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# On Pseudo-Amenability of Weighted Semigroup Algebras

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**Abstract.** Amenability and pseudo-amenability of $\ell^1(S,\omega)$ is characterized, where $S$ is a left (right) zero semigroup or it is a rectangular band semigroup. The equivalence conditions to amenability of $\ell^1(S,\omega)$ are provided, where $S$ is a band semigroup. The equivalence properties of amenability of $\ell^1(S,\omega)^{**}$ are described, where $S$ is an inverse semigroup. For a locally compact group $G$, pseudo-amenability of $L^1(G,\omega)$ is also discussed.

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# 1. Introduction

For a Banach algebra $\mathfrak{A}$ the projective tensor product $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule in a natural manner and the multiplication map $\pi : \mathfrak{A}\widehat{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ defined by $\pi(a \otimes b) = ab$ for $a, b \in \mathfrak{A}$ is a Banach $\mathfrak{A}$-bimodule homomorphism.

Amenability for Banach algebras introduced by B. E. Johnson [8]. Let $\mathfrak{A}$ be a Banach algebra and $E$ be a Banach $\mathfrak{A}$-bimodule. A continuous linear operator $D : \mathfrak{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) \, . \, b + a \, . \, D(b)$ for all $a, b \in \mathfrak{A}$. Given $x \in E$, the *inner* derivation $ad_x : \mathfrak{A} \longrightarrow E$ is defined by

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$ad_x(a) = a \,.\, x - x \,.\, a$. A Banach algebra $\mathfrak{A}$ is *amenable* if for every Banach $\mathfrak{A}$-bimodule $E$, every derivation from $\mathfrak{A}$ into $E^*$, the dual of $E$, is inner.

An *approximate diagonal* for a Banach algebra $\mathfrak{A}$ is a net $(m_i)_i$ in $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $a \,.\, m_i - m_i \,.\, a \longrightarrow 0$ and $a\pi(m_i) \longrightarrow a$, for each $a \in \mathfrak{A}$. The concept of pseudo-amenability introduced by F. Ghahramani and Y. Zhang in [4]. A Banach algebra $\mathfrak{A}$ is *pseudo-amenable* if it has an approximate diagonal. It is well-known that amenability of $\mathfrak{A}$ is equivalent to the existence of a *bounded* approximate diagonal. One may see [9, 10, 11] for more details and related notions.

The notions of biprojectivity and biflatness of Banach algebras introduced by Helemskiĭ in [6]. A Banach algebra $\mathfrak{A}$ is *biprojective* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow \mathfrak{A}\widehat{\otimes}\mathfrak{A}$ such that $\pi o \rho = I_{\mathfrak{A}}$, where $I_{\mathfrak{A}}$ is the identity map on $\mathfrak{A}$. We say that $\mathfrak{A}$ is *biflat* if there is a bounded $\mathfrak{A}$-bimodule homomorphism $\rho : \mathfrak{A} \longrightarrow (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ such that $\pi^{**} o \rho = k_{\mathfrak{A}}$, where $k_{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathfrak{A}^{**}$ is the natural embedding of $\mathfrak{A}$ into its second dual.

Let $S$ be a semigroup. A continuous function $\omega : S \longrightarrow (0, \infty)$ is a *weight* on $S$ if $\omega(st) \leqslant \omega(s)\omega(t)$, for all $s, t \in S$. Then it is standard that

$$\ell^1(S, \omega) = \left\{ f = \sum_{s \in S} f(s)\delta_s : \|f\|_\omega = \sum_{s \in S} |f(s)|\omega(s) < \infty \right\}$$

is a Banach algebra with the convolution product $\delta_s * \delta_t = \delta_{st}$. These algebras are called *Beurling algebras*.

In this note, we study the earlier mentioned properties of Banach algebras for Beurling algebras. Firstly in Section 2, we characterize amenability and pseudo-amenability of $\ell^1(S, \omega)$, for some certain class of semigroups. We prove that pseudo-amenability of $\ell^1(S, \omega)$, for a left or right zero semigroup $S$, is equivalent to it's amenability and these equivalent conditions imply that $S$ is singleton. We show that the same result holds for $\ell^1(S, \omega)$, whenever $S$ is a rectangular band semigroup and $\omega$ is separable. Further, we investigate biprojectivity of $\ell^1(S, \omega)$ whenever $S$ is either left (right) zero semigroup or a rectangular band semigroup. For a band semigroup $S$, we show that amenability of $\ell^1(S, \omega)$ is equivalent to that of $\ell^1(S)$ and these are equivalent to $S$ being a finite semilattice. We find necessary and sufficient conditions for $\ell^1(S, \omega)^{**}$ to be amenable, where $S$ is an inverse semigroup.

Finally in Section 3, we investigate pseudo-amenability of $L^1(G, \omega)$ where $G$ is a locally compact group and $\omega$ is a weight on $G$. We prove that pseudo-amenability of $L^1(G, \omega)$ implies amenability of $G$, and under a certain condition it implies diagonally boundedness of $\omega$. Next, if $L^1(G, \omega)$ is pseudo-amenable we may obtain a character $\varphi$ on $G$ for which $\varphi \leqslant \omega$.

## 2. Amenability and Pseudo-Amenability of $\ell^1(S,\omega)$

A semigroup $S$ is a *left zero semigroup* if $st = s$, and it is a *right zero semigroup* if $st = t$ for each $s,t \in S$. Then for $f,g \in \ell^1(S,\omega)$, it is obvious that $f * g = \varphi_S(f)g$ if $S$ is a right zero semigroup, and $f * g = \varphi_S(g)f$ if $S$ is a left zero semigroup, where $\varphi_S$ is the *augmentation character* on $\ell^1(S,\omega)$.

We extend the results for $\ell^1(S)$ in [1, 2] to the weighted case $\ell^1(S,\omega)$.

**Proposition 2.1.** *Suppose that $S$ is a right (left) zero semigroup and $\omega$ be a weight on $S$. Then $\ell^1(S,\omega)$ is biprojective.*

**Proof.** We only give the proof in the case $S$ is a right zero semigroup. Define $\rho : \ell^1(S,\omega) \longrightarrow \ell^1(S,\omega)\widehat{\otimes}\ell^1(S,\omega)$ by $\rho(f) = \delta_{t_\circ} \otimes f,$, where $t_0$ is an arbitrary element $S$. Then for each $f,g \in \ell^1(S,\omega)$ we have

$$\rho(f*g) = \delta_{t_\circ} \otimes (f*g) = \varphi_S(f)(\delta_{t_\circ} \otimes g) = (f*\delta_{t_0}) \otimes g = f\ .\ (\delta_{t_0} \otimes g) = f\ .\ \rho(g)$$

and similarly $\rho(f*g) = \rho(f)\ .\ g$. Further, $\pi\rho$ is the identity map on $\ell^1(S,\omega)$, as required. □

**Remark 2.2.** *It is known that every biprojective Banach algebra is biflat. Hence Proposition 2.1 shows that for every right or left zero semigroup $S$, $\ell^1(S,\omega)$ is biflat.*

Given two semigroups $S_1$ and $S_2$, we say that a weight $\omega$ on $S := S_1 \times S_2$ is *separable* if there exist two weights $\omega_1$ and $\omega_2$ on $S_1$ and $S_2$, respectively such that $\omega = \omega_1 \otimes \omega_2$. It is easy to verify that $\ell^1(S,\omega) \cong \ell^1(S_1,\omega_1)\widehat{\otimes}\ell^1(S_2,\omega_2)$.

Let $S$ be a semigroup and let $E(S) = \{p \in S : p^2 = p\}$. We say that $S$ is a *band semigroup* if $S = E(S)$. A band semigroup $S$ satisfying $sts = s$, for each $s,t \in S$ is called a *rectangular band semigroup*. For a rectangular band semigroup $S$, it is known that $S \simeq L \times R$, where $L$ and $R$ are left and right zero semigroups, respectively [7, Theorem 1.1.3].

**Proposition 2.3.** *Let $S$ be a rectangular band semigroup and $\omega$ be a separable weight on $S$. Then $\ell^1(S,\omega)$ is biprojective, and so it is biflat.*

**Proof.** In view of earlier argument, it follows From Proposition 2.1, and then from [12, Proposition 2.4]. □

**Theorem 2.4.** *Let $S$ be a rectangular band semigroup and $\omega$ be a weight on $S$. Then $\ell^1(S,\omega)$ is amenable if and only if $S$ singleton.*

**Proof.** From [13, Theorem 3.6], $\ell^1(S)$ is amenable. Then it is immediate by [1, Theorem 3.3]. □

For a semigroup $S$, we denote by $S^{op}$ the semigroup whose underlying space is $S$ but whose multiplication is the multiplication in $S$ reversed.

**Proposition 2.5.** *Let $S$ be a right (left) zero semigroup and $\omega$ be a weight on $S$. Then $\ell^1(S,\omega)$ is amenable if and only if $S$ is singleton.*

**Proof.** Suppose that $S$ is a left zero semigroup, and that $\ell^1(S,\omega)$ is amenable. Then $S^{op}$ is a right zero semigroup. It is readily seen that $S \times S^{op}$ is a rectangular band semigroup, and $\ell^1(S^{op},\omega)$ is amenable. Hence $\ell^1(S,\omega)\widehat{\otimes}\ell^1(S^{op},\omega) \cong \ell^1(S \times S^{op}, \omega \otimes \omega)$ is amenable. Now by Theorem 2.4, $S$ is singleton. $\square$

Let $\mathfrak{A}$ be Banach algebra, $\mathcal{I}$ be a *semilattice* (i.e., $\mathcal{I}$ is a commutative band semigroup) and $\{\mathfrak{A}_\alpha : \alpha \in \mathcal{I}\}$ be a collection of closed subalgebras of $\mathfrak{A}$. Then $\mathfrak{A}$ is $\ell^1$-*graded* of $\mathfrak{A}_\alpha$'s over the semilattice $\mathcal{I}$, denoted by $\mathfrak{A} = \bigoplus_{\alpha\in\mathcal{I}}^{\ell^1} \mathfrak{A}_\alpha$, if it is $\ell^1$-directsum of $\mathfrak{A}_\alpha$'s as Banach space such that $\mathfrak{A}_\alpha\mathfrak{A}_\beta \subseteq \mathfrak{A}_{\alpha\beta}$, for each $\alpha, \beta \in \mathcal{I}$.

Suppose that $S^1$ is the unitization of a semigroup $S$. An equivalence relation $\tau$ on $S$ is defined by $s\tau t \iff S^1sS^1 = S^1tS^1$, for all $s, t \in S$. If $S$ is a band semigroup, then by [7, Theorem 4.4.1], $S = \bigcup_{\alpha\in\mathcal{I}} S_\alpha$ is a semilattice of rectangular band semigroups, where $\mathcal{I} = \frac{S}{\tau}$ and for each $\alpha = [s] \in \mathcal{I}$, $S_\alpha = [s]$.

**Theorem 2.6.** *Let $S$ be a band semigroup and $\omega$ be a weight on $S$. Then the following are equivalent:*

*(i) $\ell^1(S,\omega)$ is amenable.*

*(ii) $S$ is finite and each $\tau-$class is singleton.*

*(iii) $\ell^1(S)$ is amenable.*

*(iv) $S$ is a finite semilattice.*

**Proof.** The implications $(ii)$ to $(iv)$ are equivalent [1, Theorem 3.5]. We establish $(i) \longrightarrow (ii)$ and $(iv) \longrightarrow (i)$.

$(i) \longrightarrow (ii)$ If $\ell^1(S,\omega)$ is amenable, then $E(S) = S$ is finite and so $\mathcal{I} = \frac{S}{\tau}$ is a finite semilattice. Hence $\ell^1(S,\omega) \cong \bigoplus_{\alpha\in\mathcal{I}}^{\ell^1} \ell^1(S_\alpha,\omega_\alpha)$, where $\omega_\alpha = \omega|_{S_\alpha}$. Then by [5, Proposition 3.1], each $\ell^1(S_\alpha,\omega_\alpha)$ is amenable. Now by Theorem 2.4, $S_\alpha$ is singleton for each $\alpha \in \mathcal{I}$, as required.

$(iv) \longrightarrow (i)$ In this case $\ell^1(S,\omega) \cong \ell^1(S)$, and $\ell^1(S)$ is amenable. $\square$

**Theorem 2.7.** *Let $S$ be a rectangular band semigroup, and let $\omega$ be a separable weight on $S$. Then $\ell^1(S,\omega)$ is pseudo-amenable if and only if $S$ is singleton.*

**Proof.** There is a left zero semigroup $L$ and a right zero semigroup $R$, and there are weights $\omega_L$ and $\omega_R$ on $L$ and $R$, respectively such that $S \cong L \times R$ and $\omega = \omega_L \otimes \omega_R$. We have $\ell^1(S,\omega) \cong \ell^1(L,\omega_L)\widehat{\otimes}\ell^1(R,\omega_R)$. Hence the map $\theta : \ell^1(S,\omega) \longrightarrow \ell^1(L,\omega_L)$ defined by $\theta(f \otimes g) = \varphi_R(g)f$ for $f \in \ell^1(L,\omega_L)$ and $g \in \ell^1(R,\omega_R)$, is an epimorphism of Banach algebras, whereas $\varphi_R$ is the *aug-*

*mentation* character on $\ell^1(R,\omega_R)$. Whence $\ell^1(L,\omega_L)$ has left and right approximate identity. Therefore $L$ is singleton, because it is left zero semigroup. Similarly $R$ is singleton, so is $S$. $\square$

**Corollary 2.8.** *Let $S$ be a right (left) zero semigroup and $\omega$ be a weight on $S$. Then the following are equivalent:*

*(i) $\ell^1(S,\omega)$ is pseudo-amenable.*

*(ii) $S$ is singleton.*

*(iii) $\ell^1(S,\omega)$ is amenable.*

**Proof.** The implication $(ii) \longleftrightarrow (iii)$ is Proposition 2.5. For $(i) \longrightarrow (ii)$, we apply Theorem 2.7 for the rectangular band semigroup $S \times S^{op}$ with $\omega_L = \omega_R = \omega$. $\square$

The following is a combination of Theorems 2.4 and 2.7. Notice that in Theorem 2.4, we need not $\omega$ to be separable.

**Corollary 2.9.** *Let $S$ be a rectangular band semigroup, and let $\omega$ be a separable weight on $S$. Then the following are equivalent:*

*(i) $\ell^1(S,\omega)$ is pseudo-amenable.*

*(ii) $S$ is singleton.*

*(iii) $\ell^1(S,\omega)$ is amenable.*

For the left cancellative semigroups we have the following.

**Theorem 2.10.** *Suppose that $S$ is a left cancellative semigroup and $\omega$ is a weight on $S$. If $\ell^1(S,\omega)$ is pseudo-amenable, then $S$ is a group.*

**Proof.** This is a more or less verbatim of the proof of [2, Theorem 3.6 $(i) \longrightarrow (ii)$]. $\square$

Let $(P,\leqslant)$ is a partially ordered set. Then $(P,\leqslant)$ is *locally finite* if $(x] = \{y \in S : y \leqslant x\}$ is finite for every $x \in S$, and it is *uniformly locally finite* if $\sup\{|(x]| : s \in S\} < \infty$.

We recall that a semigroup $S$ is an *inverse semigroup* if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. The maximal subgroup of $S$ at $p \in E(S)$ is denoted by $G_p$. It is known that $G_p = \{s \in S : ss^* = s^*s = p\}$.

**Theorem 2.11.** *Let $S$ be an inverse semigroup, let $\omega$ be a weight on $S$, and let $\ell^1(S,\omega)$ has a bounded approximate identity. Then the following are equivalent:*

*(i) $\ell^1(S,\omega)^{**}$ is amenable;*

*(ii) $\ell^1(S)$ is biprojective and $S$ is finite;*

*(iii)* $\ell^1(S,\omega)^{**}$ *is biprojective.*

**Proof.** $(i) \to (ii)$ Suppose that $\ell^1(S,\omega)^{**}$ is amenable. Hence $\ell^1(S)$ is amenable and $S$ is finite [13, Theorem 3.7]. Then by [12, Thorem 3.7 $(i)$], $S$ is uniformly locally finite and for each $p \in E(S)$, $G_p$ is an amenable group. Finiteness of $S$ implies that $G_p$ is finite and then [12, Theorem 3.7 $(ii)$] shows that $\ell^1(S)$ is biprojective.

$(ii) \to (iii)$ Since $S$ is finite, $\ell^1(S) \cong \ell^1(S,\omega)$, and $\ell^1(S)$ is finite-dimensional. Therefore $\ell^1(S)^{**} \cong \ell^1(S)$, and so $\ell^1(S,\omega)^{**}$ is biprojective.

$(iii) \to (i)$ Biprojectivity of $\ell^1(S,\omega)^{**}$ implies its biflatness. Thus, since $\ell^1(S,\omega)^{**}$ has a bounded approximate identity, $\ell^1(S,\omega)^{**}$ is amenable. □

## 3. Pseudo-Amenability of $L^1(G,\omega)$

Throughout $G$ is a locally compact group and $\omega$ is a weight on $G$. The weight $\omega$ is *diagonally bounded* if $\sup_{g\in G}\omega(g)\omega(g^{-1}) < \infty$. It seems to be a *right* conjecture that $L^1(G,\omega)$ will fail to be pseudo-amenable whenever $\omega$ is not diagonally bounded. Although we are not able to prove (or disprove) the conjecture, we have the following.

**Theorem 3.1.** *Suppose that there exists an approximate diagonal* $(m_i)_i$ *for* $L^1(G,\omega)$ *such that* $m_i - \delta_g \,.\, m_i \,.\, \delta_{g^{-1}} \longrightarrow 0$ *uniformly on* $G$*. Then* $\omega$ *is diagonally bounded.*

**Proof.** Notice that our assumption is stronger than pseudo-amenability of $L^1(G,\omega)$. We follow the standard argument in [3, Proposition 8.7]. Choose $f \in L^1(G,\omega)$ such that $K := suppf$ is compact and $\int f \neq 0$. Putting $F := f \,.\, \chi_K \in L^\infty(G,\omega^{-1})$, we see that $\pi^*(F) \in L^\infty(G\times G,\omega^{-1}\times\omega^{-1})$ with

$$\pi^*(F)(x,y) = F(xy) = \int \chi_K(xyt)f(t)dt\ .$$

Let $(m_i)_i \subseteq L^1(G\times G,\omega\times\omega)$ be an approximate diagonal for $L^1(G,\omega)$ such that $\delta_g \,.\, m_i \,.\, \delta_{g^{-1}} - m_i \longrightarrow 0$ uniformly on $G$, and $\pi(m_i)f - f \longrightarrow 0$. Then for each $i$

$$\langle \pi^*(F), m_i\rangle = \langle F, \pi(m_i)\rangle = \langle \chi_K, \pi(m_i)f\rangle \longrightarrow \langle \chi_K, f\rangle = \int f\ .$$

Consequently

$$\lim_i \langle \pi^*(F), m_i\rangle \neq 0\ . \qquad (1)$$

We define $E := KK^{-1}$, and $A := \{(x,y) \in G\times G \ :\ xy \in E\}$. For $r > 0$, we define $A_r := \{(x,y) \in A \ :\ \omega(x)\omega(y) < r\}$, and $B_r := \{(x,y) \in$

$A : \omega(x)\omega(y) \geqslant r\}$. Obviously, $\pi^*(F)\chi_{A_r}$ and $\pi^*(F)\chi_{B_r}$ both are in $L^\infty(G \times G, \omega^{-1} \times \omega^{-1})$, and $\pi^*(F) = \pi^*(F)\chi_A = \pi^*(F)\chi_{A_r} + \pi^*(F)\chi_{B_r}$. For every $i$, it is easy to see that

$$|\langle \pi^*(F)\chi_{B_r}, m_i\rangle| \leqslant ||m_i||\ ||F||\ r^{-1}\ c_1$$

where $c_1 := \sup_{t\in E}\omega(t)$. Hence

$$\lim_{r\longrightarrow\infty}\langle \pi^*(F)\chi_{B_r}, m_i\rangle = 0\ . \qquad (2)$$

Next, for every $g \in G$, $r > 0$, and $i$, we obtain

$$|\langle \pi^*(F)\chi_{A_r}, \delta_g\ .\ m_i\ .\ \delta_{g^{-1}}\rangle| \leqslant ||m_i||\ ||F||\ r\ c_1\ c_2^2 1\omega(g)\omega(g^{-1})$$

where $c_2 := \sup_{t\in E^{-1}}\omega(t)$. Therefore

$$\begin{aligned}|\langle \pi^*(F)\chi_{A_r}, m_i\rangle| &\leqslant |\langle \pi^*(F)\chi_{A_r}, m_i - \delta_g\ .\ m_i\ .\ \delta_{g^{-1}}\rangle| + |\langle \pi^*(F)\chi_{A_r}, \delta_g\ .\ m_i\ .\ \delta_{g^{-1}}\rangle|\\ &\leqslant ||\pi^*(F)||\ \sup_{g\in G}||m_i - \delta_g\ .\ m_i\ .\ \delta_{g^{-1}}|| + ||m_i||\ ||F||\ r\ c_1\ c_2^2 1\omega(g)\omega(g^{-1})\ . \qquad (3)\end{aligned}$$

Towards a contradiction, we assume that $\omega$ is not diagonally bounded. Then there is a sequence $(g_n)_n$ in $G$ such that $\lim_n \omega(g_n)\omega(g_n^{-1}) = \infty$ . Whence, it follows from (3) that for each $i$ and $r > 0$

$$|\langle \pi^*(F)\chi_{A_r}, m_i\rangle| \leqslant ||\pi^*(F)||\ \sup_{g\in G}||m_i - \delta_g\ .\ m_i\ .\ \delta_{g^{-1}}||\ . \qquad (4)$$

Hence

$$|\langle \pi^*(F), m_i\rangle| \leqslant ||\pi^*(F)||\ \sup_{g\in G}||m_i - \delta_g\ .\ m_i\ .\ \delta_{g^{-1}}|| + |\langle \pi^*(F)\chi_{B_r}, m_i\rangle|\ .$$

Putting (2) and (4) together, we may see that

$$\lim_i\langle \pi^*(F), m_i\rangle = 0$$

contradicting (1). □

**Theorem 3.2.** *Suppose that $L^1(G,\omega)$ is pseudo-amenable, and that $\omega$ is bounded away from 0. Then $G$ is amenable.*

**Proof.** Since $L^1(G,\omega)$ is unital, pseudo-amenability and approximate amenability are the same [4, Proposition 3.2]. Now, it is immediate by [3, Proposition 8.1]. □

We conclude by the following which is an analogue of [3, Proposition 8.9].

**Proposition 3.3.** *Let $L^1(G,\omega)$ be pseudo-amenable. Then there is a continuous positive character $\varphi$ on $G$ such that $\varphi \leqslant \omega$.*

**Proof.** Suppose that $(m_i)_i \subseteq L^1(G\times G, \omega\times\omega)$ be an approximate diagonal for $L^1(G,\omega)$. For each $i$ and $f \in L^\infty(G\times G,\omega^{-1}\times\omega^{-1})^+$ we define

$$\widetilde{m_i}(f) := \sup\{Re\langle m_i,\psi\rangle \ : \ 0 \leqslant |\psi| \leqslant f, \ \psi \in L^\infty(G\times G,\omega^{-1}\times\omega^{-1})\}.$$

Then $\widetilde{m_i} \neq 0$ on $L^\infty(G\times G,\omega^{-1}\times\omega^{-1})^+$ and we may extend $\widetilde{m_i}$ to a bounded linear functional on $L^\infty(G\times G,\omega^{-1}\times\omega^{-1})$ in the obvious manner. It is readily seen that $\widetilde{m_i}\neq 0$, and $\langle \widetilde{m_i}, f\rangle \geqslant 0$, for every $f \in L^\infty(G\times G,\omega^{-1}\times\omega^{-1})^+$. One may also check that $\delta_{g^{-1}}\ .\ \widetilde{m_i}\ .\ \delta_g - \widetilde{m_i} \longrightarrow 0$ for every $g\in G$, (not uniformly on G).

Putting $\widetilde{\omega}(x) := \sup_{g\in G}\omega(g^{-1}xg)$, $x\in G$. Then $\widetilde{\omega}\in L^\infty(G,\omega^{-1})$, $\widetilde{\omega}(xy) = \widetilde{\omega}(yx)$, $\pi^*(\widetilde{\omega}) \in L^\infty(G\times G,\omega^{-1}\times\omega^{-1})$, and $\delta_g\ .\ \pi^*(\widetilde{\omega})\ .\ \delta_{g^{-1}} = \pi^*(\widetilde{\omega})$.

Take $f\in C_c(G)^+$ with $\int f = 1$, and then $h := f\ .\ \chi_K$, where $K := suppf$. One may see that $h$ is continuous, and there is $c>0$ such that $\pi^*(\widetilde{\omega}) \geqslant c\pi^*(h)$. Hence

$$\begin{aligned}\lim_i \langle\widetilde{m_i},\pi^*(\widetilde{\omega})\rangle \geqslant c\lim_i\langle\widetilde{m_i},\pi^*(h)\rangle \geqslant c\lim_i Re\langle m_i,\pi^*(h)\rangle = c\lim_i Re\langle\pi(m_i),h\rangle \\ = c\lim_i Re\langle\pi(m_i)\ .\ f,\chi_K\rangle = cRe\langle f,\chi_K\rangle = c > 0\ .\end{aligned}$$

Therefore there is $i_0$ for which $\langle\widetilde{m_{i_0}},\pi^*(\widetilde{\omega})\rangle > 0$. Set $F := \langle\widetilde{m_{i_0}},\pi^*(\widetilde{\omega})\rangle^{-1}\pi^*(\widetilde{\omega})$, and for $g\in G$ we put

$$A_g(x,y) := 12(\log\omega(gx)\omega(gy^{-1})\omega(x)\omega(y^{-1}))F(x,y)\ , \quad (x,y\in G)\ .$$

Finally, for each $g\in G$, we define $\varphi(g) := \exp\langle\widetilde{m_{i_0}},A_g\rangle$. A similar argument used in [3, Proposition 8.9], shows that $\varphi$ is the desired character on $G$. $\square$

# References

[1] M. Essmaili and A. Medghalchi, Biflatness of certain semigroup algebra, *Bull. Iran. Math. Soc.*, 39 (2013), 959-969.

[2] M. Essmaili, M. Rostami, and A. R. Medghalchi, Pseudo-contractibility and Pseudo-amenability of semigroup algebras, *Arch. Math.*, 97 (2011), 167-177.

[3] F. Ghahramani, R. J. Loy, and Y. Zhang, Generalized notions of amenability II, *J. Funct. Anal.*, 254 (2008), 1776-1810.

[5] M. Ghandahari, H. Hatami, and N. Spronk, Amenability constant for semilattice algebras, *Semigroup Forum*, 79 (2) (2009), 279-297.

[6] A. Ya. Helemskiĭ, Flat Banach modules and amenable algebras, *Trans. Moscow Math. Soc.*, 47 (1985), 199-224.

[7] J. Howie, *Fundemental of Semigroup Theory,* The Clarendon Press, Oxford University Press, New York 1995.

[8] B. E. Johnson, Cohomology in Banach algebras, *Mem. Amer. Math. Soc.,* 127 (1972).

[9] S. M. Maepa and O. T. Mewomo, On character amenability of semigroup algebras, *Quaest. Math.,* 39 (3) (2016), 307-318.

[10] O. T. Mewomo and S. M. Maepa, On character amenability of Beurling and second dual algebras, *Acta Univ. Apulensis Math. Inform.,* 38 (4) (2014), 67-80.

[11] O. T. Mewomo, Various notions of amenability in Banach algebras, *Expo. Math.,* 29 (2011), 283-299.

[12] P. Ramsden, Biflatness of semigroup algebras, *Semigroup Forum*, 79 (3) (2009), 515-530.

[13] M. Soroshmehr, Weighted Ress matrix semigroups and their applications, *Arch. Math.*, 100 (2) (2013), 139-147.

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