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Some Properties of Composition Operators on Cesáro Function Spaces

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Abstract. In this paper, we discuss about bounded, isometric and hypercyclic composition operators on Cesáro function spaces.

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1. Introduction

Let (X, S, μ) be a $\sigma-$finite measure space and let $L^0 = L^0(X)$ denote the set of all equivalence classes of complex valued measurable functions defined on X. For $1 \leqslant p < \infty$, $L^p(\mu) = L^p(X, S, \mu)$ is the set of all $f \in L^0$ such that

$$\|f\|_p = \Big(\int_X |f|^p \, d\mu \Big)^{\frac{1}{P}} < \infty.$$

For $1 \leqslant p < \infty$ and $X = [0, 1]$ or $X = [0, \infty)$, the *Cesáro function spaces* are denoted by $Ces_p(X)$ and are defined as

$$Ces_p(X) = \{f \in L^0(X) : \int_X (\frac{1}{x} \int_0^x |f(t)| d\mu(t))^p d\mu(x) < \infty\}.$$

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The *Cesáro function spaces* $Ces_p(X)$ are Banach spaces under the norm

$$\|f\| = (\int_X (\frac{1}{x}\int_0^x |f(t)|d\mu(t))^p d\mu(x))^{\frac{1}{p}} \quad for \ \ 1 \leqslant p < \infty$$

and for $p = \infty$,

$$\|f\|_\infty = sup_{x>0}\frac{1}{x}\int_0^x |f(t)|d\mu(t) < \infty.$$

The Cesáro function space $Ces_p[0,\infty)$ for $1 \leqslant p \leqslant \infty$ was considered by Shiue [13], Hassard and Hussein [8] and Sy, Zhang and Lee [15]. Recently Astashkin and Maligranda proved that the Cesáro function spaces $Ces_p(X)$ on both $X = [0,1]$ *and* $X = [0,\infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property [1, 2]. In [3] authers investigated Rademacher sums in $Ces_p[0,1]$ *for* $1 \leqslant p \leqslant \infty$.

A measurable transformation $T : X \to X$ is called **non − singular** if $\mu T^{-1}(E) = \mu(T^{-1}(E)) = 0$ whenever $\mu(E) = 0$ for all $E \in S$. This condition means that the measure μT^{-1} is absolutely continuous with respect to μ. (It is usually denoted $\mu T^{-1} \ll \mu$). Then the Radon-Nikodym theorem assures the existence of a unique non-negative measurable function $h \in L^1(\mu)$ on X such that $\mu T^{-1}(E) = \int_E h(t)d\mu(t)$ for all $E \in S$. h is said to be the **Radon − Nikodym derivative** and denoted by $\frac{d\mu T^{-1}}{d\mu}$. A non-singular measurable transformation T induces a well-defined composition operator C_T from $Ces_p(X)$ into itself defined by $C_T f(x) = f(T(x))$ for $x \in X$ and $f \in Ces_p(X)$. There are examples to show that if T is not non-singular transformation, then C_T is not well-defined. (See [14, Page 18]) The measurable transformation $T : X \to X$ is said to be **measure preserving** if $\mu(T^{-1}(E)) = \mu(E)$ for every $E \in S$. A set $E \in S$ is called **T − invariant** if $T^{-1}(E) = E$.

A bounded linear operator T on Banach space X is (**weakly**)**hypercyclic** if there exists a vector $x \in X$ such that the orbit of x under T,

$$Orb(T,x) := \{T^n x : n \in \mathbb{N} \cup \{0\}\}$$

is (weakly)dense in X. Every such vector x is called (weakly)hypercyclic vector for T.

The study of hypercyclic operators is in a lot of literature. The first example of a hypercyclic operator is λB for $|\lambda| < 1,\ \lambda \in \mathbb{C}$ on the space $l^2(\mathbb{N})$, where B is the backward shifts [12]. [4] is a perfect survey in this topic.

2. Boundedness of Composition Operators on $\mathbf{Ces_p}(\mathbf{X})$

In [11] the authors claimed "as theorem" that T induces a bounded composition operator C_T on $Ces_p(X)$ if and only if there exists $M > 0$ such that $\mu T^{-1}(E) \leqslant M\mu(E)$ for every $E \in S$. Moreover, $\|C_T\| = sup_{0<\mu(E)<\infty}\big((\frac{\mu(T^{-1}(E)}{\mu(E)})^p\big)^{\frac{1}{p}}$. But here we present one example to show that the above result is not correct on $Ces_2(X)$.

Let $X = [0,1]$ and μ be the lebesgue measure on $[0,1]$. Consider

$$T(x) = \begin{cases} 2x & ,0 \leqslant x \leqslant \frac{1}{2} \\ 2x-1 & ,\frac{1}{2} < x \leqslant 1 \end{cases}$$

T is non-singular transformation and $h \equiv 1$. (For more details see [9]).

For

$$f(x) = \begin{cases} 1 & ,0 \leqslant x \leqslant \frac{1}{2} \\ 2(1-x) & ,\frac{1}{2} < x \leqslant 1 \end{cases}$$

we have:

$$\begin{aligned} \|f\|^2 &= \int_0^{\frac{1}{2}} (\frac{1}{x}\int_0^x 1dt)^2 dx + \int_{\frac{1}{2}}^1 (\frac{1}{x}\int_0^x 2(1-t)dt)^2 dx \\ &= \frac{31}{24} \end{aligned}$$

and

$$\begin{aligned} \|foT\|^2 = &\int_0^{\frac{1}{4}} (\frac{1}{x}\int_0^x 1dt)^2 dx + \int_{\frac{1}{4}}^{\frac{1}{2}} (\frac{1}{x}\int_0^x 2(1-2t)dt)^2 dx \\ &+ \int_{\frac{1}{2}}^{\frac{3}{4}} (\frac{1}{x}\int_0^x 1dt)^2 dx + \int_{\frac{3}{4}}^1 (\frac{1}{x}\int_0^x 4(1-t)dt)^2 dx = \frac{13}{6}, \end{aligned}$$

so $\|C_T f\| > \|f\|$, while by their claim $\|C_T\| = 1$ and we must have $\|C_T f\| \leqslant \|f\|$.

Also we note that the authors frequently, use $\|\chi_E\| = \mu(E)$ for all $E \in S$. It is incorrect too, to see this, let $E = [\frac{1}{8}, \frac{1}{4}]$, so $\|\chi_E\|^p = \int_{\frac{1}{8}}^{\frac{1}{4}} (1 - \frac{1}{8t})^p \, dt + (2^{-3p})(\frac{2^{2(p-1)}-1}{p-1})$, but $\mu(E) = \frac{1}{8}$. For $p = 2$, we have $\|\chi_E\| = (\frac{15}{64} - \frac{1}{4} ln2)^{\frac{1}{2}} \neq \frac{1}{8}$.

In the next theorem we give necessary and sufficient conditions for boundedness of composition operators on $Ces_p(X)$ $(1 \leqslant p < \infty)$.

Theorem 2.1. *Let $T : X \to X$ be a non-singular measurable transformation such that $T[0, x] = [0, x]$ for allmost all $x \in X$. Then T induces a bounded composition operator C_T on $Ces_p(X)$ $(1 \leqslant p < \infty)$. if and only if there exists $M > 0$ such that $\mu T^{-1}(E) \leqslant M\mu(E)$ for every $E \in S$.*

Proof. For $f \in Ces_p(X)$:

$$\|C_T f\|^p = \int_X \big(\frac{1}{x} \int_0^x |(foT)(t)| d\mu(t)\big)^p \, d\mu(x).$$

Now, by change of variable formula for $x \in X$ we have:

$$\int_0^x |foT(t)| d\mu(t) = \int_{T[0,x]} |f(r)| d\mu(T^{-1}(r)).$$

Since $T[0, x] = [0, x]$ $a.e.$ and by assumption $h \leqslant M$ we have:

$$\begin{aligned}\|C_T f\|^p &= \int_X \big(\frac{1}{x} \int_0^x |(foT)(t)| d\mu(t)\big)^p \, d\mu(x) = \int_X \big(\frac{1}{x} \int_0^x |f(r)| d\mu(T^{-1}(r))\big)^p \, d\mu(x) \\ &= \int_X \big(\frac{1}{x} \int_0^x |f(r)| h \, d\mu(r)\big)^p \, d\mu(x) \leqslant M^p \|f\|^p.\end{aligned}$$

So $\|C_T f\| \leqslant M\|f\|$ and C_T is bounded on $Ces_p(X)$.
Let $\|C_T\| = M$, $for \;\; all \; n \in \mathbb{N}, E_n = \{t \in X : h(t) > M + \frac{1}{n}\}$ and $E = \{t \in X \; : h(t) > M\}$. Then

$$\|C_T \chi_{E_n}\|^p = \int_X \big(\frac{1}{x} \int_0^x \chi_{E_n} oT(t) d\mu(t)\big)^p \, d\mu(x) = \int_X \big(\frac{1}{x} \int_0^x (\chi_{E_n}(t) \; h(t) d\mu(t)\big)^p \, d\mu(x)$$

$$\geqslant (M+\frac{1}{n})^p \int_X \left(\frac{1}{x}\int_0^x \chi_{E_n}(t)d\mu(t)\right)^p d\mu(x) \geqslant M^p\|\chi_{E_n}\|^p.$$

So $\mu(E_n)=0$. Since $E=\cup E_n$, hence $\mu(E)=0$ *and* $h\leqslant M$ *a.e.*. □

Corollary 2.2. *Let $T: X\to X$ be a non-singular measurable transformation such that $[0,x]$ is $T-$invariant set for allmost all $x\in X$. Then T induces a bounded composition operator C_T on $Ces_p(X)$ if and only if there exists $M>0$ such that $\mu T^{-1}(E)\leqslant M\mu(E)$ for every $E\in S$.*

Proof. Since $[0,x]$ is $T-$ *invariant* set for allmost all $x\in X$, so $T^{-1}[0,x]=[0,x]$. Therefore $T[0,x]=[0,x]$. The rest of proof follows as previous theorem. □

3. Isometric Composition Operators on $\mathbf{Ces_p(X)}$

In this section we give an counter example to show that the result in [11] is false. They claimed that C_T is an isometry if and only if T is a non-singular measure preserving transformation. Measure preserving is not sufficient for C_T to be isometry.

Let T as defined before in the previous section and let $f(t)=t+1$. Easy calculations show that

$$\|f\|^p=\int_0^1\left(\frac{1}{x}\int_0^x(t+1)dt\right)^p dx=\frac{2(\frac{3}{2})^{p+1}-2}{p+1}$$

$$\|C_Tf\|^p=\int_0^{\frac{1}{2}}\left(\frac{1}{x}\int_0^x(2t+1)dt\right)^p dx+\int_{\frac{1}{2}}^1\left(\frac{1}{x}\int_0^x 2tdt\right)^p dx=\frac{(\frac{1}{2})^{p+1}(3^{p+1}-1)}{p+1}.$$

Hence $\|C_Tf\|\neq\|f\|$.

Theorem 3.1. *Let $T: X\to X$ be a non-singular measurable transformation such that $T[0,x]=[0,x]$ for allmost all $x\in X$. Then composition operator C_T on $Ces_p(X)$ is an isometry if and only if T is a measure preserving transformation.*

Proof. Since T is a measure preserving transformation, so $h=1$ *a.e.* We have for $f\in Ces_p(X)$:

$$\|C_Tf\|^p=\int_X\left(\frac{1}{x}\int_0^x|(foT)(t)|d\mu(t)\right)^p d\mu(x).$$

Now, by change of variable formula:

$$\int_0^x |foT(t)|d\mu(t) = \int_{T[0,x]} |f(r)|d\mu(T^{-1}(r)).$$

Since $T[0,x] = [0,x]$ *a.e.*, we have:

$$\int_X \Big(\frac{1}{x}\int_0^x |(foT)(t)|d\mu(t)\Big)^p \, d\mu(x) = \int_X \Big(\frac{1}{x}\int_0^x |f(r)|d\mu(T^{-1}(r))\Big)^p \, d\mu(x)$$

$$= \int_X \Big(\frac{1}{x}\int_0^x |f(r)|h \, d\mu(r)\Big)^p \, d\mu(x) = \|f\|^p.$$

Therefore $\|C_T f\| = \|f\|$.

Conversely, suppose C_T is an isometry. Let $E = \{t \in X : h(t) > 1\}$ *and for all* $n \in \mathbb{N}, E_n = \{t \in X : h(t) > 1 + \frac{1}{n}\}$.

$$\|C_T\chi_{E_n}\|^p = \int_X \Big(\frac{1}{x}\int_0^x (\chi_{E_n}(t) \, h(t)d\mu(t)\Big)^p \, d\mu(x)$$

$$= \int_X \Big(\frac{1}{x}\int_0^x \chi_{E_n}(t)d\mu(t)\Big)^p \, d\mu(x) = \|\chi_{E_n}\|.$$

Then $\int_0^x \big(h(t)\chi_{E_n}(t) - \chi_{E_n}(t)\big)d\mu(t) = 0$ *a.e.* So $h(t)\chi_{E_n}(t) = \chi_{E_n}$ *a.e.* We get $\mu(E_n) = 0$, since $E = \cup E_n$, $\mu(E) = 0$. Now let $F = \{t \in X : h(t) < 1\}$. Similary; $\mu(F) = 0$, and finally $h = 1$ *a.e.*. □

4. Hypercyclic Composition Operators on $\mathbf{Ces_p(X)}$

In this section we discuss about hypercyclic properties of composition operators on Cesáro function spaces.

Lemma 4.1. *If T is a non-singular transformation and $\frac{d\mu T^{-1}}{d\mu} = h$, then for all $n \in \mathbb{N}$, T^n is also non-singular transformation and $\frac{d\mu T^{-n}}{d\mu} = \prod_{j=0}^{n-1} hoT^{-j}$. Moreover; if T is invertible and its inverse is also non-singular transformation, then for all $n \in \mathbb{N}$, $\frac{d\mu T^n}{d\mu} = \big(\prod_{j=1}^{n} hoT^j\big)^{-1}$.*

Proof. We use induction on n. This is true for $n = 1$. Suppose it is true for n, we prove for $n+1$. Let $E \in S$ and $\mu(E) < \infty$, so

$$\mu T^{-(n+1)}(E) = \mu T^{-n}(T^{-1}(E)) = \int_{T^{-1}(E)} \prod_{j=0}^{n-1} hoT^{-j} d\mu.$$

By change of variable:

$$\mu T^{-(n+1)}(E) = \int_E \prod_{j=1}^{n} hoT^{-j} d\mu T^{-1} = \int_E h \prod_{j=1}^{n} hoT^{-j} d\mu = \int_E \prod_{j=0}^{n} hoT^{-j} d\mu.$$

By uniqness of Radon-Nikodym derivative theorem, we have:

$$\frac{d\mu T^{-n}}{d\mu} = \prod_{j=0}^{n-1} hoT^{-j}.$$

For $E \in S$, we have:

$$\int_E h(T(x)) d\mu T(x) = \int_{T(E)} h(r) d\mu(r) = \mu(T^{-1}(T(E)) = \mu(E).$$

So $\frac{d\mu}{d\mu T} = hoT$ *or* $\frac{d\mu T}{d\mu} = \frac{1}{hoT}$. The rest of proof follows as before. □

Kitai [10] presented the hypercyclicity criterion. This criterion was independently improved by Gethner and Shapiro [6]. Eventually, the most general example of this criterion was expressed by B*é*s [5].

Hyperciclicity Criterion. ([4]) Suppose T is a continuous linear operator on a separable Banach space X, for which the sequence of non-negative powers (T^n) tends pointwise to zero on a dense subset of X. Suppose further that there is a (possibly different) dense subset Y of X, and a (possibly discontinuous) map $S : Y \to Y$ such that TS = identity on Y, and (S^n) tends point-wise to zero on Y. Then T is hypercyclic.

Note: Hassard and Hussein [8] proved that $Ces_p(X)$ are separable Banach spaces for $1 < p < \infty$ and in the case $p = \infty$ is not separable. Since hypercyclicity make sense only if the underlying space is separable, then we omit the case $p = \infty$. Also, $Ces_1[0,\infty) = \{0\}$ *and* $Ces_1[0,1] = L^1_\omega[0,1]$ *with* $\omega(t) = ln\frac{1}{t}$, *for* $0 < t \leqslant 1$. So we consider $1 < p < \infty$.

Theorem 4.2. *Suppose $T : X \to X$ is a non-singular measurable transformation, its inverse is measurable and non-singular transformation and $C_T \in B(Ces_p(X))$, $(1 < p < \infty)$.*

If $\prod_{j=0}^{n-1} hoT^{-j} \to 0$ a.e as $n \to \infty$ and $\left(\prod_{j=1}^{n} hoT^{j}\right)^{-1} \to 0$ a.e. as $n \to \infty$, then C_T is hypercyclic on $Ces_p(X)$.

Proof. For $E \in S$ and $\mu(E) < \infty$, we have:

$$\|C_T^n \chi_E\| = \|\chi_{T^{-n}(E)}\|.$$

Using the Hardy inequality [7, Theorem 327], we obtain that

$$\|\chi_{T^{-n}(E)}\| \leqslant \acute{p}\, \|\chi_{T^{-n}(E)}\|_p = \acute{p}\, \mu(T^{-n}(E)),$$

where $\acute{p} = \frac{p}{p-1}$.
Therefore by previous lemma, we have:

$$\|C_T^n \chi_E\| \leqslant \acute{p} \int_E \prod_{j=0}^{n-1} hoT^{-j}(t) d\mu(t).$$

Consider that $\prod_{j=0}^{n-1} hoT^{-j} \in L^1(\mu)$, by Lebesgue Convergence Theorem we have:

$$\|C_T^n \chi_E\| \to 0 \ \ as \ \ n \to \infty.$$

So for all simple function s we have:

$$\|C_T^n s\| \to 0 \ \ as \ \ n \to \infty.$$

Similarly we have:

$$\|C_T^{-n} s\| \to 0 \ \ as \ \ n \to \infty.$$

Since simple functions are dense in $Ces_p(X)$, thus by Hypercyclic Criterion, C_T is hypercyclic on $Ces_p(X)$. □

Corollary 4.3. *On the condition of the above theorem, C_T is hypercyclic on $L^p(\mu)$.*

Proof. Since simple functions are dense in $L^p(\mu)$ and $\|C_T^n \chi(E)\|_p = \mu(T^{-n}(E))$ the proof follows as before. □

Proposition 4.4. *If C_T is (weakly)hypercyclic on $Ces_p(X)$. Then*
i) *$T^{-1}(S) = S$ i.e. $S = \{T^{-1}(E) : E \in S\}$.*
ii) *If $h \leqslant M$ a.e. and $X = [0,1]$, then $M \geqslant \frac{p-1}{p}$.*
iii) *If $T[0,x] = [0,x]$ a.e., then $h > 1$.*

Proof. i) Suppose $T^{-1}(S) \neq S$. By [11, Theorem 10], since C_T has not dense range, so is not (weakly)hypercyclic.

ii) Let $M < \frac{p-1}{p}$ *and for all* $E_i \in S$ *with* $\mu(E_i) < \infty$, we have:

$$\|C_T \sum_{i=1}^{n} \chi_{E_i}\| = \| \sum_{i=1}^{n} C_T \chi_{E_i}\|$$

$$\leqslant \acute{p} \sum_{i=1}^{n} \mu(T^{-1}E_i) \leqslant M\acute{p} \sum_{i=1}^{n} \mu(E_i) \leqslant 1,$$

where $\acute{p} = \frac{p}{p-1}$.

So for all simple function s we have $\|C_T s\| \leqslant 1$, since simple functions are dense in $Ces_p(X)$; we get $\|C_T\| \leqslant 1$. Thus C_T can not be (weakly)hypercyclic on $Ces_p(X)$.

iii) Suppose $h \leqslant 1$, by theorem 2.1, we have $\|C_T\| \leqslant 1$, thus C_T is not (weakly)hypercyclic. □

Lemma 4.5. *If* $\Lambda : \ Ces_p[0,1] \to \mathbb{F}$ *is defined by* $\Lambda(g) = \int_0^a g(t)d\mu(t)$ *for* $0 < a < 1$, *then* Λ *is a bounded linear functional.*

Proof. Let $0 < a < b \leqslant 1$ *and* $p > 1$. By [2, lemma 1] and for $g \in Ces_p([0,1])$ we have:

$$|\Lambda(g)| = |\int_0^a g(t)d\mu(t)| \leqslant \left(\frac{p-1}{b^{1-p}-1}\right)^{\frac{1}{p}} \|g\|.$$

Hence, Λ is bounded linear functional. □

Theorem 4.6. *Let* $X = [0,1]$ *and* $T : X \to X$ *is a non-singular measure preserving transformation and* $C_T \in B(Ces_p(X))$. *If there exists* $a \in (0,1)$ *such that* $T[0,a] = [0,a]$, *then* C_T *is not weakly hypercyclic.*

Proof. Assume to reach a contradiction. Let f be a weakly hypercyclic vector for C_T. Define $\Lambda : \ Ces_p(X) \to \mathbb{F}$ such that $\Lambda(g) = \int_0^a g(t)d\mu(t) \quad (g \in Ces_p(X))$. Then Λ is a bounded linear functional. Suppose $\varepsilon > 0$ is given.

$$U = \{g \in Ces_p(X) : \ |\Lambda(g)| < \varepsilon\} \ , \ V = \{g \in Ces_p(X) : \ |\Lambda(g) - \Lambda(1)| < \varepsilon\}$$

are weak neighborhoods of zero and 1 respectively. So there exists $n \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $C_T^n f \in V, \ C_T^m f \in U$.

$$|\Lambda(C_T^n f) - \Lambda(1)| = |\int_0^a f o T^n d\mu(t) - \int_0^a 1 d\mu(t)| = |\int_0^a f(t)d\mu(t) - a| < \varepsilon.$$

Therefore $\int_0^a f(t)d\mu(t) = a$.

Similary, for $|\Lambda(C_T^m f)| < \varepsilon$ implies that $\int_0^a f(t)d\mu(t) = 0$. It is a contradiction. $\square$

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