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On the Coincidence Point in Ordered Partial Metric Spaces

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Abstract. In this paper we obtain the coincidence and common fixed point of two mappings via R-functions in the setting of ordered partial metric spaces. The result is supported by given an example. An application to the integral equation is also provided.

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1. Introduction

In 1976, the notion of coincidence and common fixed point of commuting mappings are introduced by G. Jungck [6]. Several authors have contributed to the development of the existence and uniqueness of coincidence points of operators in different spaces [2, 3, 5, 12, 14]. Khojaste et al. [7], introduced simulation function and new contraction depending simulation function. Recently, Roldan et a.l [13], modified this concept and proved the existence and uniqueness of coincidence points of two operators in the setting of complete metric spaces.

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On the other hand, in 1992, G. Mathews [8] introduced the notion of the partial metric which can be applied to the study of denotational semantics of data for network. In [9], A. Nastasi et. al proved the existence and uniqueness of fixed points by using R-functions and lower semi-continuous functions in the setting of metric spaces and partial metric spaces.

In this paper, inspired by [6, 9, 13] we deduce some coincidence point results in the setting of ordered partial metric spaces by using R-functions and an application to integral equations is given.

2. Preliminaries

We start by recalling some definitions and properties of partial metric spaces which will be needed during the paper.

Definition 2.1. *[8] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$;*
(i) $p(x, x) = p(x, y) = p(y, y) \Leftrightarrow x = y$.
(ii) $p(x, x) \leqslant p(x, y)$.
(iii) $p(x, y) = p(y, x)$.
(iv) $p(x, z) \leqslant p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. Clearly, a metric p on a set X is a partial metric such that $p(x, x) = 0$ for all $x \in X$.

Each partial metric p on X generates a T_0-topology τ_p on X which has as a base, the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Let (X, p) be a partial metric space. Then
(i) (X, τ_p) is first countable.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. A sequence

$\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$.

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Every partial metric p on X, induces a metric $p^s : X \times X \longrightarrow \mathbb{R}^+$ defined by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x, y \in X$, such that $\tau(p)$ is finer than $\tau(p^s)$ [8].

To see some examples of partial metric spaces refer to [8, 11].

Lemma 2.2. *[10] A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\longrightarrow\infty} p^s(a, x_n) = 0$ if and only if $p(a,a) = \lim_{n\longrightarrow\infty} p(a, x_n) = \lim_{n,m\longrightarrow\infty} p(x_n, x_m)$.*

Lemma 2.3. *[8] Let (X,p) be a partial metric space. Then the following hold:*
(i) If $p(x,y) = 0$, then $x = y$.
(ii) If $x \neq y$, then $p(x,y) > 0$.

Lemma 2.4.*[13] Let (X,p) be a partial metric space and let $\lambda : X \longrightarrow [0,\infty)$ be defined by $\lambda(x) = p(x,x)$ for all $x \in X$. Then the function λ is continuous in the metric space (X,p^s).*

Recently, fixed point theory has developed in metric spaces and partial metric spaces endowed with a partial ordering [5, 1].

Definition 2.5. *Let X be a nonempty set. Then $(X, \preceq, p)$ is called an ordered partial metric space if $(X, \preceq)$ is a partially ordered set, and (X,p) is a partial metric space.*

Two elements x and y of X are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.6. *[3] Two self mappings f and g on a set X have a coincidence point, say x, if $y = f(x) = g(x)$ and y is called a point of coincidence of f and g. Also f and g are said to be weakly compatible if $f(g(x)) = g(f(x))$ whenever $f(x) = g(x)$.*

Lemma 2.7. *[3] Let X be a nonempty set and the mappings $f, g : X \longrightarrow X$ have a unique point of coincidence y in X. If f and g are weakly compatible, then f and g have a unique common fixed point.*

Definition 2.8. *[1] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \longrightarrow X$. Then f is said to be g-nondecreasing if for $x, y \in X$,*

$$g(x) \preceq g(y) \Longrightarrow f(x) \preceq f(y).$$

3. Main Results

We begin this section by giving the concept of R-function (see [13]).

Definition 3.1. *A function $\varphi : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is called $R-$function if the following conditions hold:*
(i) for each sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq (0, \infty)$ with $\varphi(a_{n+1}, a_n) > 0$, for all $n \in \mathbb{N}$, then $\lim_{n\longrightarrow\infty} a_n = 0$;
(ii) for every two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $(0, \infty)$ converging to the same limit $L \geqslant 0$, then $L = 0$ whenever $L < a_n$ and $\varphi(a_n, b_n) > 0$ for all $n \in \mathbb{N}$.

In the sequel $(X, \preceq, p)$ is an ordered partial metric space where $(X, \preceq)$ is a partially ordered set and (X, p) is a partial metric space.

In the main result, we suppose that the following property holds.

Property (C). If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a nondecreasing (noncreasing) sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ $(x \preceq x_n)$ for all $n \in \mathbb{N}$. Also, assume that f and g are two self mappings on X such that f, g are comparable at some $x_0 \in X$ and f is g-nondecreasing, $f(X) \subseteq g(X)$ and one of the sets $f(X)$ or $g(X)$ is closed.

Theorem 3.2. *Let f, g be two self mappings on an ordered complete partial metric space $(X, \preceq, p)$ and the Property (C) be fulfilled. Suppose that f satisfying*

$$\varphi(p(f(x), f(y)), p(g(x), g(y))) > 0, \tag{1}$$

for all comparable $g(x), g(y)$ with $g(x) \neq g(y)$, $x, y \in X$ and some R-function φ. Also assume that for any two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty} b_n = 0$ and $\varphi(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $\lim_{n\longrightarrow\infty} a_n = 0$. Then f and g have a coincidence point $x \in X$ such that $p(g(x), g(x)) = 0$.

Moreover, if all the points of coincidence of f and g are comparable and f, g are weakly compatible, then f and g have a unique common fixed point.

Proof. By Property (C), $g(x_0) \preceq f(x_0)$ or $f(x_0) \preceq g(x_0)$. Without lose of generality, suppose $g(x_0) \preceq f(x_0)$ and choose $\{x_n\}_{n\in\mathbb{N}}$ in X such that $f(x_n) = g(x_{n+1})$ and

$$g(x_0) \preceq f(x_0) = g(x_1) \preceq f(x_1) = g(x_2) \preceq \cdots \preceq f(x_n) \preceq g(x_{n+1}),$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\{x_n\}_{n\in\mathbb{N}}$ contains a coincidence point x_j, $j \in \mathbb{N} \cup \{0\}$, of f and g, then $g(x_{j+1}) = f(x_j) = g(x_j)$. So $a_n = p(g(x_j), g(x_j)) = 0$. If not, then by cotractive condition

$$\varphi(p(f(x_j), f(x_j)), p(g(x_j), g(x_j))) > 0$$

with $a_n = p(g(x_j), g(x_j)) \neq 0$, $n \in \mathbb{N}$. By Definition 3.1 (i) and $f(x_j) = g(x_j)$, we have $\lim_{n\longrightarrow\infty} a_n = 0$ and then $p(g(x_j), g(x_j)) = 0$.

Now, assume that $\{x_n\}_{n\in\mathbb{N}}$ does not contain any coincidence point of f and g, that is $g(x_n) \neq f(x_n) = g(x_{n+1})$ for all $n \geqslant 0$. Then $a_n = p(g(x_n), g(x_{n+1})) > 0$ for all $n \geqslant 0$ and so by contraction condition, for all $n \geqslant 0$

$$\begin{aligned}
\varphi(a_{n+1}, a_n) &= \varphi(p(g(x_{n+1}), g(x_{n+2})), p(g(x_n), g(x_{n+1}))) \\
&= \varphi(p(f(x_n), f(x_{n+1})), p(g(x_n), g(x_{n+1}))) \\
&> 0.
\end{aligned}$$

Therefore $\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} p(g(x_n), g(x_{n+1})) = 0$.
But $\lim_{n\longrightarrow\infty} p(g(x_{n+1}), g(x_{n+1})) = 0$ and then

$$\lim_{n\longrightarrow\infty} p^s(g(x_n), g(x_{n+1})) = 0.$$

Claim. The sequence $\{g(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose not, then there exists subsequences $\{g(x_{m(k)})\}_{k\in\mathbb{N}}, \{g(x_{n(k)})\}_{k\in\mathbb{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that $k \leqslant n(k) < m(k)$ and

$$p^s(g(x_{n(k)}), g(x_{m(k)-1})) \leqslant \varepsilon_0 \leqslant p^s(g(x_{n(k)}), g(x_{m(k)})$$

for all $k \in \mathbb{N}$. But $\lim_{n\longrightarrow\infty} p^s(g(x_{n+1}), g(x_n)) = 0$, then

$$\lim_{k\longrightarrow\infty} p^s(g(x_{n(k)}), g(x_{m(k)})) = \lim_{n\longrightarrow\infty} p^s(g(x_{n(k)-1}), g(x_{m(k)-1})) = \varepsilon_0.$$

Suppose that $p(g(x_{n(k)-1}), g(x_{m(k)-1})) > 0$ for all $k \in \mathbb{N}$. By the contraction condition (1), for sequences $\{a_k\}_{k\in\mathbb{N}} = \{p(g(x_{n(k)}), g(x_{m(k)}))\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}} = \{p(g(x_{n(k)-1}), g(x_{m(k)-1}))\}_{k\in\mathbb{N}}$, we have

$$\varphi(a_k, b_k) \quad = \quad \varphi(p(g(x_{n(k)-1}), g(x_{m(k)})), p(g(x_{n(k)-1}), g(x_{m(k)-1}))) > 0$$

for all $k \in \mathbb{N}$. But for all $k \in \mathbb{N}$,

$$\varepsilon_0 < p(g(x_{n(k)}), g(x_{m(k)})) = a_k$$

then by Definition 3.1 (ii),

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = 0$$

and so $\varepsilon_0 = 0$, which is a contracdiction. Therefore $\{g(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in complete metric space (X, p^s). By closedness of $f(X)$ or $g(X)$, there exists $x \in X$ such that

$$\lim_{n\longrightarrow\infty} p^s(g(x_n), g(x)) = 0.$$

Using Lemma 2.4 and Lemma 2.2, then

$$0 \leqslant p(g(x), g(x)) \leqslant \liminf_{n\longrightarrow\infty} p(g(x_n), g(x_n)) \leqslant \lim_{n\longrightarrow\infty} p(g(x_n), g(x_n)) = 0.$$

Therefore $p(g(x), g(x)) = 0$ and this implies that

$$\lim_{n\longrightarrow\infty} p(g(x), g(x_n)) = 0.$$

At last, we show that x is a coincidence point of f and g.

If $\{g(x_n)\}_{n\in\mathbb{N}}$ has a subsequence $\{g(x_{n(k)})\}_{k\in\mathbb{N}}$ such that $g(x_{n(k)}) = f(x)$ for all $k \in \mathbb{N}$, then by uniqueness of the limit in (X, p^s), we have $f(x) = g(x)$. Otherwise, if there exists subsequence $\{g(x_{n(k)})\}_{k\in\mathbb{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that $g(x_{n(k)}) = g(x)$ for all $k \in \mathbb{N}$ and $g(x_{n(k_0)+1}) = g(x_{n(k_0)})$ for some $k_0 \in \mathbb{N}$, then $f(x_{n(k_0)}) = g(x_{n(k_0)})$.

If for all $k \in \mathbb{N}$, $g(x_{n(k)+1}) \neq g(x_{n(k)})$, then we can consider the sequence $\{g(x_n)\}_{n\in\mathbb{N}} \setminus \{g(x)\}_{n\in\mathbb{N}}$ instead of $\{g(x_n)\}_{n\in\mathbb{N}}$. Assume $g(x_n) \neq g(x)$ and $g(x_n) \neq f(x)$ for all $n \in \mathbb{N}$. Put $a_n = p(g(x_n), g(x))$ and $b_n = p(f(x_n), f(x))$ for all $n \in \mathbb{N}$. Clearly $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subseteq (0, \infty)$ and

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} p(g(x_n), g(x)) = 0.$$

By using Property (C), we have $g(x_n) \preceq g(x)$ for all $n \in \mathbb{N}$ and by contraction condition

$$\varphi(b_n, a_n) = \varphi(p(f(x_n), f(x)), p(g(x_n), g(x))) > 0.$$

Then by Definition 3.1 (ii), $\lim_{n\longrightarrow\infty} b_n = p(f(x_n), f(x)) = 0$. Therefore by partial metric property we have

$$\lim_{n\longrightarrow\infty} p^s(f(x_n), f(x)) = 0.$$

But $f(x_n) = g(x_{n+1})$ for all $n \in \mathbb{N}\cup\{0\}$, then $\lim_{n\longrightarrow\infty} p^s(g(x_n), f(x)) = 0$ and by uniqueness of the limit in the metric space (X, p^s), we have $f(x) = g(x)$.

Now assume all the points of coincidence of f and g are comparable and f and g are weakly compatible. Then for $y \in X$ with $f(y) = g(y)$ we have $g(y) = g(x)$. If not, then for all $n \in \mathbb{N}$, $a_n = p(g(y), g(x)) > 0$ and

$$\varphi(a_{n+1}, a_n) = \varphi(p(f(y), f(x)), p(g(y), g(x))) > 0.$$

Thus $\lim_{n\longrightarrow\infty} a_n = 0$ and $g(y) = g(x)$.

Finally, Lemma 2.7 implies that f and g have a unique common fixed point. $\square$

Following example illustrates Theorem 3.2.

Example 3.3. Let $X = \mathbb{R}^+$ with natural ordering $\leqslant$ and define the partial metric p on X by $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. So $(X,\leqslant,p)$ is an ordered partial metric space. Consider the R-function $\varphi : [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ defined by $\varphi(t,s) = s-2t$ for all $t,s \in \mathbb{R}$. Clearly $\varphi(t,s) \leqslant s-t$. Define two mappings $f,g : X \longrightarrow X$ by

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1 \\ \sqrt{x}, & x > 1. \end{cases}, \qquad g(x) = 3x.$$

Obviously, f,g are comparable on $\mathbb{R}^+$, mapping f is g-nondecreasing, $f(X) \subseteq g(X)$ and $f(X)$ is closed. For all $x \neq y$ in X, (except $x = 0$ or $y = 0$) $g(x)$ and $g(y)$ are comparable and the contraction condition

$$\varphi(p(f(x),f(y)),p(g(x),g(y))) > 0$$

holds. In fact, for $0 \leqslant x \leqslant 1$ with $x \geqslant y$ we have $\varphi(x,3x) = x > 0$ and for $x > 1$, we have $\varphi(\sqrt{x},3x) = 3x - \sqrt{x} > 0$. So all the conditions of Theorem 3.2 hold and f,g have a unique coincidence point. In fact, $f(0) = g(0) = 0$ and $p(f(0),g(0)) = 0$.

The fact that, for any R-function φ which satisfies the relation

$$\varphi(t,s) \leqslant s-t$$

for any $t,s \in [0,\infty)$ Theorem 3.2 holds, assures that Theorem 3.2 is an extension of Geraghty's fixed point theorem [4] to the coincidence point in the setting of ordered partial metric spaces.

Corollary 3.4. *Let f,g be two self mappings on an ordered complete partial metric space $(X,\preceq,p)$ and Prpperty (C) be fulfiled. Suppose that*

$$p(f(x),f(y)) \leqslant \psi(p(g(x),g(y))) \cdot p(g(x),g(y)),$$

for all comparable $g(x),g(y)$ with $g(x) \neq g(y)$, $x,y \in X$ and $\psi : [0,\infty) \longrightarrow [0,1)$ is a function with the property that $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq [0,\infty)$ whenever $\lim_{n\longrightarrow\infty} \psi(\alpha_n) = 1$. Then f and g have a coincidence point $x \in X$ such that $p(g(x),g(x)) = 0$.

Proof. Define $\varphi : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ by

$$\varphi(t,s) = \psi(s)s - t \qquad (t,s \in \mathbb{R}).$$

Clearly $\varphi(t,s) \leqslant s-t$ for all $t,s \in [0,\infty)$ and φ is a R-function. The desired result can be concluded by Theorem 3.2. $\square$

The following corollary is also valid whenever we define the function φ by $\varphi(t,s)=\psi(s)s-t$.

Corollary 3.4. *Let f,g be two self mappings on an ordered complete partial metric space $(X,\preceq,p)$ and Prpperty (C) be fulfiled. Suppose that*

$$p(f(x),f(y)) \leqslant \psi(p(g(x),g(y))) \cdot p(g(x),g(y)),$$

for all comparable $g(x),g(y)$ with $g(x)\neq g(y)$, $x,y\in X$ and $\psi:[0,\infty)\longrightarrow[0,1)$ is a function that $\limsup_{t\longrightarrow r^+}\psi(t)<1$ for all $r\in(0,\infty)$. Then f and g have a coincidence point $x\in X$ such that $p(g(x),g(x))=0$.

Proof. Define $\varphi:[0,\infty)\times[0,\infty)\longrightarrow\mathbb{R}$ by

$$\varphi(t,s)=\psi(s)s-t \qquad (t,s\in\mathbb{R}).$$

Clearly $\varphi(t,s) \leqslant s-t$ for all $t,s \in [0,\infty)$ and φ is a R-function. The desired result can be concluded by Theorem 3.2. $\square$

The following corollary is also valid whenever we define the function φ by $\varphi(t,s)=\psi(s)s-t$.

Corollary 3.5. *Let f,g be two self mappings on an ordered complete partial metric space $(X,\preceq,p)$ and Prpperty (C) be fulfiled. Suppose that*

$$p(f(x),f(y)) \leqslant \psi(p(g(x),g(y))) \cdot p(g(x),g(y)),$$

for all comparable $g(x),g(y)$ with $g(x)\neq g(y)$, $x,y\in X$ and $\psi:[0,\infty)\longrightarrow[0,1)$ is a function that $\limsup_{t\longrightarrow r^+}\psi(t)<1$ for all $r\in(0,\infty)$. Then f and g have a coincidence point $x\in X$ such that $p(g(x),g(x))=0$.

By considering the function $\psi:[0,\infty)\longrightarrow[0,1)$ which is a right continuous function and $\psi(t)>0$ for all $t\in(0,\infty)$, Corollary 3.5 again is valid.

4. An Application

In this section, by using Theorem 3.2, we prove the existence and the uniqueness of the solution of the system of integral equations

$$\begin{aligned} u(x) &= \int_a^b \lambda_1 k_1(x,t) F_1(t,u(t))dt \\ v(x) &= \int_a^b \lambda_2 k_2(x,t) F_2(t,v(t))dt \end{aligned} \tag{2}$$

in the space of real continous functions $X = C(I)$, $I = [a,b]$, where $x \in I$; $\lambda_i \in \mathbb{R}$; $k_i : I \times I \longrightarrow \mathbb{R}$, $F_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1,2$ and for $u \in C(I)$, $\|u\| = \sup_{t \in I} |u(t)|$. Consider $X = C(I)$ with the following order

$$u_1 \preceq u_2 \Longleftrightarrow u_1(t) \leqslant u_2(t) \qquad (t \in I).$$

The space (X,p) with $p(u_1,u_2) = \frac{1}{2}(\|u_1 - u_2\| + \|u_1\| + \|u_2\|)$ is a partial metric space. Consider the following assumptions on the system (2):

(1) For all $u \in X$, there exists $v \in X$ such that for all $x \in I$

$$\int_a^b \lambda_1 k_1(x,t) F_1(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t) F_2(t,v(t))dt$$

(2) For all $u_1, u_2 \in X$, if

$$\int_a^b \lambda_2 k_2(x,t) F_2(t,u_1(t))dt \leqslant \int_a^b \lambda_2 k_2(x,t) F_2(t,u_2(t))dt,$$

then

$$\int_a^b \lambda_1 k_1(x,t) F_1(t,u_1(t))dt \leqslant \int_a^b \lambda_1 k_1(x,t) F_1(t,u_2(t))dt.$$

(3) There exists $\alpha \in (0,1)$ such that $|\lambda_1| \leqslant \alpha |\lambda_1|$.

(4) For all $u_1, u_2 \in X$

$$(i) \quad |\int_a^b k_1(x,t)[F_1(t,u_1(t)) - F_1(t,u_2(t))]dt| \leqslant |\int_a^b k_2(x,t)[F_2(t,u_1(t)) - F_2(t,u_2(t))]dt|$$

$$(ii) \quad |\int_a^b k_1(x,t)F_1(t,u_i(t))dt| \leqslant |\int_a^b k_2(x,t)F_2(t,u_i(t))dt| \quad (i=1,2)$$

for all comparable $\int_a^b k_2(x,t)F_2(t,u_1(t))dt \neq \int_a^b k_2(x,t)F_2(t,u_2(t))dt$.

(5) If

$$\int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt,$$

then

$$\begin{aligned} \int_a^b \quad \lambda_1 \quad & k_1(x,t)F_1\left(t, \int_a^b \lambda_2 k_2(t,z)F_2(z,u(z))dz\right) dt \\ = \quad & \int_a^b \lambda_2 k_2(x,t)F_2\left(t, \int_a^b \lambda_1 k_1(t,z)F_1(z,u(z))dz\right) dt. \end{aligned}$$

By using the above assumptions, we show that Theorem 3.2 assures that the system (2) has a unique solution when $\varphi : [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ defined by $\varphi(t,s) = \alpha s - t$ is a R-function for $t, s \in [0,\infty)$ and $\alpha \in (0,1)$.

Define two self mappings f and g by

$$\begin{aligned} (f(u))(x) &= \int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt \\ (g(u))(x) &= \int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt. \end{aligned}$$

Let $w \in f(X)$ then $w(x) = (f(u))(x) = \int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt$. By (1), there exists $v \in X$ such that for all $x \in I$

$$\int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t)F_2(t,v(t))dt = (g(v))(x).$$

So $w = g$ and $f(X) \subseteq g(X)$.

On the other hand if $g(u) \preceq g(v)$, for $u, v \in X$, then on $(C(I), \preceq, p)$ we have

$$\int_a^b \lambda_2 k_2(x,t) F_2(t, u(t))dt = \int_a^b \lambda_2 k_2(x,t) F_2(t, v(t))dt$$

for all $x \in I$. By (2), $(f(u))(x) \leqslant (f(v))(x)$ for all $x \in I$ and $f(u) \preceq f(v)$, i.e. f is g-nondecreasing.

Note that for any $x \in I$ and $u, v \in X$,

$$\begin{aligned}
| \textstyle\int_a^b & \lambda_1 k_1(x,t)[F_1(t,u(t)) - F_1(t,v(t))]dt| + |\int_a^b \lambda_1 k_1(x,t) F_1(t,u(t))dt| \\
+ & \ |\int_a^b \lambda_1 k_1(x,t) F_1(t,v(t))dt| \\
\leqslant & \ \alpha|\lambda_2|(|\int_a^b k_1(x,t)[F_1(t,u(t)) - F_1(t,v(t))]dt| + |\int_a^b k_1(x,t)F_1(t,u(t))dt| \\
+ & \ |\int_a^b k_1(x,t)F_1(t,v(t))dt|) \\
\leqslant & \ \alpha\, |\int_a^b \lambda_2 k_2(x,t)[F_2(t,u(t)) - F_2(t,v(t))]dt| + \alpha\, |\int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt| \\
+ & \ \alpha\, |\int_a^b \lambda_2 k_2(x,t)F_2(t,v(t))dt| \\
\leqslant & \ \alpha\left(\|g(u) - g(v)\| + \|g(u)\| + \|g(v)\right).
\end{aligned}$$

So the contraction condition (1) holds, i.e.,

$$\|f(u) - f(v)\| + \|f(u)\| + \|f(v)\| \leqslant \alpha\left(\|f(u) - f(v)\| + \|f(u)\| + \|f(v)\right).$$

Therefore all assumptions of Theorem 3.2 are fulfilled and so f, g have coincidence point. Suppose that $f(u) = g(u)$ or equivalently

$$\int_a^b \lambda_1 k_1(x,t) F_1(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t) F_2(t,u(t))dt.$$

Then by condition (5),

$$\begin{aligned}
&\int_a^b \lambda_1 k_1(x,t) F_1\left(t, \int_a^b \lambda_2 k_2(t,z) F_2(z,u(z))dz\right) dt \\
&= \int_a^b \lambda_2 k_2(x,t) F_2\left(t, \int_a^b \lambda_1 k_1(t,z) F_1(z,u(z))dz\right) dt.
\end{aligned}$$

This implies that f and g are weakly compatible and they have a unique fixed point. In others words the system (2) has a unique solution.

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