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Limit Summability and Gamma Type Functions of Order Two

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Abstract. Gamma type functions satisfying the difference functional equations $f(x+1) = g(x)f(x)$ and limit summability of functions were studied and introduced by R.J. Webster and M.H. Hooshmand, respectively. It is shown that the topic of gamma type functions can be considered as a subtopic of limit summability. Indeed, if $\ln f$ is limit summable, then its limit summand function $(\ln f)_\sigma$ satisfies $(\ln f)_\sigma(x) = \ln f(x) + (\ln f)_\sigma(x-1)$ and $e^{(\ln f)_\sigma(x)}$ is gamma type function of $f(x+1)$. In this paper, we introduce and study limit summability of order two, 2-limit summand function f_{σ^2} and its results as gamma type functions of order two and also limit summand of multipliers. Finally, as an application of the study, we obtain a criteria for existence of gamma type function of the function $f(x)^x$ and give some related examples and corollaries.

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1. Introduction and Preliminaries

Gamma type functions satisfying the functional equation $f(x+1) = g(x)f(x)$ were studied by Webster in 1997. Since in the special case

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$f(x) = x$, the gamma type function is the same $\Gamma(x)$, through his study [11] some generalizations of the Bohr-Mollerup Theorem (see [3]) are obtained. On the other hand, in order to study ultra exponential functions, Hooshmand in 2001 is directed toward a topic which he called "limit summability of functions". In [5] it is shown that the topic of gamma type functions can be considered as a subtopic of limit summability and its relations are explained. The limit summand function of a real or complex function f (introduced in [5]) satisfies the difference functional equation $F(x) - F(x-1) = f(x)$. Limit summability of functions was extended in [6]. In 2010, Muller and Schleicher introduced the concept of fractional sums and euler-like identities in [9]. In fact, they arrived at the functional sequence $f_{\sigma_n}(x)$ introduced by Hooshmand (of course in the special case $\sigma = 0$) while they were not aware of the limit summability topic, even though they did not notice at theorems or conditions of convergence of the functional sequence. In this paper, limit summability of order two is desired. Let us present a summary of limit summability of functions and state motivation of the topic.

Recall from [5, 6] some basic definitions and properties of limit summability (of order one). For a real or complex function f with domain D_f we set

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\},$$

where $\mathbb{N}^*$ is the set of posetive integers and $\mathbb{N} = \{0\} \cup \mathbb{N}^*$. It is easy to see that $\Sigma_f = \bigcap_{k=1}^{\infty} D_f - k$, $\Sigma_f \cap D_f = \Sigma_f + 1$ and

$$D_f \subseteq \Sigma_f \Longleftrightarrow D_f \subseteq D_f - 1 \Longleftrightarrow \Sigma_f = D_f - 1 \tag{1}$$

If $\mathbb{N}^* \subseteq D_f$, then for any positive integer n and $x \in \Sigma_f$ we set

$$R_n(f,x) = R_n(x) = f(n) - f(x+n)\ ,\ f_{\sigma_n}(x) = xf(n) + \sum_{k=1}^{n} R_k(x).$$

Definition 1.1. *The function f is called limit summable at $x_0 \in \Sigma_f$ if the sequence $\{f_{\sigma_n}(x_0)\}$ is convergent. The function f is called limit summable on the set $S \subseteq \Sigma_f$ if it is limit summable at all points of S. Now we set*

$$D_{f_\sigma} = \{x \in \Sigma_f | f \textit{ is summable at } x\},$$

Also the function $f_\sigma(x)$ is the same limit function f_{σ_n} with domain D_{f_σ} and it is referred to as the limit summand function of f. If $x \in D_f$, we may use the notation $\sigma_n(f(x))$ instead of $\sigma_n(f,x)$.

It is proved in [5] that if $R_n(f,1)$ is convergent then $D_{f_\sigma} \cap D_f = D_{f_\sigma}+1$ which is similar to the identity $\Sigma_f \cap D_f = \Sigma_f + 1$.

Note that in general $\{0\} \subseteq D_{f_\sigma} \subseteq \Sigma_f$, $f_\sigma(0) = 0$ and if $0 \in D_f$ then $\{-1,0\} \subseteq D_{f_\sigma} \subseteq \Sigma_f$, $f_\sigma(-1) = -f(0)$. But $1 \in D_{f_\sigma}$ if and only if $R_n(f,1)$ is convergent. Now, if $D_{f_\sigma} = \Sigma_f$ (i.e., D_{f_σ} takes the maximum own amount), then f is called "weak limit summable". As it is explained in [5], a necessary condition for the summability of f at x is

$$\lim_{n\to\infty} \overline{R}_n(f,x) := \lim_{n\to\infty} R_n(f,x) - xR_{n-1}(f,1) = 0.$$

Also, if $1 \in D_{f_\sigma}$, then the functional sequence $R_n(f,x)$ is convergent on D_{f_σ} and $R(f,x) := \lim_{n\to\infty} R_n(f,x) = R(f,1)x$ (for all $x \in D_{f_\sigma}$), and

$$f_\sigma(x) = f(x) + f_\sigma(x-1) + R(1)x \quad ; \quad x \in D_{f_\sigma}+1 \tag{2}$$

So if $R(1) = 0$, then

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{j=1}^{m} f(j) \quad ; \quad m \in \mathbb{N}^*. \tag{3}$$

It is proved that the followings are equivalent:
(a) $D_f \subseteq D_{f_\sigma}$, $R(f,1) = 0$;
(b) $\Sigma_f = D_{f_\sigma}$, $D_f \subseteq D_f - 1$, $R(f,1) = 0$;
(c) f satisfies the functional equation (e.g., see [4])

$$f_\sigma(x) = f(x) + f_\sigma(x-1) \quad ; \quad x \in D_f. \tag{4}$$

Every function satisfying the above equivalent conditions is called "limit summable". Hence, if f is limit summable then $D_{f_\sigma} = D_f - 1 = \Sigma_f$.

Remark 1.2. *Since most of the functions which are used in this topic are defined on $[M,+\infty)$ or $(M,+\infty)$ where $M \leqslant 1$ is a fixed real number. We note that for these functions, the initial condition $D_f \subseteq D_f - 1$ (or equivalently $D_f \subseteq \Sigma_f$) holds, indeed, if $D_f = [M,+\infty)$ then $\Sigma_f =$*

$[M-1,+\infty)$. *Hence, these functions are limit summable if and only if* $f_n - f_{n-1} \to 0$ *and* $D_{f_\sigma} = [M-1,+\infty)$ *(i.e., the functional sequence* $f_{\sigma_n}(x)$ *is convergent at all defined points), so we have*

$$f_\sigma(x) = f(x) + f_\sigma(x-1) \quad ; \quad x \geqslant M-1.$$

Example 1.3. The real functions $f(x) = x^{\frac{-3}{2}}$ and $g(x) = \frac{1}{x} + \ln(x)$ are limit summable with $D_f = D_g = (0,+\infty)$, $\Sigma_f = \Sigma_g = (-1,+\infty) = D_{f_\sigma} = D_{g_\sigma}$, and

$$f_\sigma(x) = \zeta(\frac{3}{2}) - \zeta(\frac{3}{2}, x+1) \quad ; \quad x \geqslant -1$$

$$g_\sigma(x) = \psi(x+1) + \gamma + \ln\Gamma(x+1) \quad ; \quad x \geqslant -1$$

where $\gamma = 0.577215664901532...$ denotes the Euler-Mascheroni constant, ψ di-gamma function, $\zeta(s)$ and $\zeta(s,a)$ are Riemann and Hurwits zeta functions, respectively (see [2, 8]).

In [5], the same connections between limit summand and gamma type functions (if there exist) are stated. Also, it is shown that gamma type functions can be considered as a subtopic of limit summability. A main theorem in [11] states existence of gamma type functions as limit function of the following functional sequence f_n^*.

Theorem 1.4. *Let the function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *have the property that for each* $w > 0$, $\lim_{x\to\infty} \frac{f(x+w)}{f(x)} = 1$. *Suppose that* $F : \mathbb{R}^+ \to \mathbb{R}^+$ *is an eventually log-convex function satisfying the functional equation* $F(x+1) = f(x)F(x)$ *for* $x > 0$ *and the initial condition* $F(1) = 1$. *Then* F *is uniquely determined by* f *through the equation* $F(x) = \lim_{n\to\infty} f_n^*(x)$ *where*

$$f_n^*(x) = \frac{f(n)...f(1)f^x(n)}{f(n+x)...f(x)} \quad ; \quad x > 0 \tag{5}$$

Proof. See Theorem 3.1 of [11]. □

The limit function f^* defined by the following equation is called "gamma type function of f"

$$f^*(x) := \lim_{n\to\infty} f_n^*(x) = \lim_{n\to\infty} \frac{f(n)...f(1)f(n)^x}{f(n+x)...f(x)} \quad ; \quad x > 0. \tag{6}$$

Hence, the above theorem shows that the unique solution is the same gamma type function f^* respect to f. It is proved that Theorem 1.4 is a result of Corollary 3.3 of [5] for the special case $M = 0$. Moreover, a main relation between gamma type function of $f : \mathbb{R}^+ \to \mathbb{R}^+$ and limit summand function of $\ln f$ is proved as follow

$$f^*(x+1) = e^{(\ln f)_\sigma(x)} \quad ; \quad x > 0$$

Indeed, By using the identity $f_n^*(x+1) = \frac{f(n)}{f(n+x+1)} f(x) f_n^*(x)$ (for every function $f : \mathbb{R}^+ \to \mathbb{R}^+$) we have

$$f_n^*(x) = \frac{1}{f(x)} e^{(\ln f)_{\sigma_n}(x)} \ , \ f_n^*(x+1) = \frac{f(n)}{f(n+x+1)} e^{(\ln f)_{\sigma_n}(x)} \ ; x > 0. \quad (7)$$

Thus, the limit summability of $\ln f$ is equivalent to the existence of gamma type function of f. Therefore, we can restate the gamma type function of f as a definition.

Definition 1.5. *We say a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ has gamma type function if the functional sequence $f_n^*(x)$ (defined by (5)) is convergent on $\mathbb{R}^+$, and we call f^* (limit function of f_n^*) gamma type function of f.*

If f has gamma type function, then f satisfies the following functional equation

$$f^*(x+1) = f(x) f^*(x) \quad ; \quad x > 0. \quad (8)$$

Example 1.6. The real functions $f(x) = x$ and $g(x) = 1 + \frac{1}{x}$ have gamma type functions on $(0, +\infty)$ as follow

$$f^*(x) = e^{(\ln x)_\sigma(x-1)} = \Gamma(x) = (x-1)! \ , \ g^*(x) = x \quad ; \quad x > 0$$

2. Limit Summability of Order Two and 2-Gamma Type Functions

Let f be a real or complex function such that $D_f \supseteq \mathbb{N}^*$. We know that its limit summand function f_σ with the domain $D_{f_\sigma} \supseteq \{0\}$ exists. Now by putting $f_\sigma = g$, a natural question which is arisen here is that whether g

is summable or not (the idea for limit summability of order two). Thus, the initial condition is that $\mathbb{N}^* \subseteq D_g$ which is equivalent to $1 \in D_{f_\sigma}$ or equivalently $R_n(f,1)$ is convergent. Hence, the minimum necessary condition to study limit summability of order two is convergence of the sequence $R_n(f,1) = f_n - f_{n+1}$ (that in the case we have $D_{g_\sigma} \neq \emptyset$). But it is not necessary that $1 \in D_{g_\sigma}$ even though $1 \in D_{f_\sigma}$ (see example of page 76 of [6]). Therefore, we arrive at the following basic definitions.

Definition 2.1. *Let f be a real or complex function with domain $D_f \supseteq \mathbb{N}^*$ such that $R_n(f,1)$ is convergent. Then, we call f limit summable of order two at x_0 if f_σ (i.e., the limit sumand function of f) is limit summable at x_0, and denote the limit by $f_{\sigma^2}(x_0)$. We say that f is limit summable of order two on a set $E \subseteq \Sigma_{f_\sigma}$ if f_σ is limit summable at all points of E.*

It is worth noting that if f is limit summable of order two at x_0, then

$$\begin{aligned} f_{\sigma^2}(x_0) &:= \lim_{n\to\infty} f_{\sigma^2{}_n}(x_0) = \lim_{n\to\infty} (f_\sigma)_{\sigma_n}(x_0) \\ &= \lim_{n\to\infty} \Big(x_0 f_\sigma(n) + \sum_{k=1}^{n} f_\sigma(k) - f_\sigma(k+x_0) \Big). \end{aligned} \tag{9}$$

Note that with the condition $R_n(f,1) \to R(f,1)$, the function f is limit summable of order two at least at the points -1 and 0. Thus $\{-1,0\} \subseteq D_{f_{\sigma^2}} \subseteq D_{f_\sigma} - 1$. Also, since $f_\sigma(0) = 0$ then we can write

$$f_{\sigma^2}(x) = \sum_{n=1}^{\infty} \overline{R}_n(f_\sigma, x) = \sum_{n=1}^{\infty} \Big(R_n(f_\sigma, x) - x R_{n-1}(f_\sigma, 1) \Big) \quad ; \ x \in D_{f_{\sigma^2}} \tag{10}$$

and f_{σ^2} is the 2-limit summand function of f.

Definition 2.2. *We say that f is weak limit summable (resp. limit summable) of order two if both f and f_σ are weak limit summable (resp. limit summable).*

Note that if f is limit summable of order two, then we also call it 2-limit summable and we have

$$f_{\sigma^2}(x) = f_\sigma(x) + f_{\sigma^2}(x-1) \quad ; \quad x \in D_{f_{\sigma^2}}.$$

Example 2.3. The function $f(x) = \sqrt{x}$ is limit summable, but it is not weak limit summable of order two (indeed, $D_{f_{\sigma^2}} = \{-1, 0\}$ that is the least case).

Proposition 2.4. *For every real or complex function f, the followings are equivalent:*
(a) *f is weak limit summable of order two;*
(b) $D_{f_{\sigma^2}} = D_{f_\sigma} - 1 = \Sigma_f - 1$*;*
(c) *f is weak limit summable and f_σ is limit summable on $\Sigma_f - 1$.*

Moreover, each the above equivalent conditions follows that $R_n(f, 1)$ and $R_n(f_\sigma, 1)$ are convergent and

$$\lim_{n\to\infty} f(n+1) + R(f,1)(n+1) = -R(f_\sigma, 1).$$

Hence, if $R(f, 1) = 0$ then a necessary condition for weak limit summability of order two is convergence of the sequence f_n.

Proof. First, note that if $1 \in D_{f_\sigma}$ then $R_n(f, 1)$ is convergent and $D_{f_\sigma} \subseteq D_{f_\sigma} - 1$ (by Lemma 1.2 of [5]), hence

$$\Sigma_{f_\sigma} = D_{f_\sigma} - 1. \tag{11}$$

$(a) \Rightarrow (b)$ Since f is weak limit summable of order two, both f and f_σ are weak limit summable and by using (11) we have $D_{f_{\sigma^2}} = \Sigma_{f_\sigma} = D_{f_\sigma} - 1 = \Sigma_f - 1$. $(b) \Rightarrow (c)$ Obviously $D_{f_\sigma} = \Sigma_f$ and so $D_{f_{\sigma^2}} = \Sigma_f - 1$. $(c) \Rightarrow (a)$ Since f_σ is limit summable on $\Sigma_f - 1$ and according to (11) we have $D_{f_{\sigma^2}} = \Sigma_f - 1$, so f_σ is weak limit summable.
Finally note that (2) implies that

$$R_n(f_\sigma, 1) = f_\sigma(n) - f_\sigma(n+1) = -\Big(f(n+1) + R(f,1)(n+1)\Big) \tag{12}$$

and this identity completes the proof. □

Corollary 2.5. *For every real or complex function f, the followings are equivalent:*
(a) *f is limit summable of order two;*
(b) $D_{f_{\sigma^2}} = D_{f_\sigma} - 1 = \Sigma_f - 1 = D_f - 2$ *and* $f_n \to 0$*;*
(c) *f is weak limit summable of order two, $f_n \to 0$ and $D_f \subseteq D_f - 1$.*

Proof. $(a) \Rightarrow (b)$ Since both f and f_σ are limit summable, $\Sigma_f = D_{f_\sigma}$, $D_{f_\sigma} = D_f - 1$ and $D_{f_{\sigma^2}} = D_{f_\sigma} - 1$, thus by applying (11) and (12) we get the result.

$(b) \Rightarrow (c)$ The assumption follows that $\Sigma_f = D_{f_\sigma}$ and $\Sigma_f = D_f - 1$, so by using (1) we conclude that $D_f \subseteq D_f - 1$ and Proposition 2.4 completes the proof.

$(c) \Rightarrow (a)$ It is obvious that f is limit summable, now according to (12), $R(f_\sigma, 1) = 0$ and consequently f_σ is limit summable. $\square$

If f is limit summable of order two then

$$f_{\sigma^2}(x) = f(x) + f_\sigma(x-1) + f_{\sigma^2}(x-1) \quad ; \; x \in D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1 \tag{13}$$

But, if f is weak limit summable of order two then

$$\begin{aligned} f_{\sigma^2}(x) = f(x) + f_\sigma(x-1) + f_{\sigma^2}(x-1) + R(f,1)x \\ + R(f_\sigma, 1)x \;\; ; \; x \in (\Sigma_f \cap \Sigma_{f_\sigma}) + 1 \end{aligned} \tag{14}$$

Conversely, if the equation (13) (resp. (14)) holds then f is limit summable (resp. weak limit summable) of order two. Also, if f is 2-limit summable then by using (10) and the identity

$$\begin{aligned} R_n(f_\sigma, x) = f_\sigma(n) - f_\sigma(n+x) &= \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n} f(k+x) - f_\sigma(x) \\ &= \sum_{k=1}^{n} R_k(f, x) - f_\sigma(x) \end{aligned}$$

We arrive at

$$\begin{aligned} f_{\sigma^2}(x) = \sum_{n=1}^{\infty} \overline{R}_n(f_\sigma, x) &= \lim_{n\to\infty} \Big(\sum_{k=1}^{n} f_{\sigma_k}(x) - n f_\sigma(x) \Big) \\ &= \sum_{n=1}^{\infty} \Big(f_{\sigma_n}(x) - f_\sigma(x) \Big) \quad ; \quad x \in D_f - 2 = D_{f_\sigma} - 1. \end{aligned} \tag{15}$$

Example 2.6. The signum function $sign(x)$ is limit summable and we have

$$sign_\sigma(x) = \begin{cases} x & ; x > -1 \\ -x-1 & ; x \leqslant -1 \text{ and } x \in \mathbb{Z}^- \\ x + 2[x] & ; x \leqslant -1 \text{ and } x \notin \mathbb{Z}^-. \end{cases}$$

Also, sign(x) is weak limit summable of order two, but not 2-limit summable.

For every real number $0 < a < 1$, the real function $f(x) = a^x$ is limit summable of order two. Since $f_\sigma(x) = \frac{a}{a-1}a^x - \frac{a}{a-1}$, by applying the linearity property of σ-oprator (see Lemma 2.6 of [5]) we can write

$$f_{\sigma^2}(x) = \frac{a}{a-1}\sigma(a^x) + \frac{a}{a-1}\sigma(1) = (\frac{a}{a-1})^2(a^x - 1) - \frac{a}{a-1}x.$$

Since, gamma type function is a subtopic of limit summability, the idea of gamma type function of order two is induced as follow.

Definition 2.7. *A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ has gamma type function of order two (or 2-gamma type function) if both $f^* = g$ (i.e., gamma type function of f) and $(f^*)^* = g^*$ exist. If this is the case, we denote $(f^*)^*$ by f^{**}.*

Note that f^{**} is the limit function of

$$f_n^{**}(x) := (f^*)_n^*(x) = \frac{f^*(n)...f^*(1)f^*(n)^x}{f^*(n+x)...f^*(x)},$$

if f has gamma type function of order two, then f^* satisfies the functional equation

$$f^{**}(x+1) = f^*(x)f^{**}(x) \quad ; \quad x > 0. \tag{16}$$

Example 2.8. For every real numbers $0 < a < 1$ and $0 < b \neq 1$ the real function $f(x) = b^{a^x}$ has 2-gamma type function and

$$f^*(x) = \lim_{n\to\infty} \frac{b^a b^{a^2} ... b^{a^n} (b^{a^n})^x}{b^{a^x} b^{a^{x+1}} ... b^{a^{x+n}}} = b^{\frac{1}{a-1}a^x - \frac{a}{a-1}} \ , \ f^{**}(x) = b^{\frac{a^x + a^2 - 2a}{(a-1)^2} - \frac{ax}{(a-1)}}.$$

But the function $f(x) = x$ does not have 2-gamma type function, because the functional sequence $\Gamma_n^*(x)$ is divergent.

Remark 2.9. *Note that the gamma type functions of order two are obtainable from the topic and so all of conclusions and properties for limit summability of order two can be concluded for them. Indeed, by using (7) we get*

$$\ln f_n^*(x) = (\ln f)_{\sigma_n}(x) - \ln f(x) \quad ; \quad x > 0.$$

Now, putting f^* instead of f in the above equality yields the relation between 2-gamma type function of f and $(\ln f)_{\sigma^2}$ (the second limit summand function of $\ln f$) as follow

$$\begin{aligned}\ln f_n^{**}(x) &= (\ln f^*)_{\sigma_n}(x) - \ln f^*(x) = \Big((\ln f)_\sigma - \ln f\Big)_{\sigma_n}(x) - \ln f^*(x)\\ &= (\ln f)_{\sigma_n^2}(x) - (\ln f)_{\sigma_n}(x) - \ln f^*(x) \quad ; \quad x > 0\end{aligned}$$

So, we have

$$f_n^{**}(x) = \frac{1}{f^*(x)} e^{(\ln f)_{\sigma_n^2}(x) - (\ln f)_{\sigma_n}(x)} = \frac{f^*(n+x+1)}{f^*(n)f^*(x)} f_n^{**}(x+1) \ ; \ x > 0$$

Thus

$$f_n^{**}(x+1) = \frac{f^*(n)}{f^*(n+x+1)} e^{(\ln f)_{\sigma_n^2}(x) - (\ln f)_{\sigma_n}(x)} \ ; \ x > 0. \tag{17}$$

Hence, if $\ln f$ is limit summable of order two, then by letting $n \to \infty$ we get

$$f^{**}(x+1) = e^{(\ln f)_{\sigma^2}(x) - (\ln f)_\sigma(x)} \quad ; \quad x > 0. \tag{18}$$

This important and fundamental relation shows that every conclusion for limit summability of order two could be also used in 2-gamma type functions and vice versa, if $f : \mathbb{R}^+ \to \mathbb{R}^+$. For example, the logarithm of function $f(x) = b^{a^x}$ is 2-limit summable and

$$f^{**}(x+1) = e^{(\ln f)_{\sigma^2}(x) - (\ln f)_\sigma(x)} = b^{\frac{a}{(a-1)^2}a^x - \frac{a}{(a-1)}x - \frac{a}{(a-1)^2}}$$

Hence

$$f^{**}(x) = b^{\frac{a^x + a^2 - 2a}{(a-1)^2} - \frac{ax}{(a-1)}}$$

that is the same solution of Example 2.8.

2.1 Some criteria for limit summability of order two

In [6], it is shown that the convexity, concavity and monotonic conditions play important roles in limit summability of real functions. Our aim is to show that they can be also used for limit summability of order two (and consequently the existence of 2-gamma type functions), but it happens by adding some suitable conditions. Firstly, for this purpose, we state two results of Theorems 3.1 and 3.3 of [6] by which we will prove some tests to examine 2-limit summablity.

Theorem 2.1.1. *Let $M \leqslant 1$ be a fixed real number and $f : [M, +\infty) \to \mathbb{R}$ be a function such that the sequence $f_n := f(n)$ is bounded.*

(a) *If f is increasing (resp. decreasing) on $[M, +\infty)$ from a number on, then f is limit summable. In addition, f is uniformly limit summable on every bounded subset of $[M - 1, +\infty)$.*

(b) *If f is increasing (resp. decreasing) on $[M, +\infty)$ and $f(\infty) \leqslant 0$(resp. $f(\infty) \geqslant 0$) then f_σ is decreasing (resp. increasing) (on its domain $D_{f_\sigma} = [M - 1, +\infty)$).*

Theorem 2.1.2. *Let $M \leqslant 1$ be a fixed real number and $f : [M, +\infty) \to \mathbb{R}$ be a function such that the sequence $R_n(f, 1)$ is bounded.*

(a) *If f is convex (resp. concave) on $[M, +\infty)$ from a number on, then f is weak limit summable. Moreover, f is uniformly limit summable on every bounded subset of $[M - 1, +\infty)$.*

(b) *If f is convex (resp. concave) on $[M, +\infty)$ then the summand function of f (i.e., f_σ) is concave (resp. convex) on its domain $[M - 1, +\infty)$ and f_σ is the only function (with the domain) that is concave (resp. convex) on $[M, +\infty)$ (from a number on), $f_\sigma(0) = 0$ and satisfies the functional equation*

$$f_\sigma(x) = f(x) + f_\sigma(x - 1) + R(f, 1)x \quad ; \quad x \geqslant M.$$

Now, we can present some similar tests to examine limit summability of order two for monoton and convex (concave) functions as follow.

Theorem 2.1.3. *Let $M \leqslant 1$ be a fixed real number and $f : [M, +\infty) \to$*

$\mathbb{R}$ a function such that the sequence f_n is convergent and the partial sum sequence of f_n is bounded.

(a) *If f is increasing (resp. decreasing) on $[M, +\infty)$ from a number on and $f(\infty) \leqslant 0$(resp. $f(\infty) \geqslant 0$), then f is 2-limit summable and f_σ is uniformly summable on every bounded subset of $[M-2, +\infty)$.*

(b) *If f is increasing (resp. decreasing) on $[M, +\infty)$ and $f_\sigma(\infty) := \lim_{n\to\infty} f_\sigma(n) \geqslant 0$ (resp. $f_\sigma(\infty) \leqslant 0$) then f_{σ^2} is increasing (resp. decreasing) on its domain $D_{f_{\sigma^2}} = [M-2, +\infty)$.*

Proof. (a) Firstly, Theorem 2.1.1 causes that f is limit summable. Since $R(f, 1) = 0$ and the sequence $f_\sigma(n)$ (i.e., the partial sum sequence of f_n) is bounded, then according to Theorem 2.1.1 (b), f_σ is decreasing (resp, increasing) on $[M-1, +\infty)$. So by using part (a) of Theorem 2.1.1, f_σ is limit summable on $[M-2, +\infty)$. Now, Corollary 2.5 follows that f is limit summable of order two.

(b) Regarding part (a), and Putting f_σ instead of f in Theorem 2.1.1(b) yield the result. □

Corollary 2.1.4. *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $\prod_{n=1}^{\infty} f_n$ is convergent, then f has the 2-gamma type function. Moreover, f_n^{**} is uniformly convergent on every bounded subset of $\mathbb{R}^+$ (the similar conclusion is obtained for decreasing functions).*

Proof. In order to get the result, we apply Theorem 2.1.3 for the function $\ln f$ and relation (17). □

Example 2.1.5. For a given real number $0 < a < 1$ the function $f(x) = \ln(a^x + 1)$ is limit summable of order two (by Theorem 2.1.3). Now, by using q-pochhammer symbol, defined by

$$(a;q)_0 := 1,$$

$$(a;q)_k := (1-a)(1-aq)(1-a\,q^2)...(1-a\,q^{k-1})\ ;\ k = 1, 2, 3, ...,$$

$$(a;q)_\infty := \lim_{k\to\infty} (a;q)_k = \prod_{j=0}^{\infty} (1-a\,q^j),$$

We have the identity

$$\frac{(a;q)_\infty}{(a\,q^t;q)_\infty} = (a;q)_t \quad ; \quad t \in \mathbb{R}$$

(see [1, 7]). Therefore

$$f_\sigma(x) = \ln\left(\frac{\prod_{n=1}^{\infty}(1+a^n)}{\prod_{n=1}^{\infty}(1+a^{n+x})}\right) = \ln\left(\frac{(-a;a)_\infty}{(-a^{x+1};a)_\infty}\right) = \ln((-a;a)_x).$$

Hence, by (9) we obtain

$$\begin{aligned} f_{\sigma_n^2}(x) &= x\ln\left(\prod_{k=1}^{\infty}\frac{(1+a^k)}{(1+a^{k+n})}\right) + \sum_{t=1}^{n}\ln\left(\prod_{k=1}^{\infty}\frac{(1+a^k)}{(1+a^{k+t})}\right) \\ &\quad - \ln\left(\prod_{k=1}^{\infty}\frac{(1+a^k)}{(1+a^{k+t+x})}\right) \\ &= x\ln((-a;a)_n) + \sum_{t=1}^{n} ln\left(\prod_{k=1}^{\infty}\frac{(1+a^{k+t+x})}{(1+a^{k+t})}\right) \\ &= \ln\left(\prod_{t=1}^{n}\left(\frac{1}{(-a^{t+1};a)_x}\right)(-a;a)_n{}^x\right) \end{aligned}$$

then by letting $n \to \infty$ we obtain

$$f_{\sigma^2}(x) = \ln\left(\prod_{n=1}^{\infty}\left(\frac{1}{(-a^{n+1};a)_x}\right)(-a;a)_\infty{}^x\right).$$

Theorem 2.1.6. *Let $M \leqslant 1$ be a fixed real number and $f : [M,+\infty) \to \mathbb{R}$ a function such that $R(f,1) = 0$ and f_n is bounded*

(a) *If f is concave (resp. convex) on $[M,+\infty)$ then f is weak limit summable of order two and f_σ is uniformely summable on every bounded subset of $[M-2,+\infty)$*

(b) *If the concavity (resp. convexity) of f holds on $[M,+\infty)$ then f_{σ^2} is the only function (with domain $[M-2,+\infty)$) that is concave (resp. convex) on $[M-1,+\infty)$, $f_{\sigma^2}(0) = 0$ and satisfies the functional equation*

$$f_{\sigma^2}(x) = f(x) + f_{\sigma^2}(x-1) + R(f_\sigma,1)x \quad ; \quad x \geqslant M-1.$$

Proof. (a) Firstly, Theorem 2.1.2 makes f limit summable. Since $R(f,1)=0$ and sequence $R_n(f_\sigma,1)$ is bounded. Also, according to Theorem 2.1.2 (b) f_σ is concave (resp. convex) on $[M-1,+\infty)$. Hence, by part (a) of Theorem 2.1.2 f_σ is weak limit summable of order two.

(b) Applying part (a) and Putting f_σ instead of f in Theorem 2.1.2 (b) yield the result. □

Corollary 2.1.7. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that $f_n\to 1$.*
(a) *If f is log-concave (resp. log-convex) on $(0,+\infty)$ then f is 2-gamma type function and f_n^{**} is uniformely convergent on every bounded subset of $(0,+\infty)$. Also, if the log-concavity (resp. log-convexity) of f holds on $(0,+\infty)$ then f^{**} is the only function (with domain $(0,+\infty)$) that is log-concave (resp. log-convex) on $(0,+\infty)$, $f^{**}(1)=1$ and satisfies (16).*

Proof. We get the result, by using Theorem 2.1.6 for $\ln f$ and using (17). □

Example 2.1.8. The function $f(x)=\frac{\Gamma(x+\frac{1}{2})}{\sqrt{x}\Gamma(x)}$, is log-concave from $(0,+\infty)$ onto $(0,1)$ (see Theorem 1.12 of [10]). Hence, by applying Corollary 2.1.7, f has 2-gamma type function and f_n^{**} is uniformly convergent on every bounded subset of $(0,+\infty)$. Also, the function $g(x)=1+a^x$ where $0<a<1$ has 2-gamma type function and by applying Example 2.1.5 we get

$$g^{**}(x+1)=\frac{{(-a;a)_\infty}^x}{(-a;a)_x}\prod_{n=1}^{\infty}\frac{1}{(-a^{n+1};a)_x}.$$

Example 2.1.9. According to Example 3.7 of [6], the function $p(x)=x^r$ with the domain $D_p=(0,+\infty)$ is limit summable if and only if $r<1$. But, if $0<r<1$ then p is not 2-limit summable (because $R_n(p_\sigma,1)=n^r$) is divergent. Now, if $r<0$, then p is 2-limit summable, $D_{p_\sigma}=(-1,+\infty)$, $D_{p_{\sigma^2}}=(-2,+\infty)$ and we have

$$p_\sigma(x)=\begin{cases}\sum_{n=1}^{\infty}\Big(n^r-(n+x)^r\Big) & ;-1<r<0\\ \psi(x+1)+\gamma & ;r=-1\\ \zeta(-r)-\zeta(-r,x+1) & ;r<-1.\end{cases}$$

Now by (10) if $-1<r<0$, we have

$$\begin{aligned}p_{\sigma^2}(x) &= \sum_{n=1}^{\infty}\sum_{N=1}^{\infty}(N+n+x)^r-(N+n)^r+xn^r\\ &= (1+x)\Big(\zeta(-r)-\zeta(-r,x+2)\Big)-\Big(\zeta(r)-\zeta(r,x+2)\Big).\end{aligned}$$

If $r=-1$, then

$$p_{\sigma^2}(x)=\sum_{n=1}^{\infty}\Big(\psi(n+1)-\psi(n+x+1)+xn^r\Big).$$

and if $r<-1$, then

$$p_{\sigma^2}(x)=\sum_{n=1}^{\infty}\Big(\zeta(-r,n+x+1)-\zeta(-r,n+1)+xn^r\Big).$$

3. Relationships Between 2-Limit Summability and Limit Summability of Multipliers: Applications for 2-Gamma Type Functions

In this section we show that there exist some interesting relations between the limit summability of order two and limit summability of the multiplier of f (i.e., $\iota\cdot f$) where ι is the identity function and $(\iota\cdot f)(x)=xf(x)$). As a result of the study, we obtain a formula for gamma type function of $f(x)^x$ and $f^{**}(x)$.

Theorem 3.1. *Let f be a real or complex function such that $R(f,1)=0$. If f is limit summable at $x\in\Sigma_f$ and the functional sequence*

$$\begin{aligned}\delta_n(f,x):=\delta_n(x) &= n\Big(f_{\sigma_n}(x)-f_\sigma(x)\Big)-(x^2+x)f(n)\\ &= n\overline{R}_n(f_\sigma,x)-(x^2+x)f(n),\end{aligned}$$

is convergent, then 2-limit summability of f and limit summability of $\iota\cdot f$ at x are equivalent, and we have

$$f_{\sigma^2}(x)=(x+1)f_\sigma(x)-(\iota\cdot f)_\sigma(x)+\delta(x),\tag{19}$$

where $\delta(x) = \lim_{n\to\infty} \delta_n(x)$.

If $f : [1, +\infty) \to \mathbb{C}$ *is a limit summable function such that* $\delta_n(f, x)$ *is convergent and* $f_n \to 0$, *then 2-limit summabalitiy of* $f(x)$ *is equivalent to limit summabality of* $xf(x)$ *and we have*

$$f_{\sigma^2}(x) = (x+1)f_\sigma(x) - (\iota \cdot f)_\sigma(x) + \lim_{n\to\infty} n\Big(f_\sigma(x) - f_{\sigma_n}(x)\Big) \ ; \ x \geqslant 1. \quad (20)$$

Proof. Since $R(f, 1) = 0$, then (3) implies that

$$f_\sigma(k) = \sum_{j=1}^{k} f(j) \ , \ f_\sigma(k+x) = \sum_{j=1}^{k} f(j+x) + f_\sigma(x)$$

and

$$\begin{aligned}
(f_\sigma)_{\sigma_n}(x) &= xf_\sigma(n) + \sum_{k=1}^{n} f_\sigma(k) - f_\sigma(k+x) \\
&= xf_\sigma(n) + \sum_{k=1}^{n}\sum_{j=1}^{k} f(j) - f(j+x) - f_\sigma(x) \\
&= xf_\sigma(n) - nf_\sigma(x) + \sum_{k=1}^{n}\sum_{j=1}^{k} R_j(x) \\
&= xf_\sigma(n) - nf_\sigma(x) + \sum_{j=1}^{n}(n+1-j)R_j(x)
\end{aligned}$$

Therefore

$$(f_\sigma)_{\sigma_n}(x) = x\Big(f_\sigma(n) - nf(n)\Big) - n\Big(f_\sigma(x) - f_{\sigma_n}(x)\Big) + \sum_{j=1}^{n}(1-j)R_j(x) \quad (21)$$

On the other hand, we have

$$\begin{aligned}
(\iota \cdot f)_{\sigma_n}(x) &= xnf(n) + \sum_{j=1}^{n} jf(j) - (j+x)f(j+x) \\
&= xnf(n) - x\sum_{j=1}^{n} f(j+x) + \sum_{j=1}^{n} jR_j(x) \\
&= xnf(n) - x\Big(f_\sigma(x+n) - f_\sigma(x)\Big) + \sum_{j=1}^{n} jR_j(x)
\end{aligned}$$

So

$$(\iota \cdot f)_{\sigma_n}(x) = x\Big(nf(n) - f_\sigma(x+n) + f_\sigma(x)\Big) + \sum_{j=1}^{n} jR_j(x) \tag{22}$$

Combining (21) and (22), we get

$$\begin{aligned}
(f_\sigma)_{\sigma_n}(x) + (\iota \cdot f)_{\sigma_n}(x) &= x\Big(f_\sigma(n) + f_\sigma(x) - f_\sigma(x+n)\Big) \\
&\quad - n\Big(f_\sigma(x) - f_{\sigma_n}(x)\Big) + f_{\sigma_n}(x) - xf(n) \\
&= (x+1)f_{\sigma_n}(x) + n\Big(f_{\sigma_n}(x) - f_\sigma(x)\Big) \\
&\quad - (x^2+x)f(n). \qquad (23)
\end{aligned}$$

Now letting $n \to \infty$ in the above equality yeilds the result. $\square$

Example 3.2. If $|a| < 1$ then the complex function xa^x is limit summable and

$$\begin{aligned}
\sigma(xa^x) &= (x+1)\frac{a}{a-1}(a^x - 1) - (\frac{a}{a-1})^2(a^x-1) + x\frac{a}{a-1} \\
&= \frac{a}{a-1}xa^x - \frac{a}{(a-1)^2}(a^x - 1).
\end{aligned}$$

Corollary 3.3. *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $f_n \to 1$ and f has gamma type function, and put*

$$\mu_n(f,x) := e^{\delta_n(\ln f, x)} \quad ; \quad x > 0$$

(where δ_n is defined as in Theorem 3.1). If $\delta_n(\ln f, x)$ is convergent, then f has gamma type function of order two if and only if f^ι (where $f^\iota(x) := f(x)^x$) has gamma type function and we have

$$f^{**}(x+1) = \frac{f^*(x+1)^x}{(f^\iota)^*(x+1)}\mu(f,x) \tag{24}$$

Proof. Applying Theorem 3.1 for the function $\ln f$ and relation (17), we get the result as follow

$$f_n^{**}(x+1) = \frac{f^*(n)}{f^*(n+x+1)} e^{(x+1)(\ln f)_{\sigma_n}(x) - (\iota. \ln f)_{\sigma_n}(x) + \delta_n(\ln f, x) - (\ln f)_{\sigma_n}(x)}$$

Now, by using (7) we have

$$\mu_n(f,x) = e^{\delta_n(\ln f,x)} = \Big(\frac{f_n^*(x+1)f(n+x+1)}{f^*(x+1)f(n)}\Big)^n \frac{1}{f(n)^{x^2+x}} \quad ; \ x>0$$

and

$$\begin{aligned} f_n^{**}(x+1) = &\frac{f^*(n)}{f^*(n+x+1)}\Big(\frac{f_n^*(x+1)f(n+x+1)}{f(n)}\Big)^x \\ &\Big(\frac{(f^\iota)_n^*(x+1)f^\iota(n+x+1)}{f^\iota(n)}\Big)^{-1} \mu_n(f,x) \quad ; \quad x>0. \end{aligned} \tag{25}$$

Letting $n \to \infty$ yields the result. $\square$

Example 3.4. Consider the real function $f(x) = 1 + \frac{1}{x}$ on $(0, +\infty)$. It is easy to see that $f^*(x) = x$ for all $x > 0$. Since $f_n \to 1$, $\mu_n \to 1$ and $f^{**}(x) = \Gamma(x)$, then Corollary 3.3 implies that the function $f(x)^x = (1+\frac{1}{x})^x$ has gamma type function and we have

$$(f^\iota)^*(x+1) = \frac{(x+1)^x}{\Gamma(x+1)} = \frac{(x+1)^x}{x!} \quad ; \quad x>0.$$

Theorem 3.5. *Let f be a function such that f and $\iota \cdot f$ are summable at $x \in \Sigma_f$ and f_n is convergent. Then*

(a) *We have*

$$\lim_{n\to\infty}\Big(\sum_{k=1}^{n} f_{\sigma_k}(x) - n f_{\sigma_n}(x)\Big) = (x+1)f_\sigma(x) - (\iota\cdot f)_\sigma(x) - (x^2+x)f(\infty).$$

(b) *If $\delta(x) = 0$, then f is 2-limit summable and*

$$\begin{aligned} f_{\sigma^2}(x) &= (x+1)f_\sigma(x) - (\iota\cdot f)_\sigma(x) \\ &= \lim_{n\to\infty}\Big(\sum_{k=1}^{n} f_{\sigma_k}(x) - n f_\sigma(x)\Big) \\ &= \lim_{n\to\infty}\Big(\sum_{k=1}^{n} f_{\sigma_k}(x) - n f_{\sigma_n}(x)\Big) + (x^2+x)f(\infty). \end{aligned}$$

Proof. By computing $(f_\sigma)_{\sigma_n}(x)$ in another method we have

$$\begin{aligned}(f_\sigma)_{\sigma_n}(x) &= xf_\sigma(n) - nf_\sigma(x) + \sum_{k=1}^{n}\sum_{j=1}^{k} R_j(x)\\ &= xf_\sigma(n) - nf_\sigma(x) + \sum_{k=1}^{n} f_{\sigma_k}(x) - \sum_{k=1}^{n} xf(k)\\ &= (\sum_{k=1}^{n} f_{\sigma_k}(x)) - nf_\sigma(x). \end{aligned} \tag{26}$$

Now combining relations (26) and (23), then letting $n \to \infty$ get the result. $\square$

Corollary 3.6. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that both f and f^ι have gamma type function, and $f_n \to 1$ then*

(a) *We have*

$$\frac{f^*(x+1)^{x+1}}{(f^\iota)^*(x+1)} = \lim_{n\to\infty} \frac{\prod_{k=1}^{n} f_k^*(x+1)}{f_n^*(x+1)^n}.$$

(b) *If $\mu(x) = 1$, then f has 2-gamma type function and*

$$\begin{aligned}f^{**}(x+1) = \frac{f^*(x+1)^x}{(f^\iota)^*(x+1)} &= \lim_{n\to\infty} \frac{\prod_{k=1}^{n} f_k^*(x+1)}{f^*(x+1)^n}\\ &= \lim_{n\to\infty} \frac{\prod_{k=1}^{n} f_k^*(x+1)}{f_n^*(x+1)^n f^*(x+1)}.\end{aligned}$$

Proof. By using relations (17) and (26) we have

$$\begin{aligned} f_n^{**}(x+1) &= \frac{f^*(n)}{f^*(n+x+1)} e^{\sum_{k=1}^{n}(\ln f)_{\sigma_k}(x) - n(\ln f)_\sigma(x) - (\ln f)_{\sigma_n(x)}}\\ &= \frac{f^*(n)}{f^*(n+x+1)}\left(e^{\sum_{k=1}^{n}(\ln f)_{\sigma_k}(x)}\right)\left(e^{(\ln f)_\sigma(x)}\right)^{-n}\left(e^{(\ln f)_{\sigma_n}(x)}\right)^{-1}\\ &= \frac{f^*(n)}{f^*(n+x+1)}\left(e^{\ln \prod_{k=1}^{n} f_k^*(x+1)\frac{f(k+x+1)}{f(k)}}\right)\frac{f^*(x+1)^{-n} f(n)}{f_n^*(x+1) f(n+x+1)} \; ; x > 0\end{aligned}$$

Therefore

$$f_n^{**}(x+1) = \frac{f(n)f^*(n)}{f(n+x+1)f^*(n+x+1)f_n^*(x+1)f^*(x+1)^n} \prod_{k=1}^{n} \frac{f_k^*(x+1)f(k+x+1)}{f(k)} \quad ; \quad x > 0 \tag{27}$$

Now, Combining relations (27) and (25), then letting $n \to \infty$ get the result. $\square$

Example 3.7. If $0 < a < 1$ then the real function $g(x) = b^{xa^x}$ has gamma type function, for all $0 < b \neq 1$. Because by putting $f(x) = b^{a^x}$ we have $\mu(f, x) = 1$, $g = f^\iota$ and

$$g^*(x+1) = (f^\iota)^*(x+1) = \frac{f^*(x+1)^x}{f^{**}(x+1)} = \frac{\left(b^{\frac{a}{a-1}a^x - \frac{a}{a-1}}\right)^x}{b^{\frac{a}{(a-1)^2}a^x - \frac{a}{(a-1)}x - \frac{a}{(a-1)^2}}}$$
$$= b^{\frac{a}{(a-1)}xa^x - \frac{a}{(a-1)^2}(a^x - 1)}.$$

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