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# Stability of Special Functional Equations on Banach Lattices

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**Abstract.** In this paper, using direct method, we prove the Hyers-Ulam-Rassias stability of the some functional equations on Banach lattices.

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## 1. Introduction

We say that a functional equation $Q$ is stable if any function $g$ satisfying the equation $Q$ approximately is near to true solution of $Q$.

In 1940, S. M. Ulam [8], while he was giving a talk before the mathematics club of the University of Wisconsin, he proposed a number of importent unsolved problems. One of the peoblems is the stability of functional equations. In the last five decades the problem was tackled by numerous authors [3, 6]

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*It's* solutions via various forms of functional equations like additive, quadratic, Cubic and quartic and its mixed forms were discussed.

*Ulam's* stability problem states as follows:

Let $G$ be a group and let $H$ be a metric group with metric $d(.,.)$. Given $\varepsilon > 0$ dose exists a $\delta > 0$ such that if a function $f : G \longrightarrow H$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $A : G \longrightarrow H$ with $d(f(x), A(x)) < \varepsilon$ for all $x \in G$?

In 1941, Hyers [6] gave a first affirmative answer to the question of Ulam for Banach spaces as follows:

**Theorem 1.1.** *[4](Hyers) Let $E, E^{'}$ be Banach spaces and let $f : E \longrightarrow E^{'}$ be a mapping satisfying:*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon \tag{1}$$

*for some $\varepsilon > 0$ and all $x, y \in E$. Then the limit $A(x) = lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $A : E \longrightarrow E^{'}$ is the unique additive mapping satisfying:*

$$\|f(x) - A(x)\| \leqslant \varepsilon \tag{2}$$

*for all $x \in E$. Moreover, If $f(tx)$ is continuous in $t$ for each fixed $x \in E$, then $A$ is linear.*

**Proof.** See[4]. □

In 1983 , Skof proved the Hyers-Ulam-Rassias stability problem for quadratic of the following functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{3}$$

for a class of functions $f : A \longrightarrow B$ where $A$ is a normed space and $B$ is a Banach space ([1, 7] ).

In 1994, a generalization of the *Rassias's* theorem was obtained by Gavruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach .

In 2003, Cadariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [3]. They could persent a short and simple proof(different of the *direct method*, initiated by Hyers in 1941 ) for the generalized Hyers-Ulam stability of Jensen functional equation [3], for Cauchy functional equation [2].

In this paper, using direct method we investegat the Hyers-Ulam-Rassias stability of the following equations:

$$a) f(x+2y) - f(x-2y) = 2\Big(f(x+y) - f(x-y)\Big) + 2f(3y) - 6f(2y) + 6f(y),$$

$$b) f(3x+y) + f(3x-y) = 3\Big(f(x+y) + f(x-y)\Big) + 48f(x).$$

**Definition 1.2.** *The additive Cauchy equation $f(x+y) = f(x)+f(y)$ is said to have the Hyers-Ulam stability on $(E, E')$ if for every$f : E \to E'$ satisfying the inequality (1) for some $\varepsilon \geqslant 0$ and for all $x, y \in E$, there exists an additive function $A : E \to E'$ such that $f - A$ is bounded on $E$ [4].*

An ordered set $(M, \leqslant)$ is called a *lattice* if any two elements $x, y \in M$ have a least upper bound denoted by $x \vee y = sup\{x, y\}$ and a greatest lower bound denoted by $x \wedge y = inf\{x, y\}$.

Similarly, we denoted the supremum and the infimum for arbitrary subsets. if $v$ is the least upper bound of a subset $A \subset M$, then we will write

$$v = sup(A) = \bigvee_{x \in A} x = sup\{x : x \in A\}.$$

If $u$ is the greatest lower bound of $A$, then we will write

$$u = inf(A) = \bigwedge_{x \in A} x = inf\{x : x \in A\}.$$

Of course, if $sup(A)$ exists, then $A$ is bounded from above. To using the lattice notation, let $x, y \in \mathbb{R}$ ( $\mathbb{R}$ is a Banach lattice ) then we have :

$$x + y = x \vee y + x \wedge y, \tag{4}$$

and

$$x - y = x \vee (-y) + x \wedge (-y). \tag{5}$$

and using Relation (4) and (5) we obtain that:

$$x = \frac{1}{2}\Big(x \vee y + x \wedge y + x \vee (-y) + x \wedge (-y)\Big), \tag{6}$$

and

$$y = \frac{1}{2}\Big(x \vee y + x \wedge y + (-x) \vee y + (-x) \wedge y\Big). \tag{7}$$

## 2. Main Results

In this section, we deal with prove the Hyers-Ulam-Rassias stability of the following a Mixed Type Additive, Quadratic, and Cubic functional equation in Banach lattices.

$$\begin{aligned} f(x+2y) - f(x-2y) = 2\Big(f(x+y) - f(x-y)\Big) \\ +2f(3y) - 6f(2y) + 6f(y), \end{aligned} \tag{8}$$

By (4), (5), (6), (7), The above Mixed functional equation in the lattices form is the following:

$$\begin{aligned} & f\left(\frac{3}{2}\Big(x \vee y + x \wedge y\Big) - \frac{1}{2}\Big(x \vee (-y) + x \wedge (-y)\Big)\right) \\ & = 2\left(f\Big(x \vee y + x \wedge y)\Big) - f\Big(x \vee (-y) + x \wedge (-y)\right) \\ & + 2f\left(\frac{3}{2}\Big(x \vee y + x \wedge y + y \vee (-x) + y \wedge (-x)\Big)\right) \\ & - 6f\left(\Big(x \vee y + x \wedge y + y \vee (-x) + y \wedge (-x)\Big)\right) \\ & + 6f\left(\frac{1}{2}\Big(x \vee y + x \wedge y + y \vee (-x) + y \wedge (-x)\Big)\right) \end{aligned}$$

Let X and Y be two Banach lattices and, $f : X \longrightarrow Y$ define the difference operator $D_f : X \times X \longrightarrow Y$ by

$$D_f(x,y) = f(x+2y) - f(x-2y) - 2\Big(f(x+y) - f(x-y)\Big) \\ -2f(3y) + 6f(2y) - 6f(y),$$

for all $x, y \in X$. We consider the following functional inequality

$$\Big\|D_f(x,y)\Big\| \leqslant \phi\Big(x,y\Big),$$

for an upper bound $\phi : X \times X \longrightarrow [0,\infty)$.

**Theorem 2.1.** *Let $X$ and $Y$ be two Banach lattices and $s \in \{-1,1\}$be fixed. Suppose that an even mapping $f : X \longrightarrow Y$ satisfies $f(0) = 0$ and*

$$\Big\|D_f(x,y)\Big\| \leqslant \phi\Big(x,y\Big), \tag{9}$$

*for all $x, y \in X$. If the upper bound $\phi : X \times X \longrightarrow [0,\infty)$, is a mapping such that*

$$\sum_{i=0}^{\infty} 4^{si}\Big(\phi(2^{-si}x, 2^{-si}x) + \frac{1}{2}\phi(0, 2^{-si}x)\Big) < \infty,$$

*and that*

$$\lim_{n\longrightarrow\infty} 4^{sn}(\phi(2^{-si}x, 2^{-si}y) = 0,$$

*for all $x, y \in X$, the limit*

$$Q(x) = \lim_{n\longrightarrow\infty} 4^{sn} f(2^{-si}x),$$

*exists for all $x \in X$, and $Q : X \longrightarrow Y$ is a unique quadratic function satisfying (8) and*

$$\left\|\left(f(x) \vee \Big(-Q(x)\Big) + f(x) \wedge \Big(-Q(x)\Big)\right)\right\| \leqslant \frac{1}{8}\sum_{i=(s+1)/2}^{\infty} 4^{si}\Big(\phi(2^{-si}x, 2^{-si}x) + \frac{1}{2}\phi(0, 2^{-si}x)\Big), \tag{10}$$

*for all $x \in X$.*

**Proof.** Let $s = 1$. putting $x = 0$ in (9), we get

$$\Bigg\|2\Bigg(\Big(f(3y) \vee (-3f(2y) + f(3y) \wedge (-3f(2y)\Big) \vee (3f(y)\Bigg)$$
$$+ 2\Bigg(\Big(f(3y) \vee (-3f(2y) + f(3y) \wedge (-3f(2y)\Big) \wedge (3f(y)\Bigg)\Bigg\| \leqslant \phi\Big(0, y\Big),$$

for all $y \in X$. On the other hand by replacing $y$ by $x$ in (9), it follows that

$$\Bigg\|\Bigg(\Big( - f(3y)) \vee (4f(2y) + (-f(3y)) \wedge (4f(2y))\Big) \vee (-7f(y))\Bigg)$$
$$+ \Bigg(\Big( - f(3y) \vee (4f(2y)) + (-f(3y)) \wedge (4f(2y))\Big) \wedge (-7f(y))\Bigg)\Bigg\| \leqslant \phi\Big(y, y\Big),$$

for all $y \in X$.

Let $s = 1$. By combining two equations obtained by putting $x = 0$ in (9) and replacing $y$ by $x$ in (9) , it follows that :

$$\Big\|\Big(2f(2y)\Big) \vee \Big( - 8f(y)\Big) + \Big(2f(2y)\Big) \wedge \Big( - 8f(y)\Big)\Big\| \leqslant \phi\Big(0, y\Big) + 2\phi\Big(y, y\Big), \tag{11}$$

for all $y \in X$. With the sub stitution $y := \frac{x}{2}$ in (11) and then dividing both sides of inequality by 2 , we get

$$\Bigg\|\Big(f(x)\Big) \vee \Big( - 4f(\frac{x}{2})\Big) + \Big(f(x)\Big) \wedge \Big( - 4f(\frac{x}{2})\Big)\Bigg\| \leqslant \frac{1}{2}\Bigg(2\phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \phi\Big(0, \frac{x}{2}\Big)\Bigg). \tag{12}$$

Now, using methods similar, we can easily show that the function $Q : X \longrightarrow Y$ defined by

$$Q(x) = \lim_{n \longrightarrow \infty} 4^n f(2^{-n}x)$$

for all $x \in X$, is unique quadratic function satisfying in (8), (10). Let $s = -1$, replace $2x$ by $x$ and also dividing both sides of inequality by 4, using by (12), we have

$$\Bigg\|\Big( - f(x)\Big) \vee \Big(\frac{f(2x)}{4}\Big) + \Big( - f(x)\Big) \wedge \Big(\frac{f(2x)}{4}\Big)\Bigg\| \leqslant \frac{1}{8}\Bigg(2\phi\Big(x, x\Big) + \phi\Big(0, x\Big)\Bigg),$$

for all $x \in X$. And analogously, as in the case $s = -1$, we can show that the function $Q : X \longrightarrow Y$ defined by

$$Q(x) = \lim_{n \longrightarrow \infty} 4^{-n} f(2^n x)$$

is unique quadratic function satisfying in (8), (10). $\square$

**Theorem 2.2.** *Let $X$ and $Y$ be two Banach lattices. Function $f : X \longrightarrow Y$ satisfying in the following functional equation*

$$f(2x + y) + f(2x - y) = 2\Big(f(x + y) + f(x - y)\Big) + 12f(x), \tag{13}$$

*if and only if $f : X \longrightarrow Y$ satisfys in the functional equation*

$$f(mx + y) + f(mx - y) = m\Big(f(x + y) + f(x - y)\Big) + 2(m^3 - m)f(x),$$

*for any natural number $m \geqslant 3$.*

**Proof.** Let Function $f : X \longrightarrow Y$ satisfys in (13). If we put $x = y = 0$ in (13), we have $f(0) = 0$, and if we put $x = 0$ in (13), we get $f(-y) = -f(y)$, also we put $y = 0$ in (13), and we have $f(2x) = 8f(x)$. Furthermore, replacing $y$ by $x$ and $y$ by $2x$ in (13), then we have $f(3x) = 27f(x)$ and f(2x)=8f(x).

Then for all $x, y \in X$, all $k \in \mathbb{Z}^+$, replacing $y$ by $x + y$ in (13), we get

$$f(3x + y) + f(x - y) = 2\Big(f(2x + y) - f(y)\Big) + 12f(x), \tag{14}$$

then replacing $y$ by $y - x$ in (13), for $x, y \in X$, we have

$$f(x + y) + f(3x - y) = 2\Big(f(y) + f(2x - y)\Big) + 12f(x). \tag{15}$$

Combining (14) and (15), we lead to

$$f(3x + y) + f(3x - y) = 3\Big(f(x + y) + f(x - y)\Big) + 48f(x).$$

So by this method we get

$$f(mx + y) + f(mx - y) = m\Big(f(x + y) + f(x - y)\Big) + 2(m^3 - m)f(x).$$

The converse of theorem us automatically consistant. $\square$

Now, we prove the Hyers-Ulam-Rassias stability of the following Cubic functional equation in Banach lattice .

$$f(3x+y)+f(3x-y)=3\Big(f(x+y)+f(x-y)\Big)+48f(x),$$

The above Cubic functional equation in the lattices form is the following:

$$\begin{aligned} &f\left(\Big(x\vee y+x\wedge y\Big)-2\Big((-x\vee y)+(-x\wedge y)\Big)\right)\\ &+f\left(\Big(x\vee(-y)+x\wedge(-y)\Big)+2\Big(x\vee y+x\wedge y\Big)\right)\\ &=3f\left(x\vee y+x\wedge y\right)+3f\left(x\vee(-y)+x\wedge(-y)\right)\\ &+48f\left(\frac{1}{2}\Big(x\vee y+x\wedge y+x\vee(-y)+x\wedge(-y)\Big)\right)\end{aligned}$$

**Theorem 2.3.** *Let $X$ and $Y$ be two Banach lattices and $\phi : X^2 \longrightarrow [0,\infty)$ be a function satisfying in equality:*

$$\Phi(x,y)=\sum_{i=1}^{\infty}\frac{1}{27^i}\phi\Big(\frac{3^ix}{3},\frac{3^iy}{3}\Big)<\infty$$

*for all $x,y\in X$ and also, $f : X \longrightarrow Y$ satisfys the inequality:*

$$\begin{aligned} &\Bigg\|f\left(\Big(x\vee y+x\wedge y\Big)-2\Big((-x\vee y)+(-x\wedge y)\Big)\right)\\ &+f\left(\Big(x\vee(-y)+x\wedge(-y)\Big)+2\Big(x\vee y+x\wedge y\Big)\right)\\ &-3f\left(x\vee y+x\wedge y\right)-3f\left(x\vee(-y)+x\wedge(-y)\right)\\ &-48f\left(\frac{1}{2}\Big(x\vee y+x\wedge y+x\vee(-y)+x\wedge(-y)\Big)\right)\Bigg\|\\ &\leqslant\phi\Big(x,y\Big)\end{aligned}\tag{16}$$

*then there exists an unique cubic function* $C : X \longrightarrow Y$ *for all* $x, y \in X$ *such that:*

$$\Big\|C(x) \vee (-f(x)) + C(x) \wedge (-f(x))\Big\| \leqslant \Phi\Big(x, 0\Big). \tag{17}$$

**Proof.** If we put $y = 0$ in (16), since:

$$x \vee 0 = 0, \quad x \wedge 0 = x, \quad for\ x < 0$$

$$x \vee 0 = x, \quad x \wedge 0 = 0, \quad for\ x > 0,$$

for all $x \in X$, then we get:

$$\Big\|f(3x) \vee (-27f(x)) + f(3x) \wedge (-27f(x))\Big\| \leqslant \phi(x, 0),$$

hence:

$$\Bigg\|\frac{f(3x)}{27} \vee (-f(x)) + \frac{f(3x)}{27} \wedge (-f(x))\Bigg\| \leqslant \frac{1}{27}\phi\Big(x, 0\Big). \tag{18}$$

Replacing $x$ by $3x$ in (18) we get:

$$\Bigg\|\frac{f(3^2x)}{27} \vee (-f(3x)) + \frac{f(3^2x)}{27} \wedge (-f(3x))\Bigg\| \leqslant \frac{1}{27}\phi\Big(3x, 0\Big),$$

therefor

$$\Bigg\|\frac{f(3^2x)}{27^2} \vee \Big(-\frac{f(3x)}{27}\Big) + \frac{f(3^2x)}{27^2} \wedge \Big(-\frac{f(3x)}{27}\Big)\Bigg\| \leqslant \frac{1}{27^2}\phi\Big(3x, 0\Big),$$

so we have

$$\Bigg\|\frac{f(3^2x)}{27^2} \vee (-f(x)) + \frac{f(3^2x)}{27^2} \wedge (-f(x))\Bigg\| \leqslant \sum_{i=1}^{2} \frac{1}{27^i}\phi\Big(\frac{3^i x}{3}, 0\Big).$$

By induction on $n$, we will prove that

$$\Bigg\|\frac{f(3^nx)}{27^n} \vee (-f(3x)) + \frac{f(3^nx)}{27^n} \wedge (-f(3x))\Bigg\| \leqslant \sum_{i=1}^{n} \frac{1}{27^i}\phi\Big(\frac{3^i x}{3}, 0\Big) \tag{19}$$

To prove (19), let (19) holds for each $k \leqslant n$, then we want to prove it for case $n = k+1$ is hold. For this replacing $x$ by $3x$ in (19) , then we have

$$\left\| \frac{f(3^{k+1}x)}{27^k} \vee (-f(3x)) + \frac{f(3^{k+1}x)}{27^k} \wedge (-f(3x)) \right\| \leqslant \sum_{i=1}^{k} \frac{1}{27^i} \phi\Big(3^i x, 0\Big)$$

The dividing both sides of the above by 27, we get

$$\left\| \frac{f(3^{k+1}x)}{27^{k+1}} \vee \Big( - \frac{f(3x)}{27} \Big) + \frac{f(3^{k+1}x)}{27^{k+1}} \wedge \Big( - \frac{f(3x)}{27} \Big) \right\| \leqslant \frac{1}{27} \sum_{i=1}^{k} \frac{1}{27^i} \phi\Big(3^i x, 0\Big),$$

therefor

$$\left\| \frac{f(3^{k+1}x)}{27^{k+1}} \vee (-f(3x)) + \frac{f(3^{k+1}x)}{27^{k+1}} \wedge (-f(3x)) \right\| \leqslant \sum_{i=1}^{k+1} \frac{1}{27^i} \phi\Big(\frac{3^i x}{3}, 0\Big).$$

Now we show that $\left\{ \frac{f(3^n x)}{27^n} \right\}$ is a Cauchy sequence. Let $n > m > 0$, then:

$$\left\| \frac{f(3^n x)}{27^n} \vee \Big( - \frac{f(3^m x)}{27^m} \Big) + \frac{f(3^n x)}{27^n} \wedge \Big( - \frac{f(3^m x)}{27^m} \Big) \right\| \leqslant \sum_{i=m+1}^{n} \frac{1}{27^i} \phi\Big(\frac{3^i x}{3}, 0\Big) < \infty.$$

Taking the limit as $m \longrightarrow \infty$ yields:

$$\lim_{m \longrightarrow \infty} \left\| \frac{f(3^n x)}{27^n} \vee \Big( - \frac{f(3^m x)}{27^m} \Big) + \frac{f(3^n x)}{27^n} \wedge \Big( - \frac{f(3^m x)}{27^m} \Big) \right\| = 0.$$

Then $\left\{ \frac{f(3^n x)}{27^n} \right\}$ is a Cauchy sequence in $Y$ for all $x \in X$, and since $Y$ is Banach space, hence is converge to $Y$ Let:

$$C(x) := \lim_{n \longrightarrow \infty} \frac{f(3^n x)}{27^n}.$$

Replacing $x$ by $3^n x$ and $y$ by $3^n y$ in (16), and Then dividing both sides of the obtained in equalities by $27^n$ and finally taking limit as $n \longrightarrow \infty$, we have :

$$\lim_{n\longrightarrow\infty}\Bigg\|\frac{1}{27^n}f\Bigg(3^n\Big(x\vee y+x\wedge y\Big)-2\times 3^n\Big((-x\vee y)+(-x\wedge y)\Big)\Bigg)$$
$$+f\Bigg(\frac{1}{9^n}\Big(x\vee(-y)+x\wedge(-y)\Big)+\frac{2}{9^n}\Big(x\vee y+x\wedge y\Big)\Bigg)$$
$$-3f\Bigg(\frac{1}{9^n}\Big(x\vee y+x\wedge y\Big)\Bigg)-3f\Bigg(\frac{1}{9^n}\Big(x\vee(-y)+x\wedge(-y)\Big)\Bigg)$$
$$-48f\Bigg(\frac{\Big((x\vee y+x\wedge y+x\vee(-y)+x\wedge(-y)\Big)}{2\times 9^n}\Bigg)\Bigg\|$$
$$\leqslant\lim_{n\longrightarrow\infty}\frac{1}{27^n}\phi\Big(3^n x,3^n y\Big),$$

and therefor, we have

$$C\Bigg(\Big(x\vee y+x\wedge y\Big)-2\Big(-x\vee y)+(-x\wedge y)\Big)\Bigg)$$
$$+C\Bigg(\Big(x\vee(-y)+x\wedge(-y)\Big)+2\Big(x\vee y+x\wedge y\Big)\Bigg)$$
$$-3C\Bigg(x\vee y+x\wedge y\Bigg)-3C\Bigg(x\vee(-y)+x\wedge(-y)\Bigg)$$
$$-48C\Bigg(\frac{1}{2}\Big(x\vee y+x\wedge y+x\vee(-y)+x\wedge(-y)\Big)\Bigg)$$
$$=0.$$

Then $C:X\longrightarrow Y$ is a Cubic function. Let $K:X\longrightarrow Y$ is an another Cubic function with the property (17), then for all $x\in X$ we have:

$$\Bigg\|C(x)\vee(-K(x))+C(x)\wedge(-K(x))\Bigg\|\leqslant 2\times\sum_{i=n}^{\infty}\frac{1}{27^{i+1}}\phi\Big(3^i x,0\Big).$$

Taking the limit as $n\longrightarrow\infty$, we have $C(x)=K(x)$. Then $C$ is the unique Cubic function satisfying in the inequality (17), which ends the proof. □

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