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Optimal Solution Set in Interval Quadratic Programming Problem

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Abstract. here are several methods to compute the optimal bounds of the objective function for interval quadratic programming (IQP) problems, but no method has yet been suggested to calculate a set of optimal solutions of IQP problems. This paper presents an accurate set of optimal solutions for the interval quadratic programming problems. The optimal solution of the quadratic programming problem is not essentially an extreme point. We first propose conditions that make the optimal solutions of the IQP to extreme points and then, using these conditions, we compute the exact set of optimal solutions for the IQP problem. Under these conditions, we show the intersection of two regions is equal to a set of optimal solutions of IQP.

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1. Introduction

A quadratic programming (QP) problem involves a quadratic objective function and linear constraints.

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In the most real-world applications, the parameters of the problem are not well understood. This is caused the problem data to be uncertain and indicated with intervals. A model for decision making based on uncertainty is interval quadratic programming (IQP).

The interval data of the problem makes the optimal values of the objective function inaccurate (interval) that we show it by $f^{\pm}=[f^-,\ \ f^+]$. The numbers f^- and f^+ show the best and the worst optimal values of the IQP (in minimization problem). The optimal solutions for the IQP are also uncertain in a set.

In recent years, several papers have studied the subject of obtaining objective function bounds of interval quadratic programming problem. Hladik [5, 6], Li [7] and Li et al. [8, 9, 11] obtained the optimal values of IQP with nonnegative variables in different cases of constraints. The IQP problem with unrestricted variables in sign has been researched by Hladik [4]. Li et al. [8, 9, 11] proposed the duality theory for solving IQP. They determined the upper bound of the objective function in minimization case by Dorn dual quadratic program. A pair of two-level mathematical programs to calculate the objective function bounds and optimal solution of the IQP problem was suggested by Liu and Wang [12], Li and Tian [10], and Syaripuddin et al. [15, 16].

The set of optimal solutions for the IQP problem has not been researched yet. This paper aims at describing the conditions which state a set of optimal solutions in interval quadratic programming problems.

2. Preliminaries

The basic definitions and properties of interval arithmetic, including interval number and interval matrix, can be seen at [1, 13, 3, 14].

An interval number $x^{\pm}$ is shown as $[x^-, x^+]$ that $x^-\leqslant x^+$. If $x^-=x^+$ then it is a real number.

If $m, n \in \mathbb{N}$ then interval matrix $\boldsymbol{A}^{\pm}$ is as follows

$$\boldsymbol{A}^{\pm} = \left[\boldsymbol{A}^-,\ \boldsymbol{A}^+\right] = \left\{\boldsymbol{A}\in\mathbb{R}^{m\times n} : \boldsymbol{A}^-\leqslant\mathbf{A}\leqslant\boldsymbol{A}^+\right\}.$$

The matrices $\boldsymbol{A}^-$ and $\boldsymbol{A}^+$ are bounds of interval matrix $\boldsymbol{A}^{\pm}$. Two matrices

$$\boldsymbol{A}^c = \frac{1}{2}\left(\boldsymbol{A}^+ + \boldsymbol{A}^-\right) \qquad , \qquad \boldsymbol{A}^{\boldsymbol{\Delta}} = \frac{1}{2}(\boldsymbol{A}^+ - \boldsymbol{A}^-)$$

denote the center and the radius of $\boldsymbol{A}^{\pm}$, and we define

$$\boldsymbol{A}^{\pm} = \left[\boldsymbol{A}^c - \boldsymbol{A}^{\boldsymbol{\Delta}}\ ,\ \ \boldsymbol{A}^c + \boldsymbol{A}^{\boldsymbol{\Delta}}\right].$$

An interval vector is a one-column interval matrix that is considered as a special case of an interval matrix.

$$\boldsymbol{b}^{\pm} = \left[\boldsymbol{b}^{-},\ \boldsymbol{b}^{+}\right] = \left\{\boldsymbol{b}\in\mathbb{R}^{\boldsymbol{m}} \boldsymbol{:} \boldsymbol{b}^{-} \leqslant \boldsymbol{b} \leqslant \boldsymbol{b}^{+}\right\}$$

$$= \left[\boldsymbol{b}^{\boldsymbol{c}} - \boldsymbol{b}^{\boldsymbol{\Delta}},\ \boldsymbol{b}^{\boldsymbol{c}} + \boldsymbol{b}^{\boldsymbol{\Delta}}\right]$$

where

$$\boldsymbol{b}^{\boldsymbol{c}} = \frac{1}{2}\left(\boldsymbol{b}^{+}+\boldsymbol{b}^{-}\right) \quad , \quad \boldsymbol{b}^{\boldsymbol{\Delta}} = \frac{1}{2}\left(\boldsymbol{b}^{+}-\boldsymbol{b}^{-}\right).$$

Suppose $\{\pm 1\}^{m}$ is a set of all {-1,1} m-dimensional vectors. In other words

$$\{\pm 1\}^{m} = \{\boldsymbol{y}\in\mathbb{R}^{m} : |\boldsymbol{y}| = \boldsymbol{e}\}$$

where $\boldsymbol{e} = (1, 1, \dots, 1)^{t}$.

For each $\boldsymbol{y}\in\{\pm 1\}^{m}$, we define $\boldsymbol{D}_{y} = diag(\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_m)$ that it indicate the corresponding diagonal matrix.

If $\boldsymbol{x} \in\mathbb{R}^{n}$, then the vector of its sign is shown as follows

$$(sgn\ \boldsymbol{x})_i = \begin{cases} 1 & x_i \geqslant 0 \\ -1 & x_i < 0 \end{cases}$$

that i=$1,2,\dots,n$. We display $|\boldsymbol{x}| = \boldsymbol{D_s x}$ where $\boldsymbol{s} = sgn\ \boldsymbol{x}\in\{\pm 1\}^{n}$.

For each $\boldsymbol{y}\in\{\pm 1\}^{m}$ and $\boldsymbol{s}\in\{\pm 1\}^{n}$, following matrices are defined

$$\boldsymbol{A_{ys}} = \boldsymbol{A^{c}} - \boldsymbol{D_{y}} \boldsymbol{A^{\Delta}} \boldsymbol{D_{s}}.$$

Similarly, for each $\boldsymbol{y}\in\{\pm 1\}^{m}$ we define the following vectors

$$\boldsymbol{b_y} = \boldsymbol{b^{c}} + \boldsymbol{D_y} \boldsymbol{b^{\Delta}}$$

3. Interval Quadratic Programming

An IQP problem consists of a quadratic objective function and linear constraints that have contained one or more interval parameters. It can be stated in the inequality form as follows

$$\begin{aligned} \min \quad & \boldsymbol{x^{T}} \boldsymbol{Q}^{\pm} \boldsymbol{x} + \boldsymbol{C}^{\pm\boldsymbol{T}} \boldsymbol{x} \\ s.t. \quad & \boldsymbol{A}^{\pm} \boldsymbol{x} \leqslant \boldsymbol{b}^{\pm} \\ & \boldsymbol{x} \geqslant \boldsymbol{0} \end{aligned} \tag{1}$$

where $\boldsymbol{Q}^{\pm}=\left[\boldsymbol{Q}^{-},\ \boldsymbol{Q}^{+}\right]\in\mathbb{IR}^{n\times n}$, $\boldsymbol{C}^{\pm}=\left[\boldsymbol{C}^{-},\ \boldsymbol{C}^{+}\right]\in\mathbb{IR}^{n}$, $\boldsymbol{A}^{\pm}=\left[\boldsymbol{A}^{-},\boldsymbol{A}^{+}\right]\in\mathbb{IR}^{m\times n}$ and $\boldsymbol{b}^{\pm}=\left[\boldsymbol{b}^{-},\ \boldsymbol{b}^{+}\right]\in\mathbb{IR}^{m}$. Notice that the sets of all $m\times n$ interval matrices and all m-dimensional interval vectors respectively denoted as $\mathbb{IR}^{m\times n}$ and $\mathbb{IR}^{m}$.

The IQP problem is the family of quadratic programming problems called characteristic problems and displayed as follows

$$\begin{aligned} \min \quad & \boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{C}^{\boldsymbol{T}}\boldsymbol{x} \\ s.t. \quad & \boldsymbol{A}\boldsymbol{x}\leqslant\boldsymbol{b} \\ & \boldsymbol{x}\geqslant\boldsymbol{0} \end{aligned} \tag{2}$$

where $\mathbf{Q}\in\boldsymbol{Q}^{\pm}=\left[\boldsymbol{Q}^{-},\ \boldsymbol{Q}^{+}\right]$, $\boldsymbol{C}\in\boldsymbol{C}^{\pm}=\left[\boldsymbol{C}^{-},\ \boldsymbol{C}^{+}\right]$, $\boldsymbol{A}\in\boldsymbol{A}^{\pm}=\left[\boldsymbol{A}^{-},\ \boldsymbol{A}^{+}\right]$, and $\boldsymbol{b}\in\boldsymbol{b}^{\pm}=\left[\boldsymbol{b}^{-},\ \boldsymbol{b}^{+}\right]$. $\boldsymbol{Q}$ is positive semidefinite and symmetric for all $\mathbf{Q}\in\boldsymbol{Q}^{\pm}$. The optimal value of the objective function for the problem (2) is shown as $f\in[f^{-},f^{+}]$.

Equivalently, the IQP problem (1) can be written as

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left[q_{ij}^{-},\ q_{ij}^{+}\right]x_ix_j+\sum_{j=1}^{n}\left[c_j^{-},\ c_j^{+}\right]x_j \\ s.t. \quad & \\ & \sum_{j=1}^{n}\left[a_{ij}^{-},a_{ij}^{+}\right]x_j\leqslant\left[b_i^{-},b_i^{+}\right] \qquad i=1,2,\ldots,m \\ & x_j\geqslant 0 \qquad j=1,2,\ldots,n. \end{aligned} \tag{3}$$

The characteristic problem of (3) is as follows

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}x_ix_j+\sum_{j=1}^{n}c_jx_j \\ s.t. \quad & \\ & \sum_{j=1}^{n}a_{ij}x_j\leqslant b_i \qquad i=1,2,\ldots,m \\ & x_j\geqslant 0 \qquad j=1,2,\ldots,n. \end{aligned} \tag{4}$$

where $q_{ij}\in\left[q_{ij}^{-},\ q_{ij}^{+}\right]$, $\mathrm{c}_j\in\left[c_j^{-},\ c_j^{+}\right]$, $a_{ij}\in\left[a_{ij}^{-},\ a_{ij}^{+}\right]$ and $\mathrm{b}_i\in\left[b_i^{-},\ b_i^{+}\right]$ for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$.

To calculate the objective function bounds of IQP problem, we solve two characteristic problems (4) that they are described in Definition 3.1.

In minimizing IQP problem, calculating the lower bound $f^-$is usually easy but the upper bound f^+has hard calculation.

In the next section, we review the calculation for bounds of the objective function in different cases of constraints and decision variables.

We say that a real vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$ is a feasible solution for (2), if for each $\boldsymbol{A} \in \left[\boldsymbol{A}^-,\ \boldsymbol{A}^+\right]$ and $\boldsymbol{b} \in \left[\boldsymbol{b}^-,\ \boldsymbol{b}^+\right]$ then $\boldsymbol{Ax} \leqslant \boldsymbol{b}$ and $\boldsymbol{x} \geqslant 0$.

Definition 3.1. [7] *The bounds of optimal values for IQP problem* (3) *are computed by solving the following problems:*

$$\begin{aligned} f^- = \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^- x_i x_j + \sum_{j=1}^{n} c_j^- x_j \\ s.t. \quad & \\ & \sum_{j=1}^{n} a_{ij}^- x_j \leqslant b_i^+ \qquad i = 1, 2, \ldots, m \\ & x_j \geqslant 0 \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{5}$$

$$\begin{aligned} f^+ = \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^+ x_i x_j + \sum_{j=1}^{n} c_j^+ x_j \\ s.t. \quad & \\ & \sum_{j=1}^{n} a_{ij}^+ x_j \leqslant b_i^- \qquad i = 1, 2, \ldots, m \\ & x_j \geqslant 0 \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{6}$$

We call problems (5) and (6) the best problem (BP) and the worst problem (WP) of (3), respectively.

Theorem 3.3 concludes that objective bounds for IQP (3) lie in the range of $[f^-,\ f^+]$.

Theorem 3.2. *The problems* (5) *and* (6) *have the largest and the smallest feasible regions defined by* (3)*.*

Proof. Consider the constraints $\sum_{j=1}^{n} a_j x_j \leqslant b$ corresponding to the problem (4). Due to $x_j \geqslant 0$ for each j, we have

$$\sum_{j=1}^{n} a_j^- x_j \leqslant \sum_{j=1}^{n} a_j x_j \leqslant \sum_{j=1}^{n} a_j^+ x_j \leqslant \boldsymbol{b}^- \leqslant \boldsymbol{b} \leqslant \boldsymbol{b}^+$$

Therefore, $\sum_{j=1}^{n} a_j^- x_j \leqslant \boldsymbol{b}^+$ is the largest feasible region that covers all other regions, and $\sum_{j=1}^{n} a_j^+ x_j \leqslant \boldsymbol{b}^-$ is the smallest feasible region. $\square$

Theorem 3.3. *If f^* is the bound of a characteristic problem* (4)*, then $f^{*-} \leqslant f^* \leqslant f^{*+}$, so that f^{*-} and f^{*+} are the objective function bounds of IQP* (3)*.*

Proof. Let $\boldsymbol{x^*} = (x_1^*, x_2^*, \dots, x_n^*)^t$ is the optimal solution for characteristic problem (4), also
$\boldsymbol{x^{*-}} = \left(x_1^{*-}, x_2^{*-}, \dots, x_n^{*-}\right)^t$ and $\boldsymbol{x^{*+}} = \left(x_1^{*+}, x_2^{*+}, \dots, x_n^{*+}\right)^t$ are respectively optimal solutions for the BP and the WP.

The BP has the largest feasible region. Accordingly $\boldsymbol{x^*}$ is a feasible solution of the BP. Therefore we conclude

$$\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^{-} x_i^* x_j^* + \sum_{j=1}^{n} c_j^{-} x_j^* \geqslant \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^{-} x_i^{*-} x_j^{*-} + \sum_{j=1}^{n} c_j^{-} x_j^{*-} = f^{*-}.$$

For each j, $q_{ij} \geqslant q_{ij}^{-}$, and $\mathrm{c}_j \geqslant c_j^{-}$ then

$$f^* = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i^* x_j^* + \sum_{j=1}^{n} \mathrm{c}_j x_j^* \geqslant \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^{-} x_i^* x_j^* + \sum_{j=1}^{n} c_j^{-} x_j^*.$$

Thus $f^* \geqslant f^{*-}$.

Because the WP has the smallest feasible region, so $\boldsymbol{x^{*+}}$is a feasible solution for (4). Therefore

$$f^* = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i^* x_j^* + \sum_{j=1}^{n} \mathrm{c}_j x_j^* \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i^{*+} x_j^{*+} + \sum_{j=1}^{n} c_j x_j^{*}.^{+}$$

For all j, $q_{ij}^{+} \geqslant q_{ij}$ and $c_j^{+} \geqslant \mathrm{c}_j$ so

$$\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i^{*+} x_j^{*+} + \sum_{j=1}^{n} \mathrm{c}_j x_j^{*+} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^{+} x_i^{*+} x_j^{*+} + \sum_{j=1}^{n} c_j^{+} x_j^{*+} = f^{*+}$$

We derive $f^* \leqslant f^{*+}$; hence $f^{*-} \leqslant f^* \leqslant f^{*+}$. □

4. Objective Function Bounds of the IQP Problem

This section reviews determination the objective function bounds in different cases of constraints for IQP problem.

Case 1. In this case, the IQP has inequality constraints. In summary, the IQP problem (1) is written as follows

$$\min\left\{\boldsymbol{x^T Q^{\pm} x + C^{\pm T} x \ : \ A^{\pm} x \leqslant b^{\pm}, \ \ x \geqslant 0}\right\}$$

where $\boldsymbol{Q}^{\pm}\in\mathrm{IR}^{n\times n}$, $\boldsymbol{C}^{\pm}\in\mathrm{IR}^{n}$, $\boldsymbol{A}^{\pm}\in\mathrm{IR}^{m\times n}$, and $\boldsymbol{b}^{\pm}\in\mathrm{IR}^{m}$.

For this problem, we have

$$f^{-} = \min\left\{\boldsymbol{x^T Q^- x + C^{-^T} x \ : \ A^- x \leqslant b^+ \ , \ x \geqslant 0}\right\}$$

$$f^{+} = \min\left\{\boldsymbol{x^T Q^+ x + C^{+^T} x \ : \ A^+ x \leqslant b^-, \ x \geqslant 0}\right\} .$$

Description of this case is given in the previous section.

Case 2. In this case, the IQP problem has equality and inequality constraints. For this problems, the calculation of f^- is simple, but the calculation of f^+ is difficult [9].

$$\min\left\{\boldsymbol{x^T Q^{\pm} x + C^{\pm^T} x \ : \ A^{\pm} x \leqslant b^{\pm} \ , \ B^{\pm} x = d^{\pm}, \ x \geqslant 0}\right\}$$

where $\boldsymbol{Q}^{\pm}\in\mathrm{IR}^{n\times n}$, $\boldsymbol{C}^{\pm}\in\mathrm{IR}^{n}$, $\boldsymbol{A}^{\pm}\in\mathrm{IR}^{m\times n}$, $\boldsymbol{b}^{\pm}\in\mathrm{IR}^{m}$, $\boldsymbol{B}^{\pm}\in\mathrm{IR}^{k\times n}$, and $\boldsymbol{d}^{\pm}\in\mathrm{IR}^{k}$ and its characteristic problem is as follows

$$\min\left\{\boldsymbol{x^T Q x + C^T x \ : \ Ax \leqslant b, \ Bx = d, \ x \geqslant 0}\right\} \tag{7}$$

where $\boldsymbol{Q \in Q^{\pm}, \ C \in C^{\pm}, \ A \in A^{\pm}, \ b \in b^{\pm}, \ B \in B^{\pm}, \ d \in \ d^{\pm},}$ and $\boldsymbol{Q}$ is positive semidefinite and symmetric for all $\mathbf{Q}\in\boldsymbol{Q}^{\pm}$.

The Dorn dual quadratic programming can be used to calculate f^+ in this case (refer to [9]).

Dorn dual problem of the quadratic programming problem (1) is

$$\max\left\{\boldsymbol{-u^T Q u - b^T v - d^T w \ : \ 2Qu + A^T v + B^T w + C \geqslant 0, v \geqslant 0}\right\} .$$

Case 3. In case 3, IQP has unrestricted decision variables in sign

$$\min\left\{\boldsymbol{x^T Q^{\pm} x + C^{\pm^T} x \ : \ A^{\pm} x \leqslant b^{\pm}} \ \right\} \tag{8}$$

where $\boldsymbol{Q}^{\pm}\ \in\mathrm{IR}^{n\times n}$, $\boldsymbol{C}^{\pm}\in\mathrm{IR}^{n}$, $\boldsymbol{A}^{\pm}\in\mathrm{IR}^{m\times n}$, and $\boldsymbol{b}^{\pm}\in\mathrm{IR}^{m}$.

The characteristic problem of (2) is as follows

$$\min\left\{\boldsymbol{x^T Q x + C^T x \ : \ Ax \leqslant b} \ \right\} \tag{9}$$

where $\boldsymbol{Q}\in\boldsymbol{Q}^{\pm}$, $\boldsymbol{C}\in\boldsymbol{C}^{\pm}$, $\boldsymbol{A}\in\boldsymbol{A}^{\pm}$, $\boldsymbol{b}\in\boldsymbol{b}^{\pm}$, and $\boldsymbol{Q}$ is symmetric and positive semidefinite for all $\boldsymbol{Q}\in\boldsymbol{Q}^{\pm}$.

We call the union of all solutions as the solution set, so

$$F=\left\{\boldsymbol{x}\in\mathbb{R}^n:\exists\boldsymbol{A}\in\boldsymbol{A}^{\pm},\quad\exists\boldsymbol{b}\in\boldsymbol{b}^{\pm},\quad\boldsymbol{Ax}\leqslant\boldsymbol{b}\right\}. \tag{10}$$

In this case, both of the bounds f^- and f^+ have hard calculation [2].

Since the variable x is unrestricted in sign; the bound f^+ can not be obtained by solving a characteristic problem. So we consider several quadratic programming problems. If $\boldsymbol{s}\ \in\{\pm1\}^n$, then the corresponding orthant is described by $\boldsymbol{D_S}\ \boldsymbol{x}\geqslant 0$, and the quadratic program is given as follows

$$\begin{aligned}\min\quad & \boldsymbol{x^T(Q^c+Q^{\Delta}D_S)x+(C^c+D_SC^{\Delta})^T x}\\ s.t.\quad & \boldsymbol{(A^c+A^{\Delta}D_S)x\leqslant b^-}\\ & \boldsymbol{D_S\ x\geqslant 0}\end{aligned} \tag{11}$$

Optimal bound for the problem (11) is shown as f_s^+.

The above QP problem can easily be solved. The problem (11) has the smallest feasible region and is suitable for finding the worst bound of the objective function. Thus, the value of f^+ can be computed by solving 2^n ordinary quadratic programming problems.

$$f^+=\min_{\boldsymbol{s}\in\{\pm1\}^n} f_s^+\ .$$

Similar to f^+, the calculation f^- is also hard. For calculation f^-, the feasible set F of (11) must be convex [2]. The set F is not generally convex. If F is restricted to one orthant, then becomes convex [2]. Suppose $\boldsymbol{s}\in\{\pm1\}^n$, so the corresponding orthant is represented by $\boldsymbol{D_s\ x\geqslant 0}$, and its intersection with F is equal to

$$\left(\boldsymbol{A^c-A^{\Delta}D_S}\right)\boldsymbol{x\leqslant b^+},\qquad\boldsymbol{D_S\ x\geqslant 0}. \tag{12}$$

Thus, the lower bound f^- can be computed by solving 2^n ordinary quadratic programming problems with the feasible region (12), which identifies the largest feasible region.

$$\begin{aligned}\min\quad & \boldsymbol{x^T(Q^c-Q^{\Delta}D_S)x+(C^c-D_SC^{\Delta})^T x}\\ s.t.\quad & \boldsymbol{(A^c-A^{\Delta}D_S)x\leqslant b^+}\\ & \boldsymbol{D_S\ x\geqslant 0}.\end{aligned} \tag{13}$$

If f_s^- is the optimal bound for (13), then

$$f^-=\min_{\boldsymbol{s}\in\{\pm1\}^n} f_s^-\ .$$

5. The Optimal Solution Set in IQP

Now, we want to calculate a set of optimal solutions in interval quadratic problems. This item has not been investigated yet. First, we are going to introduce new theorems for determining the regions containing the optimal solutions to the IQP problems.

As we know, there is no requirement that the optimal solution of the QP problem is an extreme point like those of the linear programming problem.

To do this, we use the BP and the WP constraints, in which the optimal solutions to the IQP problem are positive. The BP has the largest region; hence if all components of the feasible solutions to the BP are positive, then components of the feasible solutions and thus the components of the optimal solution for all of the characteristic problems are positive.

We also assume that for the IQP problem, the number of constraints and the number of decision variables are equal, that is $m = n$.

The following definition is introduced to express a lemma and a theorem that specifies a set of optimal solutions to the IQP.

Definition 5.1. *Suppose* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^t$, *then* M_i *and* N_i *are defined as follows*

$$M_i = \left\{ \boldsymbol{x} : \sum_{j=1}^{n} a_{ij}^{-} x_j \leqslant b_i^{+} \right\} \qquad , \qquad N_i = \left\{ \boldsymbol{x} : \sum_{j=1}^{n} a_{ij}^{+} x_j \geqslant b_i^{-} \right\}$$

where $1 \leqslant i \leqslant m$. *The sets* M_i *and* N_i *respectively represent the constraints of BP and the constraints of WP with the inverse sign.*

Suppose $\boldsymbol{x}^{*-} = \left(x_1^{*-}, x_2^{*-}\ , \ \ldots,\ x_n^{*-}\right)^{\ t}$ is the optimal solution of the BP and $\boldsymbol{x}^{*} = (x_1^{*}, x_2^{*}\ , \ \ldots,\ x_n^{*})^{\ t}$ is an optimal solution of the characteristic problem (4). Assume that $m = n$ and all of the optimal solution components of the BP are positive. It is easy to check that each feasible region only has one extreme point, and given that the BP has the largest feasible region, therefore $x_j^{*-} \leqslant x_j^{*}$ where *j=1, 2,. . . , n.*

Lemma 5.2. *Suppose* $m = n$ *and all of the feasible solution components to the BP are positive. If the optimal solution for the best problem of the IQP is an extreme point, then the optimal solution of each problem* (4) *will be an extreme point.*

Proof. Consider the IQP problem (3). Let $\boldsymbol{x}^{*-} = \left(x_1^{*-}, x_2^{*-}\ , \ \ldots,\ x_n^{*-}\right)^{\ t}$ be the optimal solution of the BP (5). Because $\boldsymbol{x}^{*-}$ is an extreme point, all constraints of (5) are active in it and for each $i = 1,\ 2\ , \ldots,\ m = n$, we have

$$\sum_{j=1}^{n} a_{ij}^{-} x_j^{*-} = b_i^{+}.$$

Suppose $\boldsymbol{x^*} = (x_1^*,\ x_2^*\ ,\ \ldots,\ x_n^*)^t$ be an optimal solution for (4), we want to prove that $\boldsymbol{x^*}$ is an extreme point.

By using the proof of the contradiction we presume which x is not an extreme point, so at least one of the constraints of (4) is not active in $\boldsymbol{x^*}$. In other words

$$\exists\ k\ ,\ 1 \leqslant k \leqslant n \quad : \sum_{j=1}^{n} a_{kj} x_j^* < \mathrm{b}_k. \tag{1}$$

For the BP (5), we have

$$\sum_{j=1}^{n} a_{kj}^{-} x_j^{*-} = b_k^{+} \geqslant \mathrm{b}_k$$

All of the feasible solution components of the BP are positive. According to Theorem 3.2, BP has the largest feasible region; therefore the feasible region of (5) is perfectly in the positive area.

So if the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^t$ is a feasible solution for (5), then $x_j > 0$ where $j = 1, 2, \ldots, n$.

Since the feasible region of (5) consists of each feasible region of the problem (4), it is easy to verify that $x_j^{*-} \leqslant x_j^*$ for $j = 1, 2, \ldots, n$. By using the inequality (1) for r= k, we have

$$\sum_{j=1}^{n} a_{kj}^{-} x_j^{*-} \leqslant \sum_{j=1}^{n} a_{kj}^{-} x_j^{*} \leqslant \sum_{j=1}^{n} a_{kj} x_j^{*} < \mathrm{b}_k$$

In which $\sum_{j=1}^{n} a_{kj}^{-} x_j^{*-} < b_k^{+}$ is in contrast to the extreme point , and the activation of the kth constraint of (5) in optimal solution $\boldsymbol{x^{*-}}$.

Therefore, the optimal solution x^* for (4) must be an extreme point. □

Theorem 5.3. *Suppose that all of the assumptions of Lemma* 5.2 *are established. Then a set of optimal solutions to the IQP problem is obtained from the intersection of the region created by the constraints of BP and the constraints of WP with the inverse sign.*

Proof. We assume that $\boldsymbol{x}^*$is an optimal solution for a characteristic problem (4). According to Lemma 5.2, $\boldsymbol{x}^*$ is an extreme point, and all constraints of (4) are active in $\boldsymbol{x}^*$. We derive

$$\sum_{j=1}^{n} a_{ij}^{-} x_j^* \leqslant \sum_{j=1}^{n} a_{ij} x_j^* = \mathrm{b}_i \leqslant b_i^{+}$$

therefore $\boldsymbol{x}^* \in M_i$. Also

$$\sum_{j=1}^{n} a_{ij}^{+} x_j^* \geqslant \sum_{j=1}^{n} a_{ij} x_j^* = \mathrm{b}_i \geqslant b_i^{-}$$

therefore $\boldsymbol{x}^* \in N_i$ *and so*

$$\boldsymbol{x}^* \in \bigcap_{i=1}^{m} M_i \qquad , \qquad \boldsymbol{x}^* \in \bigcap_{i=1}^{m} N_i.$$

Then

$$\boldsymbol{x}^* \in (\bigcap_{i=1}^{m} M_i) \bigcap (\bigcap_{i=1}^{m} N_i)$$

and the theorem is proved. □

If all the assumptions of Lemma 5.2 are established, then the optimal solutions of IQP are the extreme points, and the best and the worst case (BWC) method is a technique to finding a region of optimal solutions to the IQP problem. This technique has been studied in the interval linear programming (ILP) problems [2]. The related algorithm for determining of accurate set of optimal solutions for IQP has been shown in Algorithm 5.4.

In the next section, we compare the solution regions with examples.

Algoritm 5.4. Determining of optimal solution set for $IQP(3)$

1- Suppose the $IQP(3)$

2- Obtain the lower and upper bounds of the objective function by models (5) and (6)

3- If $m = n$ and all of the feasible solution components to BP are positive, then

3-1. Obtain M_i and N_i

3-2. The optimal solution set of IQP is equal to

$$(\bigcap_{i=1}^{m} M_i) \cap (\bigcap_{i=1}^{m} N_i)$$

4- else, we can only obtain the lower and upper bounds of the IQP

5- end.

6. Numerical Examples

In this section, we solve two examples by using the technique mentioned in Theorem 5.3 to compute a set of optimal solutions. All calculations and drawing figures have been done by Maple software.

Example 6.1. Consider the IQP as follows

$$\begin{aligned} \min \quad & 3x_1^2 + 10x_1x_2 + 12x_2^2 + [-7,-4]\,x_1 + [10,14]\,x_2 \\ s.t. \quad & [-15,-13]\,x_1 + [1,2]\,x_2 \leqslant [-11,-10] \\ & [1,2]\,x_1 + [-9,-8]\,x_2 \leqslant [-33,-32] \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

The best problem of the above IQP is

$$\begin{aligned} \min \quad & 3x_1^2 + 10x_1x_2 + 12x_2^2 - 7x_1 + 10x_2 \\ s.t. \quad & -15x_1 + x_2 \leqslant -10 \\ & x_1 - 9x_2 \leqslant -32 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

This quadratic programming problem has the optimal value of the objective function $f^- = 226.4321$ and the optimal solution $\boldsymbol{x}^{*-} = \binom{0.9104}{3.6567}$, which $\boldsymbol{x}^{*-}$ is an extreme point of the feasible region of the above IQP. If we solve all of the characteristic problems and also the worst problem, we observe that their optimal solutions are extreme points and the accuracy of Lemma 5.2 is investigated.

The worst problem has the optimal value $f^+ = 377.63$ and the optimal solution $\boldsymbol{x}^{*+} = \binom{1.54}{4.51}$. The same optimal solutions are obtained using the proposed method at [16]. A set of optimal solutions obtained from the intersection of the region created by the constraints of BP and the constraints of WP with the inverse sign (briefly, intersection technique) is shown in Figure 4.

Figure 5 shows the solution regions by using the BWC and intersection techniques. Comparing these two techniques shows that the BWC box includes infeasible solutions, while the solution region obtained by intersection technique does not include infeasible solutions and specifies an exact set of optimal solutions.

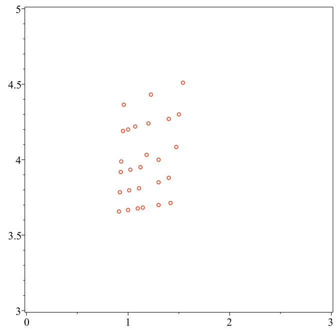

Figure 1. (Example 6.) Some of the optimal solutions of characteristic problems.

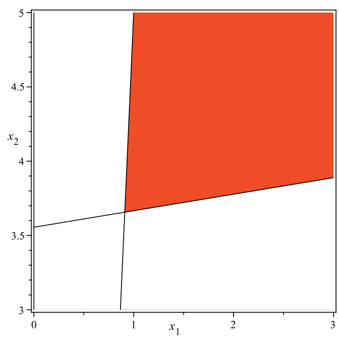

Figure 2. (Example 6.) The BP feasible region.

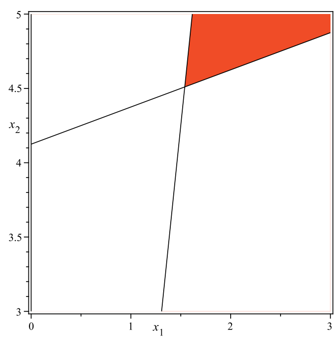

Figure 3. (Example 6.) The WP feasible region.

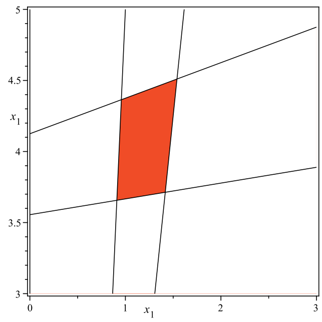

Figure 4. (Example 6.) A set of optimal solutions.

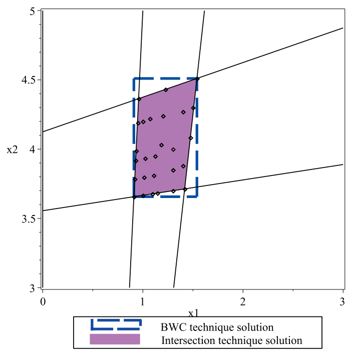

Figure 5. (Example 6.) Comparison of the solution regions.

Example 6.2. Consider the following IQP problem

$$\min \quad [11,15]\,x_1^2 + [-3,-1]\,x_1x_2 - 4x_2^2 + [-23,-15]\,x_1 + [7,9]\,x_2$$

$$s.t. \quad [-4,-2]\,x_1 + [1.8,2]\,x_2 \leqslant [15,17]$$

$$[5.5,6]\,x_1 + [-2.2,-2]\,x_2 \leqslant [-13,-11]$$

$$x_1, x_2 \geqslant 0$$

For this problem, the optimal values of the characteristic problems are located in the interval $f^{\pm} = [f^{-}\ , f^{+}] = \ [-7497,\ -191.75]$ and the optimal solutions of the best and worst problems are $\boldsymbol{x}^{*-} = \binom{16}{45}$ and $\boldsymbol{x}^{*+} = \binom{0.5}{8}$, respectively.

The optimal solution $\boldsymbol{x}^{*-}$is an extreme point, so all of the optimal solutions to the characteristic problems are extreme points.

Comparison of the optimal solution regions is shown in Figure 6.

It is noticeable that the solution region of intersection technique is far better than the BWC solution region.

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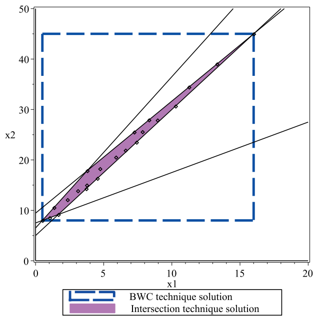

Figure 6. (Example 6.) Comparison of the solution regions.

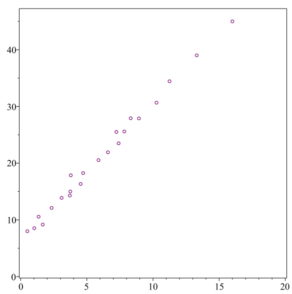

Figure 7. (Example 6.) Some of the optimal solutions of characteristic problems.

7. Conclusion

The methods so far proposed to solve IQP problems most dealing with determining the objective function bounds, and no method is presented to achieve a set of optimal solutions. In this paper, we obtained a set of optimal solutions to the IQP problem when the feasible solution components of the BP were positive and $m = n$.

Under the above conditions, if the optimal solution of the BP is an extreme point, then the optimal solution of each characteristic problem will be extreme point and also a set of optimal solutions to the IQP problem obtained from the intersection of the region created by the constraints of BP and the constraints of WP with inverse sign.

Also, the above conditions put the set of optimal solutions in the BWC box, so BWC technique for the IQP problem is used.

By examples, we showed that the optimal solution region obtained by the intersection technique is better than the optimal solution region obtained by BWC technique.

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