On the Coincidence Point in Ordered Partial Metric Spaces

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Abstract. The aim of this paper is to given the coincidence and common fixed point of two mappings via R-functin in the setting of ordered partial metric spaces. Result is supported by given an example. An application to the integral equation is also provided..

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1 Introduction

In 1976, the notion of coincidence and common fixed point of commuting mappings are introduced by G. Jungck [?]. Several authors have contributed to the development of the existence and uniqueness of coincidence points of operators in different spaces [?, ?, ?, ?, ?]. Khojaste et. al [?], introduced simulation function and new contraction depending simulation function. Recently, Roldan et. al [?], modified this concept and proved the existence and uniqueness of coincidence points of two operators in the setting of complete metric spaces.

On the other hand, in 1992, G. Mathews [?] introduced the notion of the partial metric which is a generalization of the metric and it can be applied to study of denotational semantics of data for network. In [?], A. Nastasi et. al proved the existence and uniqueness of fixed points by using R-functions and lower semi-continuous functions in the setting of metric spaces and partial metric spaces.

In this paper, inspired by [?, ?, ?] we deduce some coincidence point results in the setting of ordered partial metric spaces by using R-functions. An example is given to support the result. Section 4 is devoted to an application to integral equations.

2 Preliminaries

We start by recalling some definitions and properties of partial metric spaces which will be needed during the paper.

Definition 2.1 *[?], A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$;*
(i) $p(x, x) = p(x, y) = p(y, y) \Leftrightarrow x = y$.
(ii) $p(x, x) \leq p(x, y)$.
(iii) $p(x, y) = p(y, x)$.
(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. Clearly, a metric p on a set X is a partial metric such that $p(x, x) = 0$ for all $x \in X$.

Each partial metric p on X generates a T_0-topology τ_p on X which has as a base, the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

The following properties will be obtained from the topology τ_p on the partial metric space (X, p).

(i) (X, τ_p) is first countable.

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$.

(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
Every partial metric p on X, induces a metric $p^s : X \times X \longrightarrow \mathbb{R}^+$ defined by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, such that $\tau(p)$ is finer than $\tau(p^s)$ [**?**].

To see some examples of partial metric spaces refer to [**?**, **?**].

Lemma 2.2 *[**?**] A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\longrightarrow\infty} p^s(a, x_n) = 0$ if and only if $p(a, a) = \lim_{n\longrightarrow\infty} p(a, x_n) = \lim_{n,m\longrightarrow\infty} p(x_n, x_m)$.*

Lemma 2.3 *[**?**] Let (X, p) be a partial metric space. Then the following hold:*
(i) If $p(x, y) = 0$, then $x = y$.
(ii) If $x \neq y$, then $p(x, y) > 0$ and $p(y, x) > 0$.

Lemma 2.4 *[**?**] Let (X, p) be a partial metric space and let $\lambda : X \longrightarrow [0, \infty)$ be defined by $\lambda(x) = p(x, x)$ for all $x \in X$. Then the function λ is continuous in the metric space (X, p^s).*

Recently, fixed point theory has developed in metric spaces and partial metric spaces endowed with a partial ordering [**?**, **?**].

Definition 2.5 *Let X be a nonempty set. Then $(X, \preceq, p)$ is called an ordered partial metric space if $(X, \preceq)$ is a partially ordered set, and (X, p) is a partial metric space.*

Two elements x and y of X are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.6 *[**?**] Two self mappings f and g on a set X have a coincidence point, say x, if $y = f(x) = g(x)$ and y is called a point of coincidence of f and g. Also f and g are said to be weakly compatible if $f(g(x)) = g(f(x))$ whenever $f(x) = g(x)$*

Lemma 2.7 *[**?**] Let X be a nonempty set and the mappings $f, g : X \longrightarrow X$ have a unique point of coincidence y in X. If f and g are weakly compatible, then f and g have a unique common fixed point.*

Definition 2.8 *[**?**] Let $(X, \preceq)$ be a partially ordered set and $f, g : X \longrightarrow X$. Then f is said to be g-nondecreasing if for $x, y \in X$,*

$$g(x) \preceq g(y) \Longrightarrow f(x) \preceq f(y).$$

3 Main Results

We begin this section by giving the concept of R-function (see [**?**]).

Definition 3.1 *A function $\varphi : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is called $R-$function if the following conditions hold:*
(i) for each sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ with $\varphi(a_{n+1}, a_n) > 0$, for all $n \in \mathbb{N}$, then $\lim_{n\longrightarrow\infty} a_n = 0$;
(ii) for every two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $(0,\infty)$ converging to the same limit $L \geq 0$, then $L = 0$ whenever $L < a_n$ and $\varphi(a_n, b_n) > 0$ for all $n \in \mathbb{N}$.

In the sequel $(X, \preceq, p)$ is an ordered partial metric space where $(X, \preceq)$ is a partially ordered set and (X, p) is a partial metric space.

In the main result, we suppose that the following property holds.

Property (C). If $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a nondecreasing (noncreasing) sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ $(x \preceq x_n)$ for all $n \in \mathbb{N}$. Also, assume that f and g are two self mappings on X such that f, g are comparable at some $x_0 \in X$ and f is g-nondecreasing, $f(X) \subseteq g(X)$ and one of the sets $f(X)$ or $g(X)$ is closed.

Theorem 3.2 *Let f, g be two self mappings on an ordered complete partial metric space $(X, \preceq, p)$ and the Property (C) be fulfilled. Suppose that f satisfying*

$$\varphi(p(f(x), f(y)), p(g(x), g(y))) > 0, \tag{1}$$

for all comparable $g(x), g(y)$ with $g(x) \neq g(y)$, $x, y \in X$ and some R-function φ. Also assume that for any two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $(0,\infty)$ such that $\lim_{n\longrightarrow\infty} b_n = 0$ and $\varphi(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $\lim_{n\longrightarrow\infty} a_n = 0$. Then f and g have a coincidence point $x \in X$ such that $p(g(x), g(x)) = 0$. Moreover, if all the points of coincidence of f and g are comparable and f, g are weakly compatible, then f and g have a unique common fixed point.

Proof: By Property (C), $g(x_0) \preceq f(x_0)$ or $f(x_0) \preceq g(x_0)$. Without lose of generality, suppose $g(x_0) \preceq f(x_0)$ and choose $\{x_n\}_{n\in\mathbb{N}}$ in X such that $f(x_n) = g(x_{n+1})$ and

$$g(x_0) \preceq f(x_0) = g(x_1) \preceq f(x_1) = g(x_2) \preceq \cdots \preceq f(x_n) \preceq g(x_{n+1}),$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\{x_n\}_{n\in\mathbb{N}}$ contains a coincidence point x_j, $j \in \mathbb{N} \cup \{0\}$, of f and g, then $g(x_{j+1}) = f(x_j) = g(x_j)$. So $a_n = p(g(x_j), g(x_j)) = 0$. If not, then by cotractive condition

$$\varphi(p(f(x_j), f(x_j)), p(g(x_j), g(x_j))) > 0$$

with $a_n = p(g(x_j), g(x_j)) = 0$, $n \in \mathbb{N}$, Definition **??**(i) and $f(x_j) = g(x_j)$, we have $\lim_{n\longrightarrow\infty} a_n = 0$ and then $p(g(x_j), g(x_j)) = 0$.

Now, assume that $\{x_n\}_{n\in\mathbb{N}}$ does not contain any coincidence point of f and g, that is $g(x_n) \neq f(x_n) = g(x_{n+1})$ for all $n \geq 0$. Then $a_n = p(g(x_n), g(x_{n+1})) > 0$ for all $n \geq 0$ and so by contraction condition, for all $n \geq 0$

$$\begin{aligned}
\varphi(a_{n+1}, a_n) &= \varphi(p(g(x_{n+1}), g(x_{n+2})), p(g(x_n), g(x_{n+1}))) \\
&= \varphi(p(f(x_n), f(x_{n+1})), p(g(x_n), g(x_{n+1}))) \\
&> 0.
\end{aligned}$$

Therefore $\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} p(g(x_n), g(x_{n+1})) = 0$. But $\lim_{n\longrightarrow\infty} p(g(x_{n+1}), g(x_{n+1})) = 0$ and then

$$\lim_{n\longrightarrow\infty} p^s(g(x_n), g(x_{n+1})) = 0.$$

Claim. The sequence $\{g(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose not, then there exists subsequences $\{g(x_{m(k)})\}_{k\in\mathbb{N}}, \{g(x_{n(k)})\}_{k\in\mathbb{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that $k \le n(k) < m(k)$ and

$$p^s(g(x_{n(k)}), g(x_{m(k)-1}) \le \varepsilon_0 \le p^s(g(x_{n(k)}), g(x_{m(k)})$$

for all $k \in \mathbb{N}$. But $\lim_{n\longrightarrow\infty} p^s(g(x_{n+1}), g(x_n)) = 0$, then

$$\lim_{k\longrightarrow\infty} p^s(g(x_{n(k)}), g(x_{m(k)})) = \lim_{n\longrightarrow\infty} p^s(g(x_{n(k)-1}), g(x_{m(k)-1})) = \varepsilon_0.$$

Suppose that $p(g(x_{n(k)-1}), g(x_{m(k)-1})) > 0$ for all $k \in \mathbb{N}$. By the contraction condition (**??**), for sequences $\{a_k\}_{k\in\mathbb{N}} = \{p(g(x_{n(k)}), g(x_{m(k)}))\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}} = \{p(g(x_{n(k)-1}), g(x_{m(k)-1}))\}_{k\in\mathbb{N}}$, we have

$$\varphi(a_k, b_k) = \varphi(p(g(x_{n(k)-1}), g(x_{m(k)})), p(g(x_{n(k)-1}), g(x_{m(k)-1}))) > 0$$

for all $k \in \mathbb{N}$. But for all $k \in \mathbb{N}$,

$$\varepsilon_0 < p(g(x_{n(k)}), g(x_{m(k)})) = a_k$$

then by Definition **??**(ii),

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = 0$$

and so $\varepsilon_0 = 0$, which is a cotracdiction. Therefore $\{g(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in complete metric space (X, p^s). By closedness of $f(X)$ or $g(X)$, there exists $x \in X$ such that

$$\lim_{n\longrightarrow\infty} p^s(g(x_n), g(x)) = 0.$$

Using Lemma **??** and Lemma **??**, then

$$0 \le p(g(x), g(x)) \le \liminf_{n\longrightarrow\infty} p(g(x_n), g(x_n)) \le \lim_{n\longrightarrow\infty} p(g(x_n), g(x_n)) = 0.$$

Therefore $p(g(x), g(x)) = 0$ and this implies that

$$\lim_{n\longrightarrow\infty} p(g(x), g(x_n)) = 0.$$

At last, we show that x is a coincidence point of f and g.

If $\{g(x_n)\}_{n\in\mathbb{N}}$ has a subsequence $\{g(x_{n(k)})\}_{k\in\mathbb{N}}$ such that $g(x_{n(k)}) = f(x)$ for all $k \in \mathbb{N}$. Then by uniquness of the limit in (X, p^s), we have $f(x) = g(x)$. Otherwise, if there exists subsequence $\{g(x_{n(k)})\}_{k\in\mathbb{N}}$ of $\{g(x_n)\}_{n\in\mathbb{N}}$ such that $g(x_{n(k)}) = g(x)$ for all $k \in \mathbb{N}$ and $g(x_{n(k_0)} + 1) = g(x_{n(k_0)})$ for some $k_0 \in \mathbb{N}$, then $f(x_{n(k_0)}) = g(x_{n(k_0)})$.

If for all $k \in \mathbb{N}$, $g(x_{n(k)+1}) \neq g(x_{n(k)})$, then we can consider the sequence $\{g(x_n)\}_{n\in\mathbb{N}} \setminus \{g(x)\}_{n\in\mathbb{N}}$ insted of $\{g(x_n)\}_{n\in\mathbb{N}}$. Assume $g(x_n) \neq g(x)$ and $g(x_n) \neq f(x)$ for all $n \in \mathbb{N}$. Put $a_n = p(g(x_n), g(x))$ and $b_n = p(f(x_n), f(x))$ for all $n \in \mathbb{N}$. Clearly $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subseteq (0, \infty)$ and

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} p(g(x_n), g(x)) = 0.$$

By using Property (C), we have $g(x_n) \preceq g(x)$ for all $n \in \mathbb{N}$ and by contraction condition

$$\varphi(b_n, a_n) = \varphi(p(f(x_n), f(x)), p(g(x_n), g(x))) > 0.$$

Then by Definition **??**(ii), $\lim_{n\longrightarrow\infty} b_n = p(f(x_n), f(x)) = 0$. Therefore by partial metric property we have

$$\lim_{n\longrightarrow\infty} p^s(f(x_n), f(x)) = 0.$$

But $f(x_n) = g(x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, then $\lim_{n \longrightarrow \infty} p^s(g(x_n), f(x)) = 0$ and by uniqueness of the limit in the metric space (X, p^s), we have $f(x) = g(x)$.

Now assume all the points of coincidence of f and g are comparable and f and g are weakly compatible. Then for $y \in X$ with $f(y) = g(y)$ we have $g(y) = g(x)$. If not, then for all $n \in \mathbb{N}$ and $a_n = p(g(y), g(x)) > 0$ and

$$\varphi(a_{n+1}, a_n) = \varphi(p(f(y), f(x)), p(g(y), g(x))) > 0.$$

Thus $\lim_{n \longrightarrow \infty} a_n = 0$ and $g(y) = g(x)$.

Finally, Lemma **??** implies that f and g have a unique common fixed point. $\square$

Following example illustrates Theorem **??**.

Example 3.3 *Let $X = \mathbb{R}^+$ with natural ordering $'' \leq ''$ and define the partial metric p on X by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. So $(X, \leq, p)$ is an ordere partial metric space. Consider the R-function $\varphi : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ defined by $\varphi(t, s) = s - 2t$ for all $t, s \in \mathbb{R}$. Clearly $\varphi(t, s) \leq s - t$. Define two mappings $f, g : X \longrightarrow X$ by*

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ \sqrt{x}, & x > 1. \end{cases}, \qquad g(x) = 3x.$$

*Obviously, f, g are comparable on $\mathbb{R}^+$, mapping f is g-nondecreasing, $f(X) \subseteq g(X)$ and $f(X)$ is closed. For all $x \neq y$ in X, (except $x = 0$ or $y = 0$) $g(x)$ and $g(y)$ are comparable and the contraction condition $\varphi(p(f(x), f(y)), p(g(x), g(y))) > 0$ holds. In fact, for $0 \leq x \leq 1$ with $x \geq y$ we have $\varphi(x, 3x) = x > 0$ and for $x > 1$, we have $\varphi(\sqrt{x}, 3x) = 3x - \sqrt{x} > 0$. So all the conditions of Theorem **??** hold and f, g have a unique coincidence point. In fact, $f(0) = g(0) = 0$ and $p(f(0), g(0)) = 0$.*

The fact that, for any R-function φ which satisfies the relation

$$\varphi(t, s) \leq s - t$$

for any $t, s \in [0, \infty)$ Theorem **??** holds, assures that Theorem **??** is an extension of Geraghty's fixed point theorem [**?**] to the coincidence point in the setting of ordered partial metric spaces.

Corollary 3.4 *Let f, g be two self mappings on an ordered complete partial metric space $(X, \preceq, p)$ and Prpperty (C) be fulfiled. Suppose that*

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))) \cdot p(g(x), g(y)),$$

for all comparable $g(x), g(y)$ with $g(x) \neq g(y)$, $x, y \in X$ and $\psi : [0, \infty) \longrightarrow [0, 1)$ is a function with the property that $\lim_{n \longrightarrow \infty} \alpha_n = 0$, $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq [0, \infty)$ whenever $\lim_{n \longrightarrow \infty} \psi(\alpha_n) = 1$. Then f and g have a coincidence point $x \in X$ such that $p(g(x), g(x)) = 0$.

Proof : Define $\varphi : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ by

$$\varphi(t, s) = \psi(s)s - t \qquad (t, s \in \mathbb{R}).$$

Clearly $\varphi(t, s) \leq s - t$ for all $t, s \in [0, \infty)$ and φ is a R-function. The desired result can be concluded by Theorem **??**. $\square$

The following corollary is also valid whenever we define the function φ by $\varphi(t, s) = \psi(s)s - t$.

Corollary 3.5 *Let f, g be two self mappings on an ordered complete partial metric space $(X, \preceq, p)$ and Prpperty (C) be fulfiled. Suppose that*

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))) \cdot p(g(x), g(y)),$$

for all comparable $g(x), g(y)$ with $g(x) \neq g(y)$, $x, y \in X$ and $\psi : [0, \infty) \longrightarrow [0, 1)$ is a function that $\limsup_{t \longrightarrow r^+} \psi(t) < 1$ for all $r \in (0, \infty)$. Then f and g have a coincidence point $x \in X$ such that $p(g(x), g(x)) = 0$.

By considering the function $\psi : [0, \infty) \longrightarrow [0, 1)$ which is a right continuous function and $\psi(t) > 0$ for all $t \in (0, \infty)$, Corollary **??** again is valid.

4 An application

In this section, by using Theorem **??**, we prove the existence and unique solution of the system of integral equations

$$\begin{aligned} u(x) &= \int_a^b \lambda_1 k_1(x,t) F_1(t, u(t)) dt \\ v(x) &= \int_a^b \lambda_2 k_2(x,t) F_2(t, v(t)) dt \end{aligned} \tag{2}$$

in the space of real continous functions $X = C(I)$, $I = [a, b]$, where $x \in I$; $\lambda_i \in \mathbb{R}$; $k_i : I \times I \longrightarrow \mathbb{R}$, $F_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, 2$ and for $u \in C(I)$, $\|u\| = \sup_{t \in I} |u(t)|$. Endow $X = C(I)$ with the following order

$$u_1 \preceq u_2 \Longleftrightarrow u_1(t) \leq u_2(t) \qquad (t \in I).$$

The space (X, p) with $p(u_1, u_2) = \frac{1}{2}(\|u_1 - u_2\| + \|u_1\| + \|u_2\|)$ is a partial metric space. Consider the following assumptions on the system (**??**):

(1) For all $u \in X$, there exists $v \in X$ such that for all $x \in I$

$$\int_a^b \lambda_1 k_1(x,t) F_1(t, u(t)) dt = \int_a^b \lambda_2 k_2(x,t) F_2(t, v(t)) dt$$

(2) For all $u_1, u_2 \in X$, if

$$\int_a^b \lambda_2 k_2(x,t) F_2(t, u_1(t)) dt \leq \int_a^b \lambda_2 k_2(x,t) F_2(t, u_2(t)) dt,$$

then

$$\int_a^b \lambda_1 k_1(x,t) F_1(t, u_1(t)) dt \leq \int_a^b \lambda_1 k_1(x,t) F_1(t, u_2(t)) dt.$$

(3) There exists $\alpha \in (0, 1)$ such that $|\lambda_1| \leq \alpha |\lambda_1|$.

(4) For all $u_1, u_2 \in X$

$$(i) \quad \left| \int_a^b k_1(x,t)[F_1(t, u_1(t)) - F_1(t, u_2(t))] dt \right| \leq \left| \int_a^b k_2(x,t)[F_2(t, u_1(t)) - F_2(t, u_2(t))] dt \right|$$

$$(ii) \quad \left| \int_a^b k_1(x,t) F_1(t, u_i(t)) dt \right| \leq \left| \int_a^b k_2(x,t) F_2(t, u_i(t)) dt \right| \quad (i = 1, 2)$$

for all comparable $\int_a^b k_2(x,t) F_2(t, u_1(t)) dt \neq \int_a^b k_2(x,t) F_2(t, u_2(t)) dt$.

(5) If

$$\int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt,$$

then

$$\begin{aligned}\int_a^b \lambda_1 \quad & k_1(x,t)F_1\left(t, \int_a^b \lambda_2 k_2(t,z)F_2(z,u(z))dz\right)dt \\ &= \int_a^b \lambda_2 k_2(x,t)F_2\left(t, \int_a^b \lambda_1 k_1(t,z)F_1(z,u(z))dz\right)dt.\end{aligned}$$

By using the above assumptions, we show that Theorem **??** assures that the system (**??**) has a unique solution when $\varphi : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ defined by $\varphi(t,s) = \alpha s - t$ is a R-function for $t,s \in [0,\infty)$ and $\alpha \in (0,1)$.

Define two self mappings f and g by

$$\begin{aligned}(f(u))(x) &= \int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt \\ (g(u))(x) &= \int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt.\end{aligned}$$

Let $w \in f(X)$ then $w(x) = (f(u))(x) = \int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt$. By (1), there exists $v \in X$ such that for all $x \in I$

$$\int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t)F_2(t,v(t))dt = (g(v))(x).$$

So $w = g$ and $f(X) \subseteq g(X)$.

On the other hand if $g(u) \preceq g(v)$, for $u,v \in X$, then on $(C(I),\preceq,p)$ we have

$$\int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt = \int_a^b \lambda_2 k_2(x,t)F_2(t,v(t))dt$$

for all $x \in I$. By (2), $(f(u))(x) \leq (f(v))(x)$ for all $x \in I$ and $f(u) \preceq f(v)$, i.e. f is g-nondecreasing.

Note that for any $x \in I$ and $u,v \in X$,

$$\begin{aligned}| \int_a^b \quad & \lambda_1 k_1(x,t)[F_1(t,u(t)) - F_1(t,v(t))]dt| + |\int_a^b \lambda_1 k_1(x,t)F_1(t,u(t))dt| \\ + \quad & |\int_a^b \lambda_1 k_1(x,t)F_1(t,v(t))dt| \\ \leq \quad & \alpha|\lambda_2|(|\int_a^b k_1(x,t)[F_1(t,u(t)) - F_1(t,v(t))]dt| + |\int_a^b k_1(x,t)F_1(t,u(t))dt| \\ + \quad & |\int_a^b k_1(x,t)F_1(t,v(t))dt|) \\ \leq \quad & \alpha\,|\int_a^b \lambda_2 k_2(x,t)[F_2(t,u(t)) - F_2(t,v(t))]dt| + \alpha\,|\int_a^b \lambda_2 k_2(x,t)F_2(t,u(t))dt| \\ + \quad & \alpha\,|\int_a^b \lambda_2 k_2(x,t)F_2(t,v(t))dt| \\ \leq \quad & \alpha\left(\|g(u) - g(v)\| + \|g(u)\| + \|g(v)\right).\end{aligned}$$

So the contraction condition (**??**) holds, i.e.,

$$\|f(u) - f(v)\| + \|f(u)\| + \|f(v)\| \leq \alpha\left(\|f(u) - f(v)\| + \|f(u)\| + \|f(v)\right).$$

Therefore all assumptions of Theorem **??** are fulfilled and so f, g have coincidence point. Suppose that $f(u) = g(u)$ or equivalently

$$\int_a^b \lambda_1 k_1(x,t) F_1(t, u(t)) dt = \int_a^b \lambda_2 k_2(x,t) F_2(t, u(t)) dt.$$

Then by condition (5),

$$\int_a^b \lambda_1 k_1(x,t) F_1\left(t, \int_a^b \lambda_2 k_2(t,z) F_2(z, u(z)) dz\right) dt$$
$$= \int_a^b \lambda_2 k_2(x,t) F_2\left(t, \int_a^b \lambda_1 k_1(t,z) F_1(z, u(z)) dz\right) dt.$$

This implies that f and g are weakly compatible and they have unique fixed point. In others words the system (**??**) has a unique solution.

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