{\Gamma(\alpha)}\|x_1 - x_2\|_*[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times$$
$$\int_{E^c}(1-s)^{\alpha-1}ds$$
$$+ \quad \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\|x_1 - x_2\|_*[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times$$
$$\int_{E^c}(1-s)^{\alpha-1}ds$$
$$+ \quad \frac{1}{\Gamma(\alpha)}\|x_1 - x_2\|_*[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times$$
$$\int_{E^c}(1-s)^{\alpha-1}ds$$
$$+ \quad \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\|x_1 - x_2\|_*[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times$$
$$\int_{E^c}(1-s)^{\alpha-2}ds. \qquad (2)$$

Now let $\epsilon > 0$ be arbitary and $x_1 \to x_2$ in X. Since $\lim_{z\to 0^+}\frac{\Lambda_i(z,...,z)}{z} = q_i$ for $i = 1, ..., k_0$, so $\lim_{z\to 0^+}\frac{\Lambda_i(\Delta z,...,\Delta z)}{\Delta z} = q_i$, so for $\epsilon > 0$ there exists $0 < \delta \leqslant \epsilon$ such that $0 < z \leqslant \delta$ implies

$$0 < \Lambda_i(\Delta z, ..., \Delta z) < (q_i + \epsilon)\Delta z,$$

for all $i = 1, ..., k_0$, in particular

$$0 < \Lambda_i(\Delta\delta, ..., \Delta\delta) < (q_i + \epsilon)\Delta\delta < (q_i + \epsilon)\Delta\epsilon,$$

so when $\|x_1 - x_2\| < \delta$, by (2) we have

$$|F_{x_1}(t) - F_{x_2}(t)| \leqslant \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\delta, ..., \Delta\delta)\|\hat{a}_i\|_E$$
$$+ \quad \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\delta, ..., \Delta\delta)\|\hat{a}_i\|_E$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\delta,...,\Delta\delta)\|\hat{a}_i\|_E \\
&+ \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\delta,...,\Delta\delta)\|\hat{a}_i\|_E \\
&+ \frac{1}{\Gamma(\alpha)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\frac{M(E^c)}{\alpha} \\
&+ \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\frac{M(E^c)}{\alpha} \\
&+ \frac{1}{\Gamma(\alpha)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\frac{M(E^c)}{\alpha} \\
&+ \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\frac{M(E^c)}{\alpha-1} \\
&\leqslant \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E \\
&+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E \\
&+ \frac{M(E^c)}{\Gamma(\alpha+1)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\epsilon \\
&+ \frac{(1-t)\|g\|_{[0,\lambda]}M(E^c)}{A_\lambda\Gamma(\alpha+1)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\epsilon \\
&+ \frac{M(E^c)}{\Gamma(\alpha+1)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\epsilon \\
&+ \frac{(1-t)M(E^c)}{A_\lambda\Gamma(\alpha)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\epsilon
\end{aligned}
$$

where M is the Lebesgue measure. Hence

$$
\begin{aligned}
\|F_{x_1}-F_{x_2}\| &\leqslant ([\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}] \\
&+ M(E^c)[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\times \\
&\quad [\frac{2}{\Gamma(\alpha+1)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}])\epsilon.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
& |F'_{x_1}(t) - F'_{x_2}(t)| \\
\leqslant\ & |\frac{-1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \\
-\ & \frac{1}{A_\lambda \Gamma(\alpha)} \int_0^\lambda H(s) (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \\
+\ & \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s)) \\
& -f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds \\
+\ & \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_0^\mu (\mu-s)^{\alpha-2} (f(s, x_1(t), x'_1(s), D^\beta x_1(s), \\
& \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s))
\end{aligned}
$$

$$
-f(s, x_2(s), x'_2(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds
$$

$$
\begin{aligned}
\leqslant\ & \frac{1}{\Gamma(\alpha-1)}\int_{E\cap[0,t]}(t-s)^{\alpha-2}|f(s,x_1(t),x_1'(s),D^\beta x_1(s),\\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s))\\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds\\
+\ & \frac{1}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)|f(s,x_1(t),x_1'(s),D^\beta x_1(s),\\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s))\\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds\\
+\ & \frac{A_\lambda+B_\lambda}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}|f(s,x_1(t),x_1'(s),D^\beta x_1(s),\\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s))\\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds\\
+\ & \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}|f(s,x_1(t),x_1'(s),D^\beta x_1(s),\\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s))\\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s)))ds|\\
+\ & \frac{1}{\Gamma(\alpha-1)}\int_{E^c\cap[0,t]}(t-s)^{\alpha-2}|f(s,x_1(t),x_1'(s),D^\beta x_1(s),\\
& \int_0^s h(\xi)x_1(\xi)d\xi,\phi x_1(s))\\
& -f(s,x_2(s),x_2'(s),D^\beta x_2(s),\int_0^s h(\xi)x_2(\xi)d\xi,\phi x_2(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E^c \cap [0,\lambda]} H(s) | f(s, x_1(t), x_1'(s), D^\beta x_1(s),\\
&\quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s))\\
&\quad - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s)) | ds\\
&+ \quad \frac{A_\lambda + B_\lambda}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} | f(s, x_1(t), x_1'(s), D^\beta x_1(s),\\
&\quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s))\\
&\quad - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s)) | ds\\
&+ \quad \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_{E^c \cap [0,\mu]} (\mu - s)^{\alpha-2} | f(s, x_1(t), x_1'(s), D^\beta x_1(s),\\
&\quad \int_0^s h(\xi) x_1(\xi) d\xi, \phi x_1(s))\\
&\quad - f(s, x_2(s), x_2'(s), D^\beta x_2(s), \int_0^s h(\xi) x_2(\xi) d\xi, \phi x_2(s))) ds |\\
&\leqslant \quad \frac{1}{\Gamma(\alpha-1)} \int_{E \cap [0,t]} (t-s)^{\alpha-2} [\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|,\\
&\quad |x_1'(s) - x_2'(s)|, |D^\beta(x_1 - x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi|,\\
&\quad |\phi x_1(s) - \phi x_2(s)|)] ds\\
&+ \quad \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E \cap [0,\lambda]} H(s) [\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, |x_1'(s) - x_2'(s)|,\\
&\quad |D^\beta(x_1 - x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi|, |\phi x_1(s) - \phi x_2(s)|)] ds\\
&+ \quad \frac{\int_0^\lambda g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [\sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x_1(s) - x_2(s)|, |x_1'(s) - x_2'(s)|,\\
&\quad |D^\beta(x_1 - x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi) - x_2(\xi)) d\xi|, |\phi x_1(s) - \phi x_2(s)|)] ds
\end{aligned}
$$

$$
\begin{aligned}
+ & \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_{E\cap[0,\mu]} (\mu-s)^{\alpha-2} [\sum_{i=1}^{k_0} a_i(s)\Lambda_i(|x_1(s)-x_2(s)|, \\
& |x_1'(s)-x_2'(s)|, |D^\beta(x_1-x_2)(s)|, |\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi|, \\
& |\phi x_1(s)-\phi x_2(s)|)]ds \\
+ & \frac{1}{\Gamma(\alpha-1)} \int_{E^c\cap[0,t]} (t-s)^{\alpha-2}[l_1|x_1(s)-x_2(s)| + l_2|x_1'(s)-x_2'(s)| \\
& +l_3|D^\beta(x_1-x_2)(s)| + l_4|\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi| \\
& +l_5|\phi x_1(s)-\phi x_2(s)|]ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha)} \int_{E^c\cap[0,\lambda]} H(s)[l_1|x_1(s)-x_2(s)| + l_2|x_1'(s)-x_2'(s)| \\
& +l_3|D^\beta(x_1-x_2)(s)| + l_4|\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi| \\
& +l_5|\phi x_1(s)-\phi x_2(s)|]ds \\
+ & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1}[l_1|x_1(s)-x_2(s)| + l_2|x_1'(s)-x_2'(s)| \\
& +l_3|D^\beta(x_1-x_2)(s)| + l_4|\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi| \\
& +l_5|\phi x_1(s)-\phi x_2(s)|]ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha-1)} \int_{E^c\cap[0,\mu]} (\mu-s)^{\alpha-2}[l_1|x_1(s)-x_2(s)| \\
& +l_2|x_1'(s)-x_2'(s)| + l_3|D^\beta(x_1-x_2)(s)| + \\
& l_4|\int_0^s h(\xi)(x_1(\xi)-x_2(\xi))d\xi| + l_5|\phi x_1(s)-\phi x_2(s)|]ds \\
\leqslant & \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-2} a_i(s)\Lambda_i(\|x_1-x_2\|, \\
& \|x_1'-x_2'\|, \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}, m\|x_1-x_2\|, b_1\|x_1-x_2\| + b_2\|x_1'-x_2'\|)ds \\
+ & \frac{1}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E \|g\|_{[0,\lambda]} (\lambda-s)^{\alpha-1} a_i(s)\Lambda_i(\|x_1-x_2\|, \|x_1'-x_2'\|,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}, m\|x_1-x_2\|, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds \\
+ \quad & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\int_E (1-s)^{\alpha-1}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|, \\
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}, m\|x_1-x_2\|, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\int_E (1-s)^{\alpha-2}a_i(s)\Lambda_i(\|x_1-x_2\|,\|x_1'-x_2'\|, \\
& \frac{\|x_1'-x_2'\|}{\Gamma(2-\beta)}, m\|x_1-x_2\|, b_1\|x_1-x_2\|+b_2\|x_1'-x_2'\|)ds \\
+ \quad & \frac{1}{\Gamma(\alpha-1)}\int_{E^c}(1-s)^{\alpha-2}[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\|+\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)} \\
& +l_4m\|x_1-x_2\|+l_5b_1\|x_1-x_2\|+l_5b_2\|x_1'-x_2'\|]ds \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha)}\int_{E^c}\|g\|_{[0,\lambda]}(\lambda-s)^{\alpha-1}[l_1\|x_1-x_2\| \\
& +l_2\|x_1'-x_2'\|+\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)}+l_4m\|x_1-x_2\|+l_5b_1\|x_1-x_2\| \\
& +l_5b_2\|x_1'-x_2'\|]ds \\
+ \quad & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\int_{E^c}(1-s)^{\alpha-1}[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\|+\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)} \\
& +l_4m\|x_1-x_2\|+l_5b_1\|x_1-x_2\|+l_5b_2\|x_1'-x_2'\|]ds \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_{E^c}(1-s)^{\alpha-2}[l_1\|x_1-x_2\|+l_2\|x_1'-x_2'\| \\
& +\frac{l_3\|x_1'-x_2'\|}{\Gamma(2-\beta)}+l_4m\|x_1-x_2\|+l_5b_1\|x_1-x_2\|+l_5b_2\|x_1'-x_2'\|]ds \\
\leqslant \quad & \frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\|x_1-x_2\|_*,...,\Delta\|x_1-x_2\|_*)\int_E(1-s)^{\alpha-2}a_i(s)ds \\
+ \quad & \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\|x_1-x_2\|_*,...,\Delta\|x_1-x_2\|_*)\times \\
& \int_E(1-s)^{\alpha-1}a_i(s)ds
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x_1 - x_2\|_*, ..., \Delta\|x_1 - x_2\|_*) \times \\
& \int_E (1-s)^{\alpha-1} a_i(s) ds \\
+ \quad & \frac{1}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\|x_1 - x_2\|_*, ..., \Delta\|x_1 - x_2\|_*) \times \\
& \int_E (1-s)^{\alpha-2} a_i(s) ds \\
+ \quad & \frac{1}{\Gamma(\alpha-1)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \int_{E^c} (1-s)^{\alpha-2} ds \\
+ \quad & \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \int_{E^c} (1-s)^{\alpha-1} ds \\
+ \quad & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda \Gamma(\alpha)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \int_{E^c} (1-s)^{\alpha-1} ds \\
+ \quad & \frac{1}{A_\lambda \Gamma(\alpha-1)} \|x_1 - x_2\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] \times \\
& \int_{E^c} (1-s)^{\alpha-2} ds \\
\leqslant \quad & \frac{1}{\Gamma(\alpha-1)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\delta, ..., \Delta\delta) \|\hat{a}_i\|_E \\
& + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\delta, ..., \Delta\delta) \|\hat{a}_i\|_E \\
+ \quad & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \Lambda_i(\Delta\delta, ..., \Delta\delta) \|\hat{a}_i\|_E
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}\Lambda_i(\Delta\delta,...,\Delta\delta)\|\hat{a}_i\|_E \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]\frac{M(E^c)}{\alpha-1} \\
+ \quad & \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]\frac{M(E^c)}{\alpha} \\
+ \quad & \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]\frac{M(E^c)}{\alpha} \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\delta[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]\frac{M(E^c)}{\alpha-1} \\
\leqslant \quad & \frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E \\
+ \quad & \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E+\frac{1}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E \\
+ \quad & \frac{[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]}{A_\lambda\Gamma(\alpha)}M(E^c)\epsilon \\
+ \quad & \frac{\|g\|_{[0,\lambda]}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]}{A_\lambda\Gamma(\alpha+1)}M(E^c)\epsilon \\
+ \quad & \frac{\|g\|_{[0,\lambda]}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]}{A_\lambda\Gamma(\alpha+1)}M(E^c)\epsilon \\
+ \quad & \frac{[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]}{A_\lambda\Gamma(\alpha)}M(E^c)\epsilon \\
= \quad & [(\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}) \\
+ \quad & M(E^c)(l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2)\times \\
& (\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)})]\epsilon.
\end{aligned}
$$

So

$$
\begin{aligned}
& \|F'_{x_1} - F'_{x_2}\| \\
\leqslant\ & [(\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\epsilon\|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}) \\
+\ & M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\
& (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)} + \frac{1}{A_\lambda\Gamma(\alpha)})]\epsilon.
\end{aligned}
$$

Hence

$$
\|F_{x_1} - F_{x_2}\|_* \ \leqslant\ Q\epsilon
$$

where

$$
\begin{aligned}
Q\ =\ & \max\{[\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}] \\
+\ & M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\
& [\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)} + \frac{1}{A_\lambda\Gamma(\alpha)}], \\
& (\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}) \\
+\ & M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\
& (\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)} + \frac{1}{A_\lambda\Gamma(\alpha)})\} < \infty.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary then $\|F_{x_1} - F_{x_2}\|_* \to 0$ as $x_1 \to x_2$ in X, this shows that F is continuous in X. Since $\lim_{z\to 0^+} \frac{\Lambda_i(\Delta z,\ldots,\Delta z)}{\Delta z} = q_i$, so for $\epsilon > 0$ there is $\delta > 0$ such that $z \in (0, \delta]$ implies

$\Lambda_i(\Delta z, ..., \Delta z) < (q_i + \epsilon)\Delta z$, for all $1 \leqslant i \leqslant k_0$. Also since

$$\begin{aligned} &\max\{[\sum_{i=1}^{k_0} q_i \Delta \|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}] \\ + \quad & M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\ &[\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}], \\ &(\sum_{i=1}^{k_0} q_i \Delta \|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}) \\ + \quad & M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\ &(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)})\} < 1, \end{aligned}$$

there is $\epsilon_0 > 0$ such that

$$\begin{aligned} &\max\{[\sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta \|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}] \\ + \quad & M(E^c)[l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5(b_1 + b_2)] \times \\ &[\frac{2}{\Gamma(\alpha+1)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)}], \\ &(\sum_{i=1}^{k_0} (q_i + \epsilon_0) \Delta \|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)}) \\ + \quad & M(E^c)(l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2) \times \\ &(\frac{1}{\Gamma(\alpha)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha+1)} + \frac{1}{A_\lambda \Gamma(\alpha)})\} < 1. \end{aligned}$$

Let $\delta_0 = \delta(\epsilon_0)$, $R = \min\{\delta_0, 1\}$, $C = \{x \in X : \|x\|_* \leqslant R\}$ and define $\alpha : X^2 \to [0, \infty)$ as $\alpha(x, y) = 1$ for $x, y \in X$, otherwise let $\alpha(x, y) = 0$, so for all $1 \leqslant i \leqslant k_0$ $\Lambda_i(\Delta\|x\|_*, ..., \Delta\|x\|_*) < (q_i + \epsilon_0)\Delta R$. Let $x \in C$, then for all $t \in [0, 1]$ we have

$$
\begin{aligned}
|F_x(t)| &\leqslant |\int_0^1 G(t,s)f(s,x(t),x'(t),D^\beta x(t),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s))ds| \\
&\leqslant \int_0^1 |G(t,s)||f(s,x(t),x'(t),D^\beta x(t),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s))|ds \\
&= \int_0^1 |G(t,s)||f(s,x(t),x'(t),D^\beta x(t),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) \\
&\quad -f(s,0,...,0)|ds \\
&= \int_E |G(t,s)||f(s,x(t),x'(t),D^\beta x(t),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) \\
&\quad -f(s,0,...,0)|ds \\
&+ \int_{E^c} |G(t,s)||f(s,x(t),x'(t),D^\beta x(t),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) \\
&\quad -f(s,0,...,0)|ds \\
&\leqslant \int_E |G(t,s)|\sum_{i=1}^{k_0} a_i(s)\Lambda_i(|x(s)|,|x'(s)|,|D^\beta x(s)|, \\
&\quad |\int_0^s h(\xi)x(\xi)d\xi|,|\phi x(s)-\phi 0(s)|)ds \\
&+ \int_{E^c} |G(t,s)|[l_1|x(s)|+l_2|x'(s)|+l_3|D^\beta x(s)| \\
&+ l_4|\int_0^s h(\xi)x(\xi)d\xi|+l_5|\phi x(s)-\phi 0(s)|]ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_{E\cap[0,t]}(t-s)^{\alpha-1}\sum_{i=1}^{k_0} a_i(s)\Lambda_i(|x(s)|,|x'(s)|,|D^\beta x(s)|, \\
&\quad |\int_0^s h(\xi)x(\xi)d\xi|,|\phi x(s)-\phi 0(s)|)ds \\
&+ \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]} H(s)\sum_{i=1}^{k_0} a_i(s)\Lambda_i(|x(s)|,|x'(s)|,|D^\beta x(s)|, \\
&\quad |\int_0^s h(\xi)x(\xi)d\xi|,|\phi x(s)-\phi 0(s)|)ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
&\quad |\int_0^s h(\xi) x(\xi) d\xi|, |\phi x(s) - \phi 0(s)|) ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E\cap[0,\mu]} (\mu-s)^{\alpha-2} \sum_{i=1}^{k_0} a_i(s) \Lambda_i(|x(s)|, |x'(s)|, |D^\beta x(s)|, \\
&\quad |\int_0^s h(\xi) x(\xi) d\xi|, |\phi x(s) - \phi 0(s)|) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{E^c\cap[0,t]} (t-s)^{\alpha-1} [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
&+ l_4 |\int_0^s h(\xi) x(\xi) d\xi| + l_5 |\phi x(s) - \phi 0(s)|] ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c\cap[0,\lambda]} H(s) [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
&+ l_4 |\int_0^s h(\xi) x(\xi) d\xi| + l_5 |\phi x(s) - \phi 0(s)|] ds \\
&+ \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
&+ l_4 |\int_0^s h(\xi) x(\xi) d\xi| + l_5 |\phi x(s) - \phi 0(s)|] ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c\cap[0,\mu]} (\mu-s)^{\alpha-2} [l_1 |x(s)| + l_2 |x'(s)| + l_3 |D^\beta x(s)| \\
&+ l_4 |\int_0^s h(\xi) x(\xi) d\xi| + l_5 |\phi x(s) - \phi 0(s)|] ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_{E\cap[0,t]} (t-s)^{\alpha-1} a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
&\quad m\|x\|, b_1\|x\| + b_2\|x'\|) ds \\
&+ \frac{1-t}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_{E\cap[0,\lambda]} H(s) a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
&\quad m\|x\|, b_1\|x\| + b_2\|x'\|) ds
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
& m\|x\|, b_1\|x\| + b_2\|x'\|) ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_{E\cap[0,\mu]} (\mu-s)^{\alpha-2} a_i(s) \Lambda_i(\|x\|, \|x'\|, \frac{\|x'\|}{\Gamma(2-\beta)}, \\
& m\|x\|, b_1\|x\| + b_2\|x'\|) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{E^c\cap[0,t]} (t-s)^{\alpha-1} [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E^c\cap[0,\lambda]} H(s) [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
+ \quad & \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E^c\cap[0,\mu]} (\mu-s)^{\alpha-2} [l_1\|x\| + l_2\|x'\| + \frac{l_3\|x'\|}{\Gamma(2-\beta)} \\
& + l_4 m\|x\| + l_5 b_1\|x\| + l_5 b_2\|x'\|] ds \\
\leqslant \quad & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\Delta\|x\|_*, ..., \Delta\|x\|_*) ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E \|g\|_{[0,\lambda]} (\lambda-s)^{\alpha-1} a_i(s) \Lambda_i(\Delta\|x\|_*, ..., \Delta\|x\|_*) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-1} a_i(s) \Lambda_i(\Delta\|x\|_*, ..., \Delta\|x\|_*) ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \sum_{i=1}^{k_0} \int_E (1-s)^{\alpha-2} a_i(s) \Lambda_i(\Delta\|x\|_*, ..., \Delta\|x\|_*) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{E^c} (1-s)^{\alpha-1} \|x\|_* [l_1 + l_2 + \frac{l_3}{\Gamma(2-\beta)} + l_4 m + l_5 b_1 + l_5 b_2] ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E^c}\|g\|_{[0,\lambda]}(\lambda-s)^{\alpha-1}\|x\|_*[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)} \\
&\quad +l_4m+l_5b_1+l_5b_2]ds \\
&+ \quad \frac{1}{\Gamma(\alpha)}\int_{E^c}(1-s)^{\alpha-1}\|x\|_*[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2]ds \\
&+ \quad \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_{E^c}(1-s)^{\alpha-2}\|x\|_*[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)} \\
&\quad +l_4m+l_5b_1+l_5b_2]ds \\
&\leqslant \quad \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon_0)\Delta R\|\hat{a}_i\|_E+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon_0)\Delta R\|\hat{a}_i\|_E \\
&+ \quad \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k_0}(q_i+\epsilon_0)\Delta R\|\hat{a}_i\|_E+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{k_0}(q_i+\epsilon_0)\Delta R\|\hat{a}_i\|_E \\
&+ \quad \frac{1}{\Gamma(\alpha+1)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]M(E^c)R \\
&+ \quad \frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]M(E^c)R \\
&+ \quad \frac{1}{\Gamma(\alpha+1)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]M(E^c)R \\
&+ \quad \frac{1-t}{A_\lambda\Gamma(\alpha)}[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]M(E^c)R.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|F_x\| \quad &\leqslant \quad ((q_i+\epsilon_0)\Delta\|\hat{a}_i\|_E(\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}) \\
&+ \quad M(E^c)[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\times \\
&\quad (\frac{2}{\Gamma(\alpha+1)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}))R\leqslant R.
\end{aligned}
$$

Using similar proof we conclude that

$$\begin{aligned}\|F'_x\| &\leqslant ((\sum_{i=1}^{k_0}(q_i+\epsilon_0)\Delta\epsilon\|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)})\\ &+ M(E^c)[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\times\\ &(\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}))R\leqslant R.\end{aligned}$$

Therefore $\|F_x\|_* = \max\{\|F_x\|, \|F'_x\|\} \leqslant R$, so $\|F'_x\| \leqslant r$ so $\|F_x\|_* \leqslant r$ therefore $F_x \in C$. By similar manner we conclude that $F_y \in C$, hence $\alpha(F_x, F_y) \geqslant 1$, so F is α- admissible. It's obvious that $C \neq \phi$, hence there exists $x_0 \in C$ such that $F_{x_0} \in C$ and therefore $\alpha(x_0, F_{x_0}) \geqslant 1$. Let

$$\begin{aligned}\gamma &:= \max\{[\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a}_i\|_E][\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}]\\ &+ M(E^c)[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\times\\ &[\frac{2}{\Gamma(\alpha+1)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}],\\ &(\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a}_i\|_E)(\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)})\\ &+ M(E^c)(l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2)\times\\ &(\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)})\}<1,\end{aligned}$$

then define $\psi : [0, \infty) \to [0, \infty)$ as $\psi(t) = \gamma t$, so ψ is nondecreasing and

$$\sum_{n=1}^{\infty}\psi^n(t)=\frac{\gamma}{1-\gamma}<\infty,$$

therefore $\psi \in \Psi$. Also for $x, y \in C$ we have

$$\begin{aligned}\|F_x - F_y\| &\leqslant ([\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a_i}\|_E][\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}]\\ &+ M(E^c)[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\times\\ &[\frac{2}{\Gamma(\alpha+1)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}])\|x-y\|_* \leqslant \gamma\|x-y\|_*\end{aligned}$$

and

$$\begin{aligned}\|F'_x - F'_y\| &\leqslant (\sum_{i=1}^{k_0} q_i\Delta\|\hat{a_i}\|_E)(\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)})\\ &+ M(E^c)(l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2)\times\\ &(\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)})\|x-y\|_*\\ &\leqslant \gamma\|x-y\|_*\end{aligned}$$

so

$$\|F_x - F_y\|_* \leqslant \gamma\|x-y\|_*.$$

Hence for all $x, y \in X$ we have

$$\alpha(x,y)\|F_x - F_y\|_* \leqslant \gamma\|x-y\|_* = \Psi(\|x-y\|_*).$$

Now, using lemma (1.3) we conclude that F has a fixed point in X which is a solution for the problem. □

Example 2.3. Consider the problem

$$D^{\frac{5}{2}}x(t) + f(t, x(t), x'(t), D^{\frac{1}{2}}x(t), \int_0^t \xi x(\xi)d\xi, D^{\frac{1}{3}}x(t)) = 0, \qquad (3)$$

with boundary conditions $x'(\frac{1}{3}) = x'(0) + \int_0^{\frac{1}{2}} sx(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $2 \leqslant j \leqslant 3$, where

$$f(t, x_1, ..., x_5) = \begin{cases} \frac{1}{10\sqrt{1-t}p(t)}\Sigma_{i=1}^5|x_i & t \in E := [0, \frac{1}{3}] \\ \\ \frac{1}{20}(1-t)\Sigma_{i=1}^5|x_i| & t \in E^c := (\frac{1}{3}, 1] \end{cases}$$

and $p(t) = 0$ whenever $t \in E \cap \mathcal{Q}$ and $p(t) = 1$ whenever $t \in E \cap \mathcal{Q}^c$. Put $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\Lambda_i(x_1, ..., x_5) = \frac{1}{10}\Sigma_{i=1}^5 |x_i|$, $a_i(t) = \frac{1}{\sqrt{1-tp(t)}}$, $g(t) = h(t) = t$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, $m = \frac{1}{2}$, $\phi x(t) = D^{\frac{1}{3}} x(t)$, $b_1 = 0$, $b_2 = \frac{1}{\Gamma(\frac{5}{3})}$, $q_i = 5$, $l_1 = ... = l_5 = \frac{2}{60}$ and $k_0 = 1$ for all $1 \leqslant i \leqslant k_0$. Note that for $t \in E^c$

$$|f(t, x_1, ..., x_5) - f(t, y_1, ..., y_5)| = \frac{1}{20}(1-t)|\Sigma_{i=1}^5 |x_i| - |y_i| \leqslant \frac{2}{60}\Sigma_{i=1}^5 |x_i - y_i|,$$

and for $t \in E$ we have

$$\begin{aligned} |f(t, x_1, ..., x_5) \quad &- \quad f(t, y_1, ..., y_5)| = \frac{1}{\sqrt{1-tp(t)}} |\Sigma_{i=1}^5 |x_i| - |y_i|| \\ &\leqslant \quad \frac{1}{10\sqrt{1-tp(t)}} |x_i - y_i| = a_i(t)\Sigma_{i=1}^{k_0} \Lambda_i(|x_i - y_i|), \end{aligned}$$

$\lim_{z \to 0^+} \frac{\Lambda_i(z,z,z,z,z)}{z} = \frac{1}{2} = q_i$, $(1-t)^{\alpha-2} a_i(t) \in L^1(E)$, $\|\hat{a}_i\| = \frac{1}{3}$, $\|g\|_{[0,\lambda]} = \frac{1}{18}$, for all $1 \leqslant i \leqslant k_0$, $|\phi x(t) - \phi y(t)| \leqslant D^{\frac{1}{3}} |x(t) - y(t)| \leqslant \frac{1}{\Gamma(\frac{5}{3})} |x(t) - y(t)|$, $\int_0^1 h(t)dt = \frac{1}{2} = m$,

$$\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1 + b_2\} = \max\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{2}, \frac{1}{\Gamma(\frac{5}{3})}\} = \frac{2}{\sqrt{\pi}},$$

$$A_\lambda = \int_0^\lambda (1-t)g(t)dt = \int_0^{\frac{1}{2}} (1-t)tdt = \frac{1}{12},$$

$M(E^c) = \frac{2}{3}$ and

$$
\begin{aligned}
& \max\{[\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a_i}\|_E][\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}] \\
+ & M(E^c)[l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5(b_1+b_2)]\times \\
& [\frac{2}{\Gamma(\alpha+1)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}], \\
& (\sum_{i=1}^{k_0}(q_i+\epsilon)\Delta\|\hat{a_i}\|_E)(\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}) \\
+ & M(E^c)(l_1+l_2+\frac{l_3}{\Gamma(2-\beta)}+l_4m+l_5b_1+l_5b_2)\times \\
& (\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)})\} \\
\leqslant & \max\{(\frac{1}{2}.\frac{2}{\sqrt{\pi}}.\frac{1}{3})[\frac{2}{\Gamma(\frac{5}{2})}+\frac{\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{5}{2})}+\frac{1}{\frac{1}{2}\Gamma(\frac{3}{2})}] \\
+ & \frac{2}{3}[\frac{2}{60}+\frac{2}{60}+\frac{\frac{2}{60}}{\Gamma(\frac{3}{2})}+\frac{2}{60}.\frac{1}{2}+\frac{2}{60}.\frac{1}{\Gamma(\frac{5}{3})}][\frac{2}{\Gamma(\frac{7}{2})}+\frac{\frac{1}{18}}{\frac{1}{2}.\Gamma(\frac{7}{2})}+\frac{1}{\frac{1}{2}.\Gamma(\frac{5}{2})}], \\
& (\frac{1}{2}.\frac{2}{\sqrt{\pi}}.\frac{1}{3})(\frac{1}{\Gamma(\frac{3}{2})}+\frac{2.\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{5}{2})}+\frac{1}{\frac{1}{2}\Gamma(\frac{3}{2})}) \\
+ & \frac{2}{3}[\frac{2}{60}+\frac{2}{60}+\frac{\frac{2}{60}}{\Gamma(\frac{3}{2})}+\frac{2}{60}.\frac{1}{2}+\frac{2}{60}.\frac{1}{\Gamma(\frac{5}{3})}]\times \\
& (\frac{1}{\Gamma(\frac{5}{2})}+\frac{2.\frac{1}{18}}{\frac{1}{2}.\Gamma(\frac{7}{2})}+\frac{1}{\frac{1}{2}.\Gamma(\frac{5}{2})})\}<1.
\end{aligned}
$$

Now by using Theorem (2.2), problem (3) has a solution.

Theorem 2.4. Let $\alpha \geqslant 2$, $\beta,\mu,\lambda \in (0,1)$ $g \in L^1[0,\lambda]$, $g(t) > 0$ for a.e. $t \in [0,\lambda]$, $\phi : [0,1] \to \mathbb{R}^+$ is such that for all $x, y \in C^1[0,1]$, $|\phi x(t)-\phi y(t)| \leqslant b_1|x(t)-y(t)|+b_2|x'(t)-y'(t)|$ for some $b_1, b_2 \in [0,\infty)$, $h \in L^1[0,1]$ and $f : [0,1]\times X^5 \to \mathbb{R}$ be a mapping which is singular on some set $E \subset [0,1]$ such that for $t \in E$ we have

$$|f(t,x_1,x_2,...,x_5)-f(t,y_1,y_2,...,y_5)| \leqslant \sum_{i=1}^{5} a_i(t)\Lambda_i(|x_i-y_i|)$$

and is continuous on $E^c \subset [0,1]$ for all $x_1, ..., x_5, y_1, ..., y_5 \in X$, where $a_i : [0,1] \to \mathbb{R}^+$, $\hat{a_i} \in L^1(E)$, $\hat{a_i}(s) = (1-s)^{\alpha-2} a_i(s)$, $\Lambda_i : X \to \mathbb{R}^+$ is a nondecreasing mapping such that $\lim_{z\to 0^+} \frac{\Lambda_i(z)}{z^{\gamma_i}} = q_i$ for some $\gamma_i, q_i \in [0,\infty)$, and all $1 \leqslant i \leqslant 5$. Also let $q > 0$, $\Delta = \max\{1, \frac{2}{\Gamma(2-\beta)}, m, b_1 + b_2\}$ and for almost all $t \in [0,1]$ and $(x_1, ..., x_5) \in X^5$ we have

$$|f(t, x_1, x_2, ..., x_5)| \leqslant b(t)L(x_1, x_2, ..., x_5) + K(x_1, x_2, ..., x_5)$$

where $b : [0,1] \to \mathbb{R}^+$, $L, K : \mathbb{R}^5 \to [0,\infty)$ are such that $(1-t)^{\alpha-2} b(t) \in L^1[0,1]$, L, K are such that

$$\lim_{z\to\infty} \frac{L(z,z,z,z,z)}{z} = q$$

and

$$\lim_{z\to\infty} K(z,z,z,z,z) < \infty.$$

If

$$\max\{\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)} , \frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda \Gamma(\alpha)} + \frac{1}{A_\lambda \Gamma(\alpha-1)})\} q \|\hat{b}\|_{[0,1]} \in [0, \frac{1}{\Delta}),$$

then

$$D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(s)x(s)ds, \phi x(t)) = 0$$

with boundary conditions $x'(\mu) = x'(0) + \int_0^\lambda g(s)x(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for$j \geqslant 2$ has a solution.

Proof. Define $F_1, F_2 : X \to \mathbb{R}$ as
$F_1 x(t) = \int_{E^c} G(t,s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds$ and
$F_2 x(t) = \int_E G(t,s) f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))ds$,
so $Fx = F_1 X + F_2 x$. Now to show that F is continuous, we will prove that F_1, F_2 are continuous. Let $\epsilon > 0$ be arbitary and $t \in E^c$ be fixed

for a moment, then there exists $\delta > 0$ such that $\sqrt{(x_1 - y_1)^2 + ... + (x_5 - y_5)^2} < \delta$ implies that

$$|f(t, x_1, ..., x_5) - f(t, y_1, ..., y_5)| < \epsilon. \tag{4}$$

Now let $\{x_n\}$ be a sequence such that $x_n \to x_0$ in X for some $x_0 \in X$, then there exists $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies that $\|x_n - x_0\|_* < \frac{\delta}{\Delta\sqrt{5}}$ where $\Delta = \max\{1, \frac{2}{\Gamma(2-\beta)}, m, b_1 + b_2\}$, then $\|x_n - x_0\| < \frac{\delta}{\Delta\sqrt{5}} \leqslant \frac{\delta}{\sqrt{5}}$ and $\|x'_n - x'_0\| < \frac{\delta}{\Delta\sqrt{5}} \leqslant \frac{\delta}{\sqrt{5}}$, hence for $t \in E^c$, $|x_n(t) - x_0(t)| < \frac{\delta}{\sqrt{5}}$ and $|x'_n(t) - x'_0(t)| < \frac{\delta}{\sqrt{5}}$, so if $n \geqslant n_0$, then since $\|D^\beta x_n - D^\beta x_0\| \leqslant \frac{\|x'_n - x'_0\|}{\Gamma(2-\beta)}$ we have

$$|D^\beta x_n(t) - D^\beta x_0(t)| \leqslant \frac{\frac{\delta}{\Delta\sqrt{5}}}{\Gamma(2-\beta)} \leqslant \frac{\delta}{\sqrt{5}}$$

and

$$\begin{aligned} |\int_0^t h(\xi)x_n(\xi)d\xi - \int_0^t h(\xi)x_0(\xi)d\xi| &\leqslant \int_0^t |h(\xi)||x_n(\xi) - x_0(\xi)|d\xi \\ &< \frac{\delta}{\Delta\sqrt{5}} \int_0^t |h(\xi)|d\xi = m\frac{\delta}{\Delta\sqrt{5}} \leqslant \frac{\delta}{\sqrt{5}}. \end{aligned}$$

Also we have

$$\begin{aligned} |\phi x_n(t) - \phi x_0(t)| &\leqslant b_1|x_n(t) - x_0(t)| + b_2|x'_n(t) - x'_0(t)| \\ &< (b_1 + b_2)\frac{\delta}{\Delta\sqrt{5}} \leqslant \frac{\delta}{\sqrt{5}}, \end{aligned}$$

hence

$$\begin{aligned} &(|x_n(t) - x_0(t)|^2 + |x'_n(t) - x'_0(t)|^2 + |D^\beta x_n(t) - D^\beta x_0(t)|^2 \\ &+|\int_0^t h(\xi)x_n(\xi)d\xi - \int_0^t h(\xi)x_0(\xi)d\xi|^2 + |\phi x_n(t) - \phi x_0(t)|^2)^{\frac{1}{2}} \\ &< \sqrt{\frac{\delta^2}{5} + ... + \frac{\delta^2}{5}} = \sqrt{\delta^2} = \delta \end{aligned}$$

so by (4) we conclude that

$$\begin{aligned} &|f(t, x_n(t), x'_n(t), D^\beta x_n(t), \int_0^t h(\xi)x_n(\xi)d\xi, \phi x_n(t)) \\ &-f(t, x_0(t), x'_0(t), D^\beta x_0(t), \int_0^t h(s)x_0(s)ds, \phi x_0(t))| < \epsilon \end{aligned}$$

as $n \geqslant n_0$. So

$$f(t, x_n(s), x_n'(s), D^\beta x_n(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_n(s)) \to$$
$$f(t, x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_0(s))$$

as $x_n(t) \to x_0(t)$ for $s \in E^c$. In other hand $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ are bonded and in $L^1(E^c)$ respect to s, hence
$F_1x_n(t) = \int_{E^c} G(t,s)f(s, x_n(s), x_n'(s), D^\beta x_n(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_n(s))ds$
tends to
$F_1x_0(t) = \int_{E^c} G(t,s)f(s, x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s h(\xi)x_0(\xi)d\xi, \phi x_0(s))ds$
and
$F_1'x_n(t) = \int_{E^c} \frac{\partial G(t,s)}{\partial t} f(s, x_n(s), x_n'(s), D^\beta x_n(s), \int_0^s h(\xi)x_n(\xi)d\xi, \phi x_n(s))ds$
tends to
$F_1x_0(t) = \int_{E^c} \frac{\partial G(t,s)}{\partial t} f(s, x_0(s), x_0'(s), D^\beta x_0(s), \int_0^s h(\xi)x_0(\xi)d\xi, \phi x_0(s))ds$
as $n \to \infty$, so F_1 is continuous in X. Now we will prove that F_2 is continuous in X. Let $x, y \in X$, then for all $t \in [0,1]$ we have

$$\begin{aligned}
& |F_2x(t) - F_2y(t)| \\
\leqslant\ & |\int_E G(t,s)[f(s, x(s), x'(s), D^\beta x(s), \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))]ds| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)} \int_{E\cap[0,t]} (t-s)^{\alpha-1} |f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi, \phi x(s)) \\
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))|ds \\
+\ & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E\cap[0,\lambda]} H(s)|f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi, \phi x(s))
\end{aligned}$$

$$
\begin{aligned}
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s))| ds \\
+ \quad & \frac{|t \int_0^\lambda g(\xi) d\xi - \int_0^\lambda \xi g(\xi) d\xi|}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} |f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) \\
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s))| ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E\cap[0,\mu]} (\mu - s)^{\alpha-2} |f(s, x(s), x'(s), D^\beta x(s), \\
& \int_0^s h(\xi) x(\xi) d\xi, \phi x(s)) \\
& -f(s, y(s), y'(s), D^\beta y(s), \int_0^s h(\xi) y(\xi) d\xi, \phi y(s))| ds \\
\leqslant \quad & \frac{1}{\Gamma(\alpha)} \int_{E\cap[0,t]} (t-s)^{\alpha-1} [a_1(s)\Lambda_1(|x(s) - y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s) - y'(s)|) + a_3(s)\Lambda_3(|D^\beta (x-y)(s)|) \\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi) - y(\xi)) d\xi|) + a_5(s)\Lambda_5(|\phi x(s) - \phi y(s)|)] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha)} \int_{E\cap[0,\lambda]} H(s) [a_1(s)\Lambda_1(|x(s) - y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s) - y'(s)|) + a_3(s)\Lambda_3(|D^\beta (x-y)(s)|) \\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi) - y(\xi)) d\xi|) + a_5(s)\Lambda_5(|\phi x(s) - \phi y(s)|)] ds \\
+ \quad & \frac{\int_0^\lambda |t-\xi| g(\xi) d\xi}{A_\lambda \Gamma(\alpha)} \int_E (1-s)^{\alpha-1} [a_1(s)\Lambda_1(|x(s) - y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s) - y'(s)|) + a_3(s)\Lambda_3(|D^\beta (x-y)(s)|) \\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi) - y(\xi)) d\xi|) + a_5(s)\Lambda_5(|\phi x(s) - \phi y(s)|)] ds \\
+ \quad & \frac{1-t}{A_\lambda \Gamma(\alpha-1)} \int_{E\cap[0,\mu]} (\mu - s)^{\alpha-2} [a_1(s)\Lambda_1(|x(s) - y(s)|) \\
& +a_2(s)\Lambda_2(|x'(s) - y'(s)|) + a_3(s)\Lambda_3(|D^\beta (x-y)(s)|)
\end{aligned}
$$

$$
\begin{aligned}
&+a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_{E\cap[0,t]}(t-s)^{\alpha-1}[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|)\\
&+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|)\\
&+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|)\\
&+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|)\\
&+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds\\
+\ & \frac{\int_0^\lambda|t-\xi|g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(\|x-y\|)\\
&+a_2(s)\Lambda_2(\|x'-y'\|)+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})\\
&+a_4(s)\Lambda_4(m\|x-y\|)+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}[a_1(s)\Lambda_1(\|x-y\|)\\
&+a_2(s)\Lambda_2(\|x'-y'\|)+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})\\
&+a_4(s)\Lambda_4(m\|x-y\|)+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)\\
&+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_E\|g\|_{[0,\lambda]}(\lambda-s)^{\alpha-1}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)\\
&+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds\\
+\ & \frac{1}{\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)\\
&+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds\\
+\ & \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_E(1-s)^{\alpha-2}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)
\end{aligned}
$$

$$+ ... + a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds.$$

Now Let $0 < \epsilon < 1$ be arbitry and $\|x-y\|_* < \epsilon$. Since for each $i = 1, ..., 5$, $\lim_{z\to 0^+} \frac{\Lambda_i(\Delta z)}{(\Delta z)^{\gamma_i}} = q_i$, hence there exists $\delta > 0$ such that $0 < z < \delta$ implies that $\frac{\Lambda_i(\Delta z)}{(\Delta z)^{\gamma_i}} - q_i < \epsilon$, so
$\Lambda_i(\Delta z) < \Delta^{\gamma_i}(q_i + \epsilon)z^{\gamma_i}$. Let $\|x-y\|_* < \min\{\epsilon, \delta\}$, then we have

$$\Lambda_i(\Delta\|x-y\|_*) < \Delta^{\gamma_i}(q_i+\epsilon)\|x-y\|_*^{\gamma_i} < \Delta^{\gamma_i}(q_i+\epsilon)\epsilon^{\gamma_i}$$

for each $i = 1, ..., 5$. So $\|x-y\|_* < \min\{\epsilon, \delta\}$ implies that

$$\begin{aligned} |F_2x(t) &- F_2y(t)| \leqslant \frac{1}{\Gamma(\alpha)}\int_E (1-s)^{\alpha-1}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1} \\ &+ ... + a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds \\ &+ \frac{1-t}{A_\lambda\Gamma(\alpha)}\int_E \|g\|_{[0,\lambda]}(1-s)^{\alpha-1}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1} \\ &+ ... + a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_E (1-s)^{\alpha-1}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1} \\ &+ ... + a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds \\ &+ \frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_E (1-s)^{\alpha-2}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1} \\ &+ ... + a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds. \end{aligned}$$

Let $\gamma_0 := \min\{\gamma_1, ..., \gamma_5\}$, then for all $1 \leqslant i \leqslant 5$, $\epsilon^{\gamma_i} \leqslant \epsilon^{\gamma_0}$, so we have

$$\begin{aligned} \|F_2x &- F_2y\| \leqslant [\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds \\ &+ \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds \end{aligned}$$

$$+ \quad \frac{1}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds]\epsilon^{\gamma_0}. \quad (5)$$

Now since $(1-s)^{\alpha-2}a_i(s) \in L^1(E)$, so $(1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon) \in L^1(E)$ and since $\epsilon > 0$ was arbitary, by (5) we conclude that $\|F_2x - F_2y\| \to 0$ as $x \to y$. By the similar way we conclude that

$$\begin{aligned}
& |F_2'x(t) - F_2'y(t)| \\
\leqslant\ & |\int_E \frac{\partial G}{\partial t}(t,s)[f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) \\
& -f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))]ds| \\
\leqslant\ & \frac{1}{\Gamma(\alpha-1)}\int_{E\cap[0,t]}(t-s)^{\alpha-2}|f(s,x(s),x'(s),D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) \\
& -f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))|ds \\
+\ & \frac{1}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]} H(s)|f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) \\
& -f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))|ds \\
+\ & \frac{A_\lambda+B_\lambda}{A_\lambda\Gamma(\alpha)}\int_E (1-s)^{\alpha-1}|f(s,x(s),x'(s),D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi,\phi x(s)) - f(s,y(s),y'(s),D^\beta y(s), \\
& \int_0^s h(\xi)y(\xi)d\xi,\phi y(s))|ds \\
+\ & \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}|f(s,x(s),x'(s),D^\beta x(s), \\
& \int_0^s h(\xi)x(\xi)d\xi,\phi x(s))
\end{aligned}$$

$$
\begin{aligned}
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}|f(s,x(s),x'(s),D^\beta x(s),\\
& \int_0^s h(\xi)x(\xi)d\xi,\phi x(s))\\
& -f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))|ds\\
\leqslant \quad & \frac{1}{\Gamma(\alpha-1)}\int_{E\cap[0,t]}(t-s)^{\alpha-2}[a_1(s)\Lambda_1(|x(s)-y(s)|)\\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|)\\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds\\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)[a_1(s)\Lambda_1(|x(s)-y(s)|)\\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|)\\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds\\
+ \quad & \frac{A_\lambda+B_\lambda}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(|x(s)-y(s)|)\\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|)\\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds\\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}[a_1(s)\Lambda_1(|x(s)-y(s)|)\\
& +a_2(s)\Lambda_2(|x'(s)-y'(s)|)+a_3(s)\Lambda_3(|D^\beta(x-y)(s)|)\\
& +a_4(s)\Lambda_4(|\int_0^s h(\xi)(x(\xi)-y(\xi))d\xi|)+a_5(s)\Lambda_5(|\phi x(s)-\phi y(s)|)]ds\\
\leqslant \quad & \frac{1}{\Gamma(\alpha-1)}\int_{E\cap[0,t]}(t-s)^{\alpha-2}[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|)\\
& +a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|)\\
& +a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds\\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha)}\int_{E\cap[0,\lambda]}H(s)[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|)
\end{aligned}
$$

$$+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|)$$
$$+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds$$
$$+\quad \frac{\int_0^\lambda g(\xi)d\xi}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-1}[a_1(s)\Lambda_1(\|x-y\|)+a_2(s)\Lambda_2(\|x'-y'\|)$$
$$+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|)$$
$$+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds$$
$$+\quad \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_{E\cap[0,\mu]}(\mu-s)^{\alpha-2}[a_1(s)\Lambda_1(\|x-y\|)$$
$$+a_2(s)\Lambda_2(\|x'-y'\|)+a_3(s)\Lambda_3(\frac{\|x'-y'\|}{\Gamma(2-\beta)})+a_4(s)\Lambda_4(m\|x-y\|)$$
$$+a_5(s)\Lambda_5(b_1\|x-y\|+b_2\|x'-y'\|)]ds$$
$$\leqslant\quad \frac{1}{\Gamma(\alpha-1)}\int_E(1-s)^{\alpha-2}(1-s)^{\alpha-2}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)$$
$$+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds$$
$$+\quad \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-2}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)$$
$$+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds$$
$$+\quad \frac{\|g\|_{[0,\lambda]}}{\Gamma(\alpha)}\int_E(1-s)^{\alpha-2}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)$$
$$+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds$$
$$+\quad \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_E(1-s)^{\alpha-2}[a_1(s)\Lambda_1(\Delta\|x-y\|_*)$$
$$+...+a_5(s)\Lambda_5(\Delta\|x-y\|_*)]ds$$
$$\leqslant\quad \frac{1}{\Gamma(\alpha-1)}\int_E(1-s)^{\alpha-2}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1}$$
$$+...+a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds$$
$$+\quad \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\int_E(1-s)^{\alpha-2}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1}$$
$$+...+a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds$$
$$+\quad \frac{\|g\|_{[0,\lambda]}}{\Gamma(\alpha)}\int_E(1-s)^{\alpha-2}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1}$$

$$
\begin{aligned}
&+ ... + a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\int_E (1-s)^{\alpha-2}[a_1(s)\Delta^{\gamma_1}(q_1+\epsilon)\epsilon^{\gamma_1} \\
&+ ... + a_5(s)\Delta^{\gamma_5}(q_5+\epsilon)\epsilon^{\gamma_5}]ds.
\end{aligned}
$$

So we conclude that

$$
\begin{aligned}
\|F_2x \quad - \quad & F_2y\| \leqslant [\frac{1}{\Gamma(\alpha-1)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds \\
+ \quad & \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds \\
+ \quad & \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds \\
+ \quad & \frac{1}{A_\lambda\Gamma(\alpha-1)}\sum_{i=1}^{5}\int_E (1-s)^{\alpha-2}a_i(s)\Delta^{\gamma_i}(q_i+\epsilon)ds]\epsilon^{\gamma_0}.
\end{aligned}
$$

Therefore $\|F_2'x - F_2'y\| \to 0$ as $x \to y$, hence

$$\|F_2x - F_2y\|_* = \max\{\|F_2x - F_2y\|, \|F_2'x - F_2'y\|\} \to 0$$

as $x \to y$, so we conclude that F_2 is continuous in X, hence $F = F_1 + F_2$ is continuous in X. Now we have $\lim_{z\to\infty}\frac{L(\Delta z,\Delta z,\Delta z,\Delta z,\Delta z)}{\Delta z} = q$, so for each $\epsilon > 0$ there is $r > 0$ such that $z \geqslant r$ implies that $\frac{L(\Delta z,...,\Delta z)}{\Delta z} - q < \epsilon$ hence $L(\Delta z, ..., \Delta z) < (q+\epsilon)\Delta z$ for $z \geqslant r$. In other hand $\lim_{z\to\infty}\frac{K(\Delta z,\Delta z,\Delta z,\Delta z,\Delta z)}{\Delta z} = 0$, so for each $\epsilon > 0$ there is $r' > 0$ such that $z \geqslant r'$ implies that $\frac{K(\Delta z,...,\Delta z)}{\Delta z} < \epsilon$ hence $K(\Delta z, ..., \Delta z) < \Delta z\epsilon$ for $z \geqslant r'$. Now since

$$
\begin{aligned}
&\max\{\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}, \\
&\frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)})\}q\|\hat{b}\|_{[0,1]} \in [0, \frac{1}{\Delta}),
\end{aligned}
$$

there is an ϵ_0 such that

$$\begin{aligned}
&\max\{[\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}](q+\epsilon_0)\|\hat{b}\|_{[0,1]}\\
+\quad&[\frac{2}{\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}]\epsilon_0,\\
&[\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)}](q+\epsilon_0)\|\hat{b}\|_{[0,1]}\\
+\quad&[\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)}]\epsilon_0\}\in[0,\frac{1}{\Delta}).
\end{aligned}$$

Let $r_1 := r(\epsilon_0)$ and $r_2 := r'(\epsilon_0)$ and put $r_0 := \max\{r_1, r_2\}$ and define $\Omega = \{x \in X : \|x\|_* < r_0\}$. If there exists y $\partial\Omega$ such that $y(t) = \lambda_0 F_y(t)$ for some $\lambda_0 \in (0,1)$ and all $t \in [0,1]$, then $\|y\|_* = r_0$,

$$y(t)=\lambda_0\int_0^1 G(t,s)f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s))ds$$

and

$$y'(t)=\lambda_0\int_0^1 \frac{\partial G}{\partial t}(t,s)f(s,x(s),x'(s),D^\beta x(s),\int_0^s h(\xi)x(\xi)d\xi,\phi x(s))ds.$$

Hence

$$\begin{aligned}
&|y(t)|\\
=\quad&|\lambda_0\int_0^1 G(t,s)f(s,y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds|\\
\leqslant\quad&\lambda_0[\int_0^1 |G(t,s)|b(s)L(y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds\\
&+\int_0^1 |G(t,s)|b(s)K(y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds]\\
\leqslant\quad&\lambda_0[\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b(s)L(y(s),y'(s),D^\beta y(s),\int_0^s h(\xi)y(\xi)d\xi,\\
&\phi y(s))ds+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\int_0^1(\lambda-s)^{\alpha-1}b(s)L(y(s),y'(s),D^\beta y(s),\\
&\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b(s)L(y(s),y'(s),
\end{aligned}$$

$$
\begin{aligned}
& D^{\beta}y(s), \int_0^s h(\xi)y(\xi)d\xi, \phi y(s))ds \\
& +\frac{1-t}{A_\lambda\Gamma(\alpha)}\int_0^1(\mu-s)^{\alpha-1}b(s)L(y(s),y'(s),D^{\beta}y(s),\int_0^s h(\xi)y(\xi)d\xi, \\
& \phi y(s))ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}K(y(s),y'(s),D^{\beta}y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds \\
& +\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\int_0^1(\lambda-s)^{\alpha-1}K(y(s),y'(s),D^{\beta}y(s), \\
& \int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}K(y(s),y'(s),D^{\beta}y(s), \\
& \int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_0^1(\mu-s)^{\alpha-2}K(y(s),y'(s), \\
& D^{\beta}y(s),\int_0^s h(\xi)y(\xi)d\xi,\phi y(s))ds] \\
\leqslant\ & \lambda_0[\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b(s)L(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds \\
& +\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b(s)L(r,r_0,\frac{r_0}{\Gamma(2-\beta)}, \\
& mr_0,(b_1+b_2)r_0)ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}b(s)L(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds \\
& +\frac{1-t}{A_\lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-2}b(s)L(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}K(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds \\
& +\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}K(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}K(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}K(r,r_0,\frac{r_0}{\Gamma(2-\beta)},mr_0,(b_1+b_2)r_0)ds]\\
\leqslant\ & \lambda_0[\frac{1}{\Gamma(\alpha)}L(\Delta r,...,\Delta r)\int_0^1(1-s)^{\alpha-2}b(s)ds\\
&+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}L(\Delta r,...,\Delta r)\int_0^1(1-s)^{\alpha-2}b(s)ds\\
&+\frac{1}{\Gamma(\alpha)}L(\Delta r,...,\Delta r)\int_0^1(1-s)^{\alpha-2}b(s)ds\\
&+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}L(\Delta r,...,\Delta r)\int_0^1(1-s)^{\alpha-2}b(s)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^1K(\Delta r,...,\Delta r)(1-s)^{\alpha-1}ds\\
&+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}K(\Delta r,...,\Delta r)(1-s)^{\alpha-1}ds\\
&+\frac{1}{\Gamma(\alpha)}K(\Delta r,...,\Delta r)(1-s)^{\alpha-1}ds\\
&+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}K(\Delta r,...,\Delta r)(1-s)^{\alpha-2}ds]
\end{aligned}
$$

$$
\begin{aligned}
\leqslant\ & \lambda_0[\frac{1}{\Gamma(\alpha)}(q+\epsilon_0)\Delta r_0\|\hat{b}\|_{[0,1]}+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}(q+\epsilon_0)\Delta r_0\|\hat{b}\|_{[0,1]}\\
&+\frac{1}{\Gamma(\alpha)}(q+\epsilon_0)\Delta r_0\|\hat{b}\|_{[0,1]}+\frac{1-t}{A_\lambda\Gamma(\alpha-1)}(q+\epsilon_0)\Delta r_0\|\hat{b}\|_{[0,1]}\\
&+\frac{1}{\Gamma(\alpha+1)}\Delta r_0\epsilon_0+\frac{(1-t)\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}\Delta r_0\epsilon_0\\
&+\frac{1}{\Gamma(\alpha+1)}\Delta r_0\epsilon_0+\frac{1-t}{A_\lambda\Gamma(\alpha)}\Delta r_0\epsilon_0],
\end{aligned}
$$

so we have

$$
\begin{aligned}
\|y\|\ \leqslant\ & \lambda_0[(\frac{2}{\Gamma(\alpha)}+\frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)})(q+\epsilon_0)\|\hat{b}\|_{[0,1]}\\
&+(\frac{2}{\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)})\epsilon_0]\Delta r_0<r_0.
\end{aligned}
$$

By same manner we conclude that

$$\begin{aligned}\|y'\| &\leqslant \lambda_0[(\frac{1}{\Gamma(\alpha-1)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)}+\frac{1}{A_\lambda\Gamma(\alpha-1)})(q+\epsilon_0)\|\hat{b}\|_{[0,1]} \\ &\quad +(\frac{1}{\Gamma(\alpha)}+\frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha+1)}+\frac{1}{A_\lambda\Gamma(\alpha)})\epsilon_0]\Delta r_0 < r_0,\end{aligned}$$

hence $\|y\|_* = \max\{\|y\|, \|y'\|\} < r_0$, so $y \notin \partial\Omega$ that it's a contradiction, so by lemma ([14]) we conclude that F has a fixed point in $\bar{\Omega}$ which is a solution for the problem. □

Example 2.5. Consider the problem

$$D^{\frac{7}{2}}x(t)+f(t,x(t),x'(t),D^{\frac{1}{2}}x(t),\int_0^t \xi^2 x(\xi)d\xi, I^{\frac{1}{3}}x(t))=0, \qquad (6)$$

with boundary conditions $x'(\frac{1}{2}) = x'(0)+\int_0^{\frac{1}{2}} sx(s)ds$ and $x(1) = x^{(j)}(0) = 0$ for $2 \leqslant j \leqslant 4$, where $f(t,x_1,...,x_5) = 1+sint+g(t,x_1,...,x_5)$, $g(t,x_1,...,x_5) = frac0.1p(t)\Sigma_{i=1}^5|x_i|$ for $t \in E := [0.2, 0.6)$ and $g(t,x_1,...,x_5) = 0.1t\Sigma_{i=1}^5|x_i| + \Sigma_{i=1}^5\frac{|x_i|}{1+|x_i|}$ for $t \in E^c := [0, 0.2) \cup [0.6, 1]$, $p(t) = 0$ whenever $t \in E \cap \mathcal{Q}$ and $p(t) = 1$ whenever $t \in E \cap \mathcal{Q}^c$. Put $\alpha = \frac{7}{2}$,

$\beta = \frac{1}{2}$, $\Lambda_i(x_1,...,x_5) = L(x_1,...,x_5) = 0.1\Sigma_{i=1}^5|x_i|$, $b(t) = \frac{1}{p(t)}$, $g(t) = t$, $h(t) = t^2$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$, $m = \frac{1}{3}$, $\phi x(t) = I^{\frac{1}{3}}x(t)$, $b_1 = \frac{1}{\Gamma(\frac{1}{2})}$, $b_2 = 0$, $q = 0.5$, $K(x_1,...,x_5) = 2 + \frac{|x_i|}{1+|x_i|}$. Note that for $t \in E$

$$|f(t,x_1,...,x_5)-f(t,y_1,...,y_5)| = \frac{0.1}{p(t)}|\Sigma_{i=1}^5|x_i|-|y_i| \leqslant \frac{0.1}{p(t)}\Sigma_{i=1}^5|x_i-y_i|,$$

and for $t \in E$ we f is continuous, $\lim_{z\to\infty}\frac{\Lambda_i(z,z,z,z,z)}{z} = 0.5 = q$, for all $1 \leqslant i \leqslant 5$, $\lim_{z\to\infty}\frac{L(z,z,z,z,z)}{z} = 0$,

$$\begin{aligned}|f(t,x_1,...,x_5)| &\leqslant \frac{0.1}{p(t)}\Sigma_{i=1}^5|x_i-y_i|+1+sint+\frac{|x_i|}{1+|x_i|} \\ &\leqslant b(t)L(x_1,...,x_5)+K(x_1,...,x_5),\end{aligned}$$

for almost all $t \in [0,1]$, $(1-t)^{\alpha-2}b(t) \in L^1(E)$, $\|\hat{b}\|_E = 0.28$, $\|g\|_{[0,\lambda]} = \frac{1}{18}$,

$$\begin{aligned} |I^{\frac{1}{3}}x(t) \quad - \quad I^{\frac{1}{3}}y(t)| &\leqslant \frac{1}{\Gamma(\frac{1}{3})}\int_0^t (t-s)^{\frac{1}{3}-1}|x(s)-y(s)|ds \\ &\leqslant \quad \frac{\|x-y\|}{\Gamma(\frac{1}{3})}\int_0^t \frac{ds}{(t-s)^{\frac{2}{3}}} \leqslant \frac{\|x-y\|}{\Gamma(\frac{1}{2})} = b_1\|x-y\|, \end{aligned}$$

$\int_0^1 h(t)dt = \frac{1}{3} = m$,

$$\Delta = \max\{1, \frac{1}{\Gamma(2-\beta)}, m, b_1+b_2\} = \max\{1, \frac{1}{\Gamma(\frac{3}{2})}, \frac{1}{3}, \frac{1}{\Gamma(\frac{1}{2})}\} = \frac{2}{\sqrt{\pi}},$$

$$A_\lambda = \int_0^\lambda (1-t)g(t)dt = \int_0^{\frac{1}{2}} (1-t)tdt = \frac{1}{12}$$

and

$$\begin{aligned} & \max\{\frac{2}{\Gamma(\alpha)} + \frac{\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)}, \\ & \frac{1}{\Gamma(\alpha-1)} + \frac{2\|g\|_{[0,\lambda]}}{A_\lambda\Gamma(\alpha)} + \frac{1}{A_\lambda\Gamma(\alpha-1)})\}q\|\hat{b}\|_{[0,1]} \\ = \quad & \max\{\frac{2}{\Gamma(\frac{7}{2})} + \frac{\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{7}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{5}{2})}, \\ & \frac{1}{\Gamma(\frac{5}{2})} + \frac{2.\frac{1}{18}}{\frac{1}{2}\Gamma(\frac{7}{2})} + \frac{1}{\frac{1}{2}\Gamma(\frac{5}{2})}\} \times 0.5 \times 0.28 \in [0, \frac{\sqrt{\pi}}{2}). \end{aligned}$$

Now using Theorem (2.), the problem (6) has a solution.

References

[1] O. P. Agrawal, D. O'Regan and S. Stanek, Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations, *J. Math. Anal. Appl.* 371 (2010), 57-68.

[2] J. Sabatier, O. P. Agrawal and J. A. T. Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering* Springer, Dordrecht (2007)

[3] S. Stanek, The existence of positive solutions of singular fractional boundary value problems, *Computers and Mathematics with Applications*, 62 (2011), 13791388.

[4] Z. Bai, H. L, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* 311 (2005), 495505.

[5] D. Baleanu, Kh. Ghafarnezhad, Sh. Reazapour and M. Shabibi, On the existence of solution of a three steps crisis integro-differential equation, *Advances in difference equations*, 135 (2018), 20 p.

[6] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivative: theory and applications*, Gordon and Breach (1993).

[7] Y. Liu ans P. J. Y. Wong, Global existence of solutions for a system of singular fractional differential equations with impulse effects,*J. Appl. Math. Inform.* 33 (2015), 327-342.

[8] M. Shabibi, M. Postolache, Sh. Rezapour and S. M. Vaezpour, Investigation of a multi-singular pointwise defined fractional integro-differential equation, *J. Mathl Anal.* 7 (2016), 61-77.

[9] M. Shabibi, M. Postolache, Sh. Rezapour, Positive solutions for a singular sum fractional differential systems, *International Journal of Analysis and Applications* 13 (2017) 108-118.

[10] S. Stanek, The existence of positive solutions of singular fractional boundary value problems, *Comput. Math. with Appl.*, 62 (2011), 1379-1388.

[11] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlin. Anal.*, 75 (2012) 2154-2165.

[12] I. Podlubny, *Fractional differential euations*, Academic Press (1999)

[13] S. Abbas, M. Benchohara, A. Alsaeidi and Y. Zhou, Stability results for partial fractional differential equations with noninstaneous impulses, *Advances in Difference Equations*, 75 (2017),

[14] E. Zeidler, *Nonlinear Functional Analysis and Its Applications-I: Fixed-point Theorems*, Springer, New York, NY, USA, (1986)

Ali Mansouri
Assistant Professor of Mathematics
Department of Mathematics
Arak Branch, Islamic Azad University
Arak, Iran
E-mail: *arakmath2@yahoo.com*

Mehdi Shabibi
Assistant Professor of Mathematics
Department of Mathematics
Meharn Branch, Islamic Azad University
Mehran, Iran
E-mail: *mehdi_math1983@yahoo.com*